MATH1510
Financial Mathematics I

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Description of the module

This is the description of the module as it appears in the module catalogue.

Objectives

Introduction to mathematical modelling of financial and insurance markets with particular emphasis on the time-value of money and interest rates. Introduction to simple financial instruments. This module covers a major part of the Faculty and Institute of Actuaries CT1 syllabus (Financial Mathematics, core technical).

Learning outcomes

On completion of this module, students should be able to understand the time value of money and to calculate interest rates and discount factors. They should be able to apply these concepts to the pricing of simple, fixed-income financial instruments and the assessment of investment projects.

Syllabus

- Interest rates. Simple interest rates. Present value of a single future payment. Discount factors.
- Effective and nominal interest rates. Real and money interest rates. Compound interest rates. Relation between the time periods for compound interest rates and the discount factor.
- Compound interest functions. Annuities and perpetuities.
- Loans.
- Introduction to fixed-income instruments. Generalized cashflow model.
- Net present value of a sequence of cashflows. Equation of value. Internal rate of return. Investment project appraisal.
- Examples of cashflow patterns and their present values.
- Elementary compound interest problems.

Reading list

These lecture notes are based on the following books:

1. Samuel A. Broverman, Mathematics of Investment and Credit, 4th ed., ACTEX Publications, 2008. ISBN 978-1-56698-657-1.
2. The Faculty of Actuaries and Institute of Actuaries, Subject CT1: Financial Mathematics, Core Technical. Core reading for the 2009 examinations.
3. Stephen G. Kellison, The Theory of Interest, 3rd ed., McGraw-Hill, 2009. ISBN 978-007-127627-6.
4. John McCutcheon and William F. Scott, An Introduction to the Mathematics of Finance, Elsevier Butterworth-Heinemann, 1986. ISBN 0-7506-0092-6.
5. Petr Zima and Robert L. Brown, Mathematics of Finance, 2nd ed., Schaum's Outline Series, McGraw-Hill, 1996. ISBN 0-07-008203.

The syllabus for the MATH1510 module is based on Units 1–9 and Unit 11 of book 2. The remainder forms the basis of MATH2510 (Financial Mathematics II). The book 2 describes the first exam that you need to pass to become an accredited actuary in the UK. It is written in a concise and perhaps dry style.

These lecture notes are largely based on Book 4. Book 5 contains many exercises, but does not go quite as deep. Book 3 is written from a U.S. perspective, so the terminology is slightly different, but it has some good explanations. Book 1 is written by a professor from a U.S./Canadian background and is particularly good in making connections to applications.

All these books are useful for consolidating the course material. They allow you to gain background knowledge and to try your hand at further exercises. However, the lecture notes cover the entire syllabus of the module.

Organization for 2010/11

Lecturer	Jitse Niesen
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Office	Mathematics 8.22f
Telephone	35870 (from outside: 0113 3435870)
Lectures	Tuesdays 10:00 – 11:00 in Engineering LT B Wednesdays 12:00 – 13:00 in Roger Stevens LT 23 Fridays 14:00 – 15:00 in Roger Stevens LT 23
Example classes	These are in weeks 3, 5, 7, 9 and 11, as follows: • Weds 9:00 – 10:00 in Parkinson B22 • Weds 9:00 – 10:00 in Parkinson B09 • Thurs 9:00 – 10:00 in Mathematics 8.22d
Tutor	Huamao Wang, e-mail: bnhw@leeds.ac.uk
Office hours	Thursdays .... (to be determined) or whenever you find the lecturer and he has time.
Course work	There will be five sets of course work, to be handed in at the start of the lectures on Friday 4 February, 18 February, 4 March, 18 March and 1 April.
Late work	One mark (out of ten) will be deducted for work handed in late on Friday, and one further mark per working day.
Copying	Collaboration is allowed (even encouraged), copying not. See the student handbook for details.
Attendance	Complete and hand in a cover sheet with your course work to report attendance.
Exam	The exam will take place in the period 16 May – 3 June; exact date and location to be announced.
Assessment	The course work counts for 15%, the exam for 85%.
Lecture notes	These notes and supporting materials are available in the Blackboard VLE.

Contents

Chapter 1

The time value of money

Interest is the compensation one gets for lending a certain asset. For instance, suppose that you put some money on a bank account for a year. Then, the bank can do whatever it wants with that money for a year. To reward you for that, it pays you some interest.

The asset being lent out is called the *capital*. Usually, both the capital and the interest is expressed in money. However, that is not necessary. For instance, a farmer may lend his tractor to a neighbour, and get 10% of the grain harvested in return. In this course, the capital is always expressed in money, and in that case it is also called the *principal*.

1.1 Simple interest

Interest is the reward for lending the capital to somebody for a period of time. There are various methods for computing the interest. As the name implies, *simple interest* is easy to understand, and that is the main reason why we talk about it here. The idea behind simple interest is that the amount of interest is the product of three quantities: the rate of interest, the principal, and the period of time. However, as we will see at the end of this section, simple interest suffers from a major problem. For this reason, its use in practice is limited.

Definition 1.1.1 (Simple interest)**.** The interest earned on a capital C lent over a period n at a rate i is niC.

Example 1.1.2. How much interest do you get if you put 1000 pounds for two years in a savings acount that pays simple interest at a rate of 9% per annum? And if you leave it in the account for only half ar year?

Answer. If you leave it for two years, you get

$$2 \cdot 0.09 \cdot 1000 = 180$$

pounds in interest. If you leave it for only half a year, then you get $\frac{1}{2} \cdot 0.09 \cdot 1000 = 45$ pounds.

As this example shows, the rate of interest is usually quoted as a percentage; 9% corresponds to a factor of 0.09. Furthermore, you have to be careful that the rate of interest is quoted using the same time unit as the period. In this example, the period is measured in years, and the interest rate is quoted per annum ("per annum" is Latin for "per year"). These are the units that are used most often. In Section 1.5 we will consider other possibilities.

Example 1.1.3. Suppose you put £1000 in a savings account paying simple interest at 9% per annum for one year. Then, you withdraw the money with interest and put it for one year in another account paying simple interest at 9%. How much do you have in the end?

Answer. In the first year, you would earn $1 \cdot 0.09 \cdot 1000 = 90$ pounds in interest, so you have £1090 after one year. In the second year, you earn $1 \cdot 0.09 \cdot 1090 = 98.1$ pounds in interest, so you have £1188.10 ($= 1090 + 98.1$) at the end of the two years.

Now compare Examples 1.1.2 and 1.1.3. The first example shows that if you invest £1000 for two years, the capital grows to £1180. But the second example shows that you can get £1188.10 by switching accounts after a year. Even better is to open a new account every month.

This inconsistency means that simple interest is not that often used in practice. Instead, savings accounts in banks pay compound interest, which will be introduced in the next section. Nevertheless, simple interest is sometimes used, especially in short-term investments.

1.2 Compound interest

Most bank accounts use *compound interest*. The idea behind compound interest is that in the second year, you should get interest on the interest you earned in the first year. In other words, the interest you earn in the first year is combined with the principal, and in the second year you earn interest on the combined sum.

What happens with the example from the previous section, where the investor put £1000 for two years in an account paying 9%, if we consider compound interest? In the first year, the investor would receive £90 interest (9% of £1000). This would be credited to his account, so he now has £1090. In the second year, he would get £98.10 interest (9% of £1090) so that he ends up with £1188.10; this is the same number as we found before. The capital is multiplied by 1.09 every year: $1.09 \cdot 1000 = 1090$ and $1.09 \cdot 1090 = 1188.1$.

More generally, the interest over one year is iC, where i denotes the interest rate and C the capital at the beginning of the year. Thus, at the end of the year, the capital has grown to $C + iC = (1 + i)C$. In the second year, the principal is $(1 + i)C$ and the interest is computed over this amount, so the interest is $i(1 + i)C$ and the capital has grown to $(1 + i)C + i(1 + i)C = (1 + i)^2C$. In the third year, the interest is $i(1 + i)^2C$ and the capital has grown to $(1 + i)^3C$. This reasoning, which can be made more formal by using complete induction, leads to the following definition.

Definition 1.2.1 (Compound interest)**.** A capital C lent over a period n at a rate i grows to $(1 + i)^nC$.

Example 1.2.2. How much do you have after you put 1000 pounds for two years in a savings acount that pays compound interest at a rate of 9% per annum? And if you leave it in the account for only half ar year?

Answer. If you leave it in the account for two years, then at the end you have

$$(1 + 0.09)^2 \cdot 1000 = 1188.10,$$

as we computed above. If you leave it in the account for only half a year, then at the end you have

$$(1 + 0.09)^{1/2} \cdot 1000 = \sqrt{1.09} \cdot 1000 = 1044.03$$

pounds (rounded to the nearest penny). This is 97p less than the 45 pounds interest you get if the account would pay simple interest at the same rate (see Example 1.1.2).

Example 1.2.3. Suppose that a capital of 500 dollars earns 150 dollars of interest in 6 years. What was the interest rate if compound interest is used? What if simple interest is used?

Answer. The capital accumulated to \$650, so in the case of compound interest we have to solve the rate i from the equation

$$\begin{aligned}(1+i)^6 \cdot 500 = 650 &\iff (1+i)^6 = 1.3 \\ &\iff 1 + i = 1.3^{1/6} = 1.044698\ldots \\ &\iff i = 0.044698\ldots\end{aligned}$$

Thus, the interest rate is 4.47%, rounded to the nearest basis point (a basis point is 0.01%). Note that the computation is the same, regardless of the currency used.

In the case of simple interest, the equation to solve $6 \cdot i \cdot 500 = 150$, so $i = \frac{150}{6 \cdot 500} = 0.05$, so the rate is 5%.

Example 1.2.4. How long does it take to double your capital if you put it in an account paying compound interest at a rate of $7\frac{1}{2}\%$? What if the account pays simple interest?

Answer. The question is for what value of n does a capital C accumulate to $2C$ if $i = 0.075$. So we have to solve the equation $1.075^n C = 2C$. The first step is to divide by C to get $1.075^n = 2$. Then take logarithms:

$$\log(1.075^n) = \log(2) \iff n \log(1.075) = \log(2) \iff n = \frac{\log(2)}{\log(1.075)} = 9.58\ldots$$

So, it takes 9.58 years to double your capital. Note that it does not matter how much you have at the start: it takes as long for one pound to grow to two pounds as for a million pounds to grow to two million.

The computation is simpler for simple interest. We have to solve the equation $n \cdot 0.075 \cdot C = C$, so $n = \frac{1}{0.075} = 13\frac{1}{3}$, so with simple interest it takes $13\frac{1}{3}$ years to double your capital.

More generally, if the interest rate is i, then the time required to double your capital is

$$n = \frac{\log(2)}{\log(1+i)}.$$

We can approximate the denominator by $\log(1 + i) \approx i$ for small i; this is the first term of the Taylor series of $\log(1 + i)$ around $i = 0$ (note that, as is common in mathematics, "log" denotes the *natural* logarithm). Thus, we get $n \approx \frac{\log(2)}{i}$. If instead of the interest rate i we use the percentage $p = 100i$, and we approximate $\log(2) = 0.693\ldots$ by 0.72, we get

$$n \approx \frac{72}{p}.$$

This is known as the *rule of 72*: To calculate how many years it takes you to double your money, you divide 72 by the interest rate expressed as a percentage. Let us return to the above example with a rate of $7\frac{1}{2}\%$. We have $p = 7\frac{1}{2}$ so we compute $72/7\frac{1}{2} = 9.6$, which is very close to the actual value of $n = 9.58$ we computed before.

The rule of 72 can already be found in a Italian book from 1494: *Summa de Arithmetica* by Luca Pacioli. The use of the number 72 instead of 69.3 has two advantages: many numbers divide 72, and it gives a better approximation for rates above 4% (remember that the Taylor approximation is centered around $i = 0$; it turns out that it is slightly too small for rates of 5–10% and using 72 instead of 69.3 compensates for this).

Remember that with simple interest, you could increase the interest you earn by withdrawing your money from the account halfway. Compound interest has the desirable property that this does not make a difference. Suppose that you put your money m years in one account and then n years in another account, and that both account pay compount interest at a rate i. Then, after the first m years, your capital has grown to $(1+i)^m C$. You withdraw that and put it in another account for n years, after which your capital has grown to $(1+i)^n(1+i)^m C$. This is the same as what you would get if you had kept the capital in the same account for $m+n$ years, because

$$(1+i)^n(1+i)^m C = (1+i)^{m+n} C.$$

This is the reason why compound interest is used so much in practice. Unless noted otherwise, interest will always refer to compound interest.

1.3 Comparing simple and compound interest

Simple interest is defined by the formula "interest = inC." Thus, in n years the capital grows from C to $C + niC = (1+ni)C$. Simple interest and compound interest compare as follows:

$$\begin{aligned} \text{simple interest:} \quad & \text{capital after } n \text{ years} = (1+ni)C \\ \text{compound interest:} \quad & \text{capital after } n \text{ years} = (1+i)^n C \end{aligned}$$

These formulas are compared in Figure 1.1. The left plot shows how a principal of 1 pound grows under interest at 9%. The dashed line is for simple interest and the solid curve for compound interest. We see that compound interest pays out more in the long term. A careful comparison shows that for periods less than a year simple interest pays out more, while compound interest pays out more if the period is longer than a year. This agrees with what we found before. A capital of £1000, invested for half a year at 9%, grows to £1045 under simple interest and to £1044.03 under compound interest, while the same capital invested for

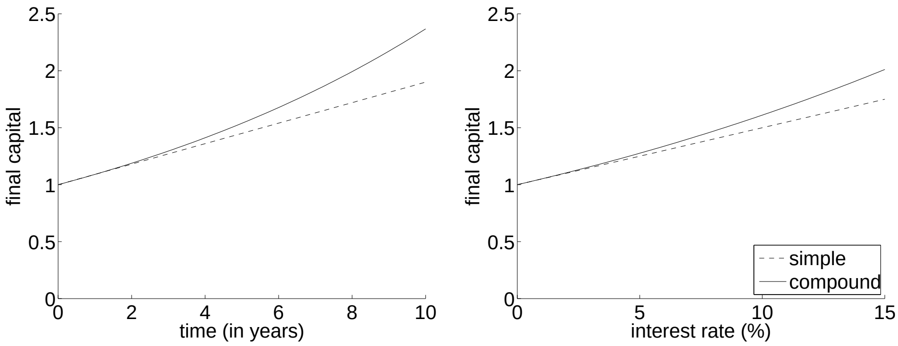

Figure 1.1: Comparison of simple interest and compound interest. The left figure plots the growth of capital in time at a rate of 9%. The right figure plots the amount of capital after 5 years for various interest rates.

two years grows to £1180 under simple interest and £1188.10 under compound interest. The difference between compound and simple interest get bigger as the period gets longer.

This follows from the following algebraic inequalities: if i is positive, then

$$(1+i)^n < 1+ni \quad \text{if} \quad n<1,$$
$$(1+i)^n > 1+ni \quad \text{if} \quad n>1.$$

These will not be proven here. However, it is easy to see that the formulas for simple and compound interest give the same results if $n = 0$ and $n = 1$. Now consider the case $n = 2$. A capital C grows to $(1 + 2i)C$ under simple interest and to $(1 + i)^2C = (1 + 2i + i^2)C$ under compound interest. We have $(1 + 2i + i^2)C > (1 + 2i)C$ (because C is positive), so compound interest pays out more than simple interest.

The right plot in Figure 1.1 shows the final capital after putting a principal of 1 pound away for five years at varying interest rates. Again, the dashed line corresponds to simple interest and the solid curve corresponds to compound interest. We see again that compound interest pays out more, as $n = 5$ is greater than 1. However, the plot also shows that the difference is smaller if the interest rate is small.

This can be explained with the theory of Taylor series. A capital C will grow in n years to $(1 + i)^nC$. The Taylor series of $f(i) = (1 + i)^nC$ around $i = 0$ is

$$f(0) + f'(0)i + \tfrac{1}{2}f''(0)i^2 + \cdots = C + niC + \tfrac{1}{2}n(n-1)i^2C + \cdots .$$

The first two terms are $C + niC = (1 + ni)C$, which is precisely the formula for simple interest. Thus, you can use the formula for simple interest as an approximation for compound interest; this approximation is especially good if the rate of interest is small. Especially in the past, people often used simple interest instead of compound interest, notwithstanding the inconsistency of simple interest, to simplify the computations.

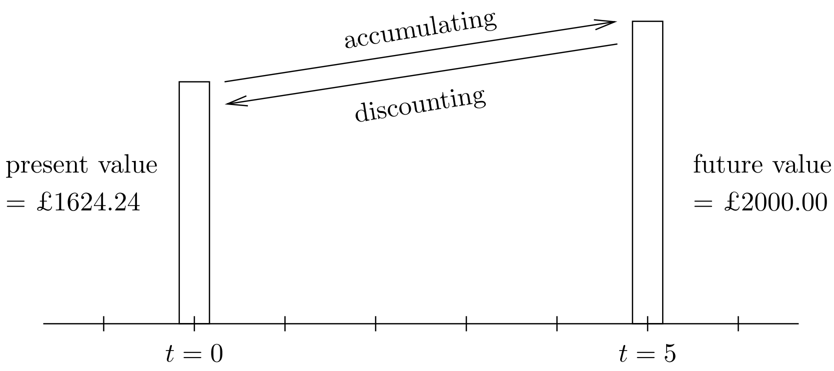

Figure 1.2: The time value of money: £1624.24 now is equivalent to £2000 in five years at a rate of $4\frac{1}{4}\%$.

1.4 Discounting

The formula for compound interest relates four quantities: the capital C at the start, the interest rate i, the period n, and the capital at the end. We have seen how to calculate the interest rate (Example 1.2.3), the period (Example 1.2.4), and the capital at the end (Example 1.2.2). The one remaining possibility is covered in the next example.

Example 1.4.1. How much do you need to invest now to get £2000 after five years if the rate of interest is $4\frac{1}{4}\%$?

Answer. One pound will accumulate to $(1 + 0.0425)^5 = 1.2313466$ in five years, so you need to invest $2000/1.2313466 = 1624.24$ pounds.

We say that £1624.24 now is equivalent to £2000 in five years at a rate of $4\frac{1}{4}\%$. We call £1624.24 the *present value* and £2000 the *future value*. When you move a payment forward in time, it *accumulates*; when you move it backward, it is *discounted* (see Figure 1.2).

This shows that money has a *time value*: the value of money depends on the time. £2000 now is worth more than £2000 in five years' time. In financial mathematics, all payments must have a date attached to them.

More generally, suppose the interest rate is i. How much do you need to invest to get a capital C after one time unit? The answer is $\frac{1}{1+i}C$. The factor

$$v = \frac{1}{1+i}. \tag{1.1}$$

is known as the discount factor. It is the factor with which you have to multiply a payment to shift it backward by one year (see Figure 1.3). If the interest rate is $4\frac{1}{4}\%$, then the discount factor is $\frac{1}{1.0425} = 0.95923$.

Provided the interest rate is not too big, the discount factor is close to one. Therefore people often use the rate of discount $d = 1 - v$, usually expressed as a percentage (compare how the interest rate i is used instead of the "accumulation factor" $1 + i$). In our example, the rate of discount is 0.04077 or 4.077%.

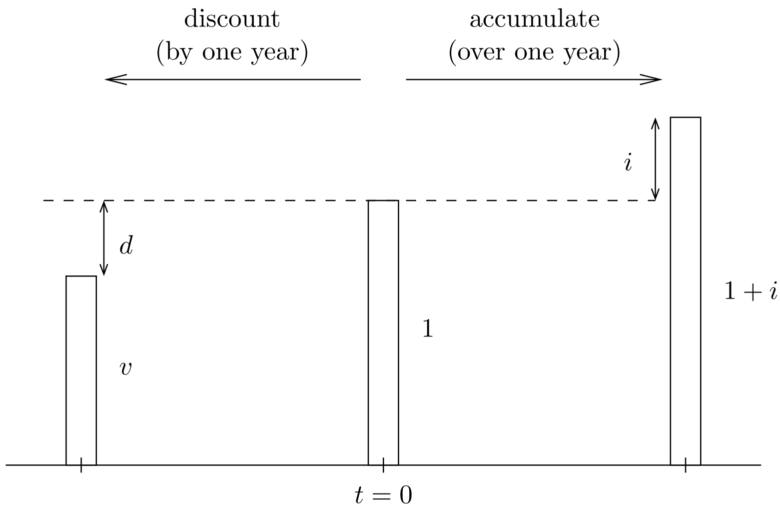

Figure 1.3: The relation between the interest rate i, the rate of discount d and the discount factor v.

Example 1.4.2. Suppose that the interest rate is 7%. What is the present value of a payment of €70 in a year's time?

Answer. The discount factor is $v = 1/1.07 = 0.934579$, so the present value is $0.934579 \cdot 70 = 65.42$ euro (to the nearest cent).

Usually, interest is paid *in arrears*. If you borrow money for a year, then at the end of the year you have to pay the money back plus interest. However, there are also some situations in which the interest is paid in advance. The rate of discount is useful in these situations, as the following example shows.

Example 1.4.3. Suppose that the interest rate is 7%. If you borrow €1000 for a year and you have to pay interest at the start of the year, how much do you have to pay?

Answer. If interest were to be paid in arrears, then you would have to pay $0.07 \cdot 1000 = 70$ euros at the end of the year. However, you have to pay the interest one year earlier. As we saw in Example 1.4.2, the equivalent amount is $v \cdot 70 = 65.42$ euros.

There is another way to arrive at the answer. At the start of the year, you get €1000 from the lender but you have to pay interest immediately, so in effect you get less from the lender. At the end of the year, you pay €1000 back. The amount you should get at the start of the year should be equivalent to the €1000 you pay at the end of the year. The discount factor is $v = 1/1.07 = 0.934579$, so the present value of the €1000 at the end of the year is €934.58. Thus, the interest you have to pay is €1000 − €934.58 = €65.42.

In terms of the interest rate $i = 0.07$ and the capital $C = 1000$, the first method calculates ivC and the second method calculates $C - vC = (1 - v)C = dC$. Both methods yield the same answer, so we arrive at the important relation

$$d = iv. \tag{1.2}$$

We can check this relation algebraically. We found before, in equation (1.1), that the discount factor is

$$v = \frac{1}{1+i}.$$

The rate of discount is

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i}. \tag{1.3}$$

Comparing these two formulas, we find that indeed $d = iv$.

We summarize this discussion with a formal definition of the three quantities d, i and v.

Definition 1.4.4. The *rate of interest* i is the interest paid at the end of a time unit divided by the capital at the beginning of the time unit. The *rate of discount* d is the interest paid at the beginning of a time unit divided by the capital at the end of the time unit. The *discount factor* v is the amount of money one needs to invest to get one unit of capital after one time unit.

This definition concerns periods of one year (assuming that time is measured in years). In Example 1.4.1, we found that the present value of a payment of £2000 due in five years is £1624.24, if compound interest is used at a rate of $4\frac{1}{4}\%$. This was computed as $2000/(1+0.0425)^5$. The same method can be used to find the present value of a payment of C due in n years if compound interest is used at a rate i. The question is: which amount x accumulates to C in n years? The formula for compound interest yields that $(1+i)^n x = C$, so the present value x is

$$\frac{C}{(1+i)^n} = v^n C = (1-d)^n C. \tag{1.4}$$

This is called *compound discounting*, analogous with compound interest.

There is another method, called *simple discounting* (analogous to simple interest) or *commercial discounting*. This is defined as follows. The present value of a payment of C due in n years, at a rate of simple discount of d, is $(1 - nd)C$.

Simple discounting is not the same as simple interest. The present value of a payment of C due in n years, at a rate of simple interest of i, is the amount x that accumulates to C over n years. Simple interest is defined by $C = (1+ni)x$, so the present value is $x = (1+ni)^{-1}C$.

Example 1.4.5. What is the present value of £6000 due in a month assuming 8% p.a. simple discount? What is the corresponding rate of (compound) discount? And the rate of (compound) interest? And the rate of simple interest?

Answer. One month is $\frac{1}{12}$ year, so the present value of is $(1 - \frac{1}{12} \cdot 0.08) \cdot 6000 =$ 5960 pounds. We can compute the rate of (compound) discount d from the formula "present value $= (1-d)^n C$":

$$\begin{aligned} 5960 = (1-d)^{1/12} \cdot 6000 &\implies (1-d)^{1/12} = \tfrac{5960}{6000} = 0.993333 \\ &\implies 1 - d = 0.993333^{12} = 0.922869 \\ &\implies d = 0.077131. \end{aligned}$$

Thus, the rate of discount is 7.71%. The rate of (compound) interest i follows from

$$\frac{1}{1+i} = 1 - d = 0.922869 \implies 1 + i = 1.083577$$

so the rate of (compound) interest is 8.36%. Finally, to find the rate of simple interest, solve $5960 = (1 + \frac{1}{12}i)^{-1}6000$ to get $i = 0.080537$, so the rate of simple interest is 8.05%.

One important application for simple discount is U.S. Treasury Bills. However, it is used even less in practice than simple interest.

1.5 Interest payable monthly, quarterly, etc.

Up to now, we assumed that interest is paid once a year. In practice interest is often paid more frequently, for instance quarterly (four times a year). This is straightforward if the interest rate is also quoted per quarter, as the following example shows.

Example 1.5.1. Suppose that you save £1000 in an account that pays 2% interest every quarter. How much do you have in one year, if the interest is paid in the same account?

Answer. We can use the formula for compound interest in Definition 1.2.1, which says that a capital C accumulates to $(1 + i)^n C$ over a period n, if the rate is i. The rate $i = 0.02$ is measured in quarters, so we also have to measure the period n in quarters. One year is four quarters, so the capital accumulates to $1.02^4 \cdot 1000 = 1082.43$ pounds.

However, interest rates are usually not quoted per quarter even if interest is paid quarterly. The rate is usually quoted per annum (p.a.). In the above example, with 2% per quarter, the interest rate would be quoted as 8% p.a. payable quarterly. This rate is called the *nominal interest rate* payable quarterly. You may also see the words "convertible" or "compounded" instead of "payable".

It may seem more logical to quote the rate as 8.243%. After all, we computed that £1000 accumulates to £1082.43 in a year. The rate of 8.243% is called the *effective interest rate*. It often appears in advertisements in the U.K. as the Annual Equivalent Rate (AER). The effective interest rate corresponds to the interest rate i as defined in Definition 1.4.4: the interest paid at the end of a time unit divided by the capital at the beginning of the time unit.

Definition 1.5.2. The *interest conversion period* is the period between two successive interest payments. Denote the quotient of the time unit and the interest conversion period by p. The *nominal interest rate*, denoted $i^{(p)}$, is p times the interest rate per conversion period.

Common values for p include $p = 365$ (interest payable daily) and $p = 12$ (interest payable monthly). The term "interest payable $p$thly" is used if we do not want to specify the conversion period. In the example, the interest conversion period is a quarter and the time unit is a year, so $p = 4$. The interest rate per quarter is 2%, so the nominal interest rate is $i^{(4)} = 4 \cdot 0.02 = 0.08$ or 8%, and the effective interest rate is $i = 0.08243$.

To compute the effective interest rate from the nominal interest rate $i^{(p)}$, remember that the interest rate per conversion period is $i^{(p)}/p$. There are p conversion periods in a time unit. Thus, by the formula for compound interest, a capital C accumulates to $(1+i^{(p)}/p)^p C$ in a time unit. However, if the effective interest rate is i, then a capital C accumulates to $(1+i)C$ in a time unit. Thus, a nominal interest rate $i^{(p)}$ payable $p$thly is equivalent to an effective interest rate i if

$$1+i = \left(1+\frac{i^{(p)}}{p}\right)^p. \tag{1.5}$$

Example 1.5.3. Suppose that an account offers a nominal interest rate of 8% p.a. payable quarterly. What is the AER? What if the nominal rate is the same, but interest is payable monthly? Weekly? Daily?

Answer. For interest payable quarterly, we put $p = 4$ and $i^{(4)} = 0.08$ in (1.5) to find

$$1+i = \left(1+\frac{0.08}{4}\right)^4 = 1.08243,$$

so the AER is 8.243%. This is the example we considered above. In the other cases, we find:

$$\begin{aligned}
&\text{monthly } (p=12): \quad 1+i = \left(1+\frac{0.08}{12}\right)^{12} = 1.08300\\
&\text{weekly } (p=52): \quad 1+i = \left(1+\frac{0.08}{52}\right)^{52} = 1.08322\\
&\text{daily } (p=365): \quad 1+i = \left(1+\frac{0.08}{365}\right)^{365} = 1.08328
\end{aligned}$$

So, the AER is 8.300% for interest payable monthly, 8.322% for interest payable weekly, and 8.328% for interest payable daily.

It looks like the numbers converge to some limit as the conversion period gets shorter. This idea will be taken up at the end of the module.

There is an alternative but equivalent definition of the symbol $i^{(p)}$, which leads naturally to the valuation of annuities described in the next chapter. In Example 1.5.1, we assumed that the interest is paid in the account so that it generates more interest. If this is not the case, but you use the interest for other purposes, then the amount in the account will remain constant at £1000. You will get £20 interest after each quarter. This is equivalent to receiving £82.43 at the end of the year, given an (effective) interest rate of 8.243% p.a., as the following computation shows:

- £20 at the end of the first quarter is equivalent to $1.08243^{3/4} \cdot 20 = 21.22$ pounds at the end of the year.
- £20 at the end of the second quarter is equivalent to $1.08243^{1/2} \cdot 20 = 20.81$ pounds at the end of the year.
- £20 at the end of the third quarter is equivalent to $1.08243^{1/4} \cdot 20 = 20.40$ pounds at the end of the year.

Thus, £20 at the end of each quarter is equivalent to $21.22 + 20.81 + 20.40 + 20.00 = 82.43$ pounds at the end of the year.

More generally, a capital of 1 generates $i^{(p)}/p$ interest per conversion period. We can either leave the interest in the account, in which case the capital accumulates to $1 + i = (1 + i^{(p)}/p)^p$ at the end of the year, as we computed above, so we get a payment of i at the end of the year. Or we can take the interest as soon as it is paid, so we get p payments of $i^{(p)}/p$ each at times $\frac{1}{p}, \frac{2}{p}, \ldots, 1$. The payment of $i^{(p)}/p$ at time $\frac{k}{p}$ is equivalent to

$$(1+i)^{(p-k)/p} \frac{i^{(p)}}{p}$$

at the end of the year, because it needs to be shifted $p - k$ periods forward. Thus, the series of p payments is equivalent to

$$\sum_{k=1}^{p} (1+i)^{(p-k)/p} \frac{i^{(p)}}{p}$$

at the end of the year. If we make the substitution $n = p - k$, we get

$$\sum_{k=1}^{p} (1+i)^{(p-k)/p} \frac{i^{(p)}}{p}. = \sum_{n=0}^{p-1} (1+i)^{n/p} \frac{i^{(p)}}{p}.$$

This sum can be evaluated with the following formula for a geometric sum:

$$1 + r + r^2 + \cdots + r^n = \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}. \tag{1.6}$$

Thus, we find that the series of p payments is equivalent to

$$\begin{aligned} \sum_{n=0}^{p-1} (1+i)^{n/p} \frac{i^{(p)}}{p} &= \frac{\big((1+i)^{1/p}\big)^p - 1}{(1+i)^{1/p} - 1} \frac{i^{(p)}}{p} \\ &= \frac{i}{\Big(1 + \frac{i^{(p)}}{p}\Big) - 1} \frac{i^{(p)}}{p} = i \end{aligned}$$

at the end of the year, where in the last line we used that $1+i = (1+i^{(p)}/p)^p$, as stated in (1.5). Thus, a series of p payments of $i^{(p)}/p$ each at times $\frac{1}{p}, \frac{2}{p}, \ldots, 1$ is equivalent to a payment of i at time 1.

This is illustrated in Figure 1.4, which shows four equivalent ways to pay interest on a principal of 1. The top two rows show that a payment of d now is equivalent to a payment of i in a year's time. Indeed, the present value of the latter payment is iv, and in Section 1.4 we found that $iv = d$. The discussion in the preceding paragraph shows that a total payment of $i^{(p)}$ in p equal installments, one at the end of every period of $1/p$ year.

A similar discussion can be had for discounting instead of accumulating interest. A rate of discount of 2% compounded quarterly gives rise to a *nominal rate of discount* of 8% per annum. However, the present value of a payment of C due in one year is $(1 - 0.02)^4 C = 0.9224C$, see (1.4). Thus, the effective rate of discount is $d = 0.0776$ or 7.76%.

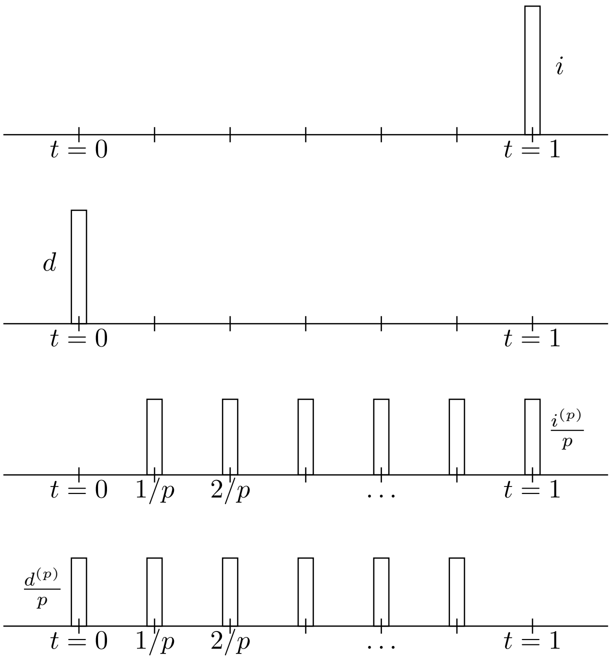

Figure 1.4: The following four situations are equivalent: A payment of i at the end of the year, a payment of d at the beginning of the year, a series of p payments of $i^{(p)}/p$ each at the end of every $1/p$ of a year, and a series of p payments of $d^{(p)}/p$ each at the beginning of every $1/p$ of a year.

Definition 1.5.4. The *nominal rate of discount* compounded $p$thly, denoted $d^{(p)}$, is p times the rate of discount per conversion period.

A similar computation as the one leading to (1.5) yields that

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p. \tag{1.7}$$

In Section 1.4, we concluded that the rate of discount arises in two situations: when computing the present value of a payment and when interest is paid in advance. Indeed, if the principal at the end of a time unit is 1 and interest is paid in advance, then the interest is d by Definition 1.4.4. Analogously to the discussion under (1.5), it can be shown that if interest is paid $p$thly in advance, then the total interest is $d^{(p)}$. In other words, p payments of $d^{(p)}/p$ each at the beginning of every period of $1/p$ time unit is equivalent to one payment of d at the beginning of the time unit. This follows from the computation

$$\sum_{k=0}^{p-1}(1-d)^{k/p}\frac{d^{(p)}}{p} = \frac{\big((1-d)^{1/p}\big)^p - 1}{(1-d)^{1/p} - 1}\,\frac{d^{(p)}}{p} = \frac{d}{\Big(1 - \frac{d^{(p)}}{p}\Big) - 1}\,\frac{d^{(p)}}{p} = d.$$

This is illustrated in the fourth row of Figure 4.1.

Example 1.5.5 (Kellison, p. 22)**.** Compare the following three loans: a loan charging an annual effective rate of 9%, a loan charging $8\frac{3}{4}\%$ compounded quarterly, and a loan charging $8\frac{1}{2}\%$ payable in advance and convertible monthly.

Answer. We will convert all rates to annual effective rates. For the second loan, we use (1.5) with $p = 4$ and $i^{(4)} = 0.0875$ to get $1+i = (1+i^{(p)}/p)^p = 1.0904$, so the annual effective rate is 9.04%. For the third loan, we use (1.7) with $p = 12$ and $d^{(12)} = 0.085$ to get $1 - d = (1 - d^{(p)}/p)^p = 0.91823$. Then, we use (1.1) and (1.3) to deduce that $1 + i = \frac{1}{v} = \frac{1}{1-d} = 1.0890$, so the annual effective rate is 8.90%. Thus, the third loan has the most favourable interest rate.

Consider again the equivalent payments in Figure 1.4. A payment of i at the end of the year is equivalent to a payment of d at the start of the year. However, a payment made later is worth less than a payment made earlier. It follows that i has to be bigger than d. Similarly, the p payments of $i^{(p)}/p$ each in the third row are done before the end of the year, with the exception of the last payment. Thus $i^{(p)}$ has to be smaller than i. Continuing this reasoning, we find that the discount and interest rates are ordered as followed.

$$d < d^{(2)} < d^{(3)} < d^{(4)} < \cdots < i^{(4)} < i^{(3)} < i^{(2)} < i.$$

Chapter 2

Annuities and loans

An *annuity* is a sequence of payments with fixed frequency. The term "annuity" originally referred to annual payments (hence the name), but it is now also used for payments with any frequency. Annuities appear in many situations; for instance, interest payments on an investment can be considered as an annuity. An important application is the schedule of payments to pay off a loan.

The word "annuity" refers in everyday language usually to a life annuity. A life annuity pays out an income at regular intervals until you die. Thus, the number of payments that a life annuity makes is not known. An annuity with a fixed number of payments is called an *annuity certain*, while an annuity whose number of payments depend on some other event (such as a life annuity) is a *contingent annuity*. Valuing contingent annuities requires the use of probabilities and this will not be covered in this module. These notes only looks at annuities certain, which will be called "annuity" for short.

2.1 Annuities immediate

The analysis of annuities relies on the formula for geometric sums:

$$1 + r + r^2 + \cdots + r^n = \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}. \tag{2.1}$$

This formula appeared already in Section 1.5, where it was used to relate nominal interest rates to effective interest rates. In fact, the basic computations for annuities are similar to the one we did in Section 1.5. It is illustrated in the following example.

Example 2.1.1. At the end of every year, you put £100 in a savings account which pays 5% interest. You do this for eight years. How much do you have at the end (just after your last payment)?

Answer. The first payment is done at the end of the first year and the last payment is done at the end of the eighth year. Thus, the first payment accumulates interest for seven years, so it grows to $(1 + 0.05)^7 \cdot 100 = 140.71$ pounds. The second payment accumulates interest for six years, so it grows to $1.05^6 \cdot 100 = 134.01$ pounds. And so on, until the last payment which does not

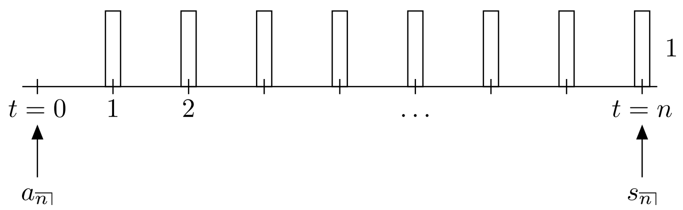

Figure 2.1: The present and accumulated value of an annuity immediate.

accumulate any interest. The accumulated value of the eight payments is

$$\begin{aligned}&1.05^7 \cdot 100 + 1.05^6 \cdot 100 + \cdots + 100 \\ &\qquad = 100\Big(1 + \cdots + 1.05^6 + 1.05^7\Big) = 100 \sum_{k=0}^{7} 1.05^k.\end{aligned}$$

This sum can be evaluated with the formula for a geometric sum. Substitute $r = 1.05$ and $n = 7$ in (2.1) to get

$$\sum_{k=0}^{7} 1.05^k = \frac{1.05^8 - 1}{1.05 - 1} = 9.5491.$$

Thus, the accumulated value of the eight payments is £954.91.

In the above example, we computed the accumulated value of an annuity. More precisely, we considered an annuity with payments made at the end of every year. Such an annuity is called an annuity immediate (the term is unfortunate because it does not seem to be related to its meaning).

Definition 2.1.2. An *annuity immediate* is a regular series of payments at the end of every period. Consider an annuity immediate paying one unit of capital at the end of every period for n periods. The accumulated value of this annuity at the end of the nth period is denoted $s_{\overline{n}|}$.

The accumulated value depends on the interest rate i, but the rate is usually only implicit in the symbol $s_{\overline{n}|}$. If it is necessary to mention the rate explicitly, the symbol $s_{\overline{n}|i}$ is used.

Let us derive a formula for $s_{\overline{n}|}$. The situation is depicted in Figure 2.1. The annuity consists of payments of 1 at $t = 1, 2, \ldots, n$ and we wish to compute the accumulated value at $t = n$. The accumulated value of the first payment is $(1+i)^{n-1}$, the accumulated value of the second payment is $(1+i)^{n-2}$, and so on till the last payment which has accumulated value 1. Thus, the accumulated values of all payments together is

$$(1+i)^{n-1} + (1+i)^{n-2} + \cdots + 1 = \sum_{k=0}^{n-1} (1+i)^k.$$

The formula for a geometric sum, cf. (2.1), yields

$$\sum_{k=0}^{n-1} (1+i)^k = \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}.$$

We arrive at the following formula for the accumulated value of an annuity immediate:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}. \tag{2.2}$$

This formula is not valid if $i = 0$. In that case, there is no interest, so the accumulated value of the annuities is just the sum of the payments: $s_{\overline{n}|} = n$.

The accumulated value is the value of the annuity at $t = n$. We may also be interested in the value at $t = 0$, the present value of the annuity. This is denoted by $a_{\overline{n}|}$, as shown in Figure 2.1.

Definition 2.1.3. Consider an annuity immediate paying one unit of capital at the end of every period for n periods. The value of this annuity at the start of the first period is denoted $a_{\overline{n}|}$.

A formula for $a_{\overline{n}|}$ can be derived as above. The first payment is made after a year, so its present value is the discount factor $v = \frac{1}{1+i}$. The present value of the second value is v^2, and so on till the last payment which has a present value of v^n. Thus, the present value of all payments together is

$$v + v^2 + \cdots + v^n = v(1 + v + \cdot + v^{n-1}) = v \sum_{k=0}^{n-1} v^k.$$

Now, use the formula for a geometric sum:

$$v \sum_{k=0}^{n-1} v^k = v \frac{v^n - 1}{v - 1} = \frac{v}{1-v}(1 - v^n).$$

The fraction $\frac{v}{1-v}$ can be simplified if we use the relation $v = \frac{1}{1+i}$:

$$\frac{v}{1-v} = \frac{\frac{1}{1+i}}{1 - \frac{1}{1+i}} = \frac{1}{(1+i) - 1} = \frac{1}{i}.$$

By combining these results, we arrive at the following formula for the present value of an annuity immediate:

$$a_{\overline{n}|} = \frac{1 - v^n}{i}. \tag{2.3}$$

Similar to equation (2.2) for $s_{\overline{n}|}$, the equation for $a_{\overline{n}|}$ is not valid for $i = 0$, in which case $a_{\overline{n}|} = n$.

There is a simple relation between the present value $a_{\overline{n}|}$ and the accumulated value $s_{\overline{n}|}$. They are value of the same sequence of payments, but evaluated at different times: $a_{\overline{n}|}$ is the value at $t = 0$ and $s_{\overline{n}|}$ is the value at $t = n$ (see Figure 2.1). Thus, $a_{\overline{n}|}$ equals $s_{\overline{n}|}$ discounted by n years:

$$a_{\overline{n}|} = v^n s_{\overline{n}|}. \tag{2.4}$$

This relation is easily checked. According to (2.2), the right-hand side evaluates to

$$v^n s_{\overline{n}|} = v^n \frac{(1+i)^n - 1}{i} = \frac{\left(\frac{1+i}{v}\right)^n - v^n}{i} = \frac{1 - v^n}{i} = a_{\overline{n}|},$$

where the last-but-one equality follows from $v = \frac{1}{1+i}$ and the last equality from (2.3). This proves (2.4).

One important application of annuities is the repayment of loans. This is illustrated in the following example.

Example 2.1.4. A loan of €2500 at a rate of $6\frac{1}{2}\%$ is paid off in ten years, by paying ten equal installments at the end of every year. How much is each installment?

Answer. Suppose that each installment is x euros. Then the loan is paid off by a 10-year annuity immediate. The present value of this annuity is $xa_{\overline{10}|}$ at $6\frac{1}{2}\%$. We compute $v = \frac{i}{1+i} = 0.938967$ and

$$a_{\overline{10}|} = \frac{1 - v^{10}}{i} = \frac{1 - 0.938967^{10}}{0.065} = 7.188830.$$

The present value should be equal to €2500, so the size of each installment is $x = 2500/a_{\overline{10}|} = 347.7617$ euros. Rounded to the nearest cent, this is €347.76.

Every installment in the above example is used to both pay interest and pay back a part of the loan. This is studied in more detail in Section 2.7. Another possibility is to only pay interest every year, and to pay back the principal at the end. If the principal is one unit of capital which is borrowed for n years, then the borrower pays i at the end of every year and 1 at the end of the n years. The payments of i form an annuity with present value $ia_{\overline{n}|}$. The present value of the payment of 1 at the end of n years is v^n. These payments are equivalent to the payment of the one unit of capital borrowed at the start. Thus, we find

$$1 = ia_{\overline{n}|} + v^n.$$

This gives another way to derive formula (2.3). Similarly, if we compare the payments at $t = n$, we find

$$(1+i)^n = is_{\overline{n}|} + 1,$$

and (2.2) follows.

2.2 Annuities due and perpetuities

The previous section considered annuities immediate, in which the payments are made in arrears (that is, at the end of the year). Another possibility is to make the payments at advance. Annuities that pay at the start of each year are called annuities due.

Definition 2.2.1. An *annuity due* is a regular series of payments at the beginning of every period. Consider an annuity immediate paying one unit of capital at the beginning of every period for n periods. The value of this annuity at the start of the first period is denoted $\ddot{a}_{\overline{n}|}$, and the accumulated value at the end of the nth period is denoted $\ddot{s}_{\overline{n}|}$.

The situation is illustrated in Figure 2.2, which should be compared to the corresponding figure for annuities immediate. Both $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ are measured at

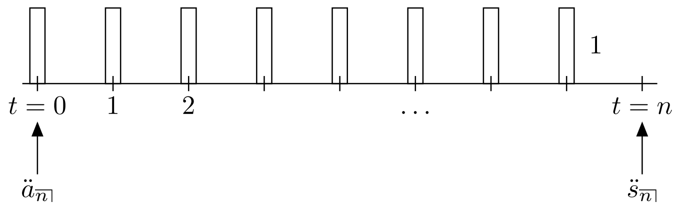

Figure 2.2: The present and accumulated value of an annuity due.

$t = 0$, while $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are both measured at $t = n$. The present value of an annuity immediate ($a_{\overline{n}|}$) is measured one period before the first payment, while the present value of an annuity due ($\ddot{a}_{\overline{n}|}$) is measured at the first payment. On the other hand, the accumulated value of an annuity immediate ($s_{\overline{n}|}$) is at the last payment, while the accumulated value of an annuity due ($\ddot{s}_{\overline{n}|}$) is measured one period after the last payment.

We can easily derive formulas for $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$. One method is to sum a geometric series. An annuity due consists of payments at $t = 0$, $t = 1$, ..., $t = n - 1$, so its value at $t = 0$ is

$$\ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \sum_{k=0}^{n-1} v^k = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}. \tag{2.5}$$

The value at $t = n$ is

$$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) = \sum_{k=1}^{n} (1+i)^k \\ &= (1+i)\frac{(1+i)^n - 1}{(1+i) - 1} = \frac{1+i}{i}\big((1+i)^n - 1\big) = \frac{(1+i)^n - 1}{d}. \end{aligned} \tag{2.6}$$

If we compare these formulas with the formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$, given in (2.3) and (2.2), we see that they are identical except that the denominator is d instead of i. In other words,

$$\ddot{a}_{\overline{n}|} = \frac{i}{d} a_{\overline{n}|} = (1+i) a_{\overline{n}|} \quad \text{and} \quad \ddot{s}_{\overline{n}|} = \frac{i}{d} s_{\overline{n}|} = (1+i) s_{\overline{n}|}.$$

There a simple explanation for this. An annuity due is an annuity immediate with all payments shifted one time period in the past (compare Figures 2.1 and 2.2). Thus, the value of an annuity due at $t = 0$ equals the value of an annuity immediate at $t = 1$. We know that an annuity immediate is worth $a_{\overline{n}|}$ at $t = 0$, so its value at $t = 1$ is $(1 + i)a_{\overline{n}|}$ and this has to equal $\ddot{a}_{\overline{n}|}$. Similarly, $\ddot{s}_{\overline{n}|}$ is not only the value of an annuity due at $t = n$ but also the value of an annuity immediate at $t = n + 1$. Annuities immediate and annuities due refer to the same sequence of payments evaluated at different times.

There is another relationship between annuities immediate and annuities due. An annuity immediate over n years has payments at $t = 1$, ..., $t = n$ and an annuity due over $n + 1$ years has payments at $t = 0$, $t = 1$, ..., $t = n$. Thus, the difference is a single payment at $t = 0$. It follows that

$$\ddot{a}_{\overline{n+1}|} = a_{\overline{n}|} + 1. \tag{2.7}$$

Similarly, $s_{\overline{n+1}|}$ is the value at $t = n + 1$ of a series of $n + 1$ payments at times $t = 1, \ldots, n + 1$, which is the same as the value at $t = n$ of a series of $n + 1$ payments at $t = 0, \ldots, n$. On the other hand, $\ddot{s}_{\overline{n}|}$ is the value at $t = n$ of a series of n payments at $t = 0, \ldots, n - 1$. The difference is a single payment at $t = n$, so

$$s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1. \tag{2.8}$$

The relations (2.7) and (2.8) can be checked algebraically by substituting (2.2), (2.3), (2.5) and (2.6) in them.

There is an alternative method to derive the formulas for $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$, analogous to the discussion at the end of the previous section. Consider a loan of one unit of capital over n years, and suppose that the borrower pays interest *in advance* and repays the principal after n years. As discussed in Section 1.4, the interest over one unit of capital is d if paid in advance, so the borrower pays an annuity due of size d over n years and a single payment of 1 after n years. These payments should be equivalent to the one unit of capital borrowed at the start. By evaluating this equivalence at $t = 0$ and $t = n$, respectively, we find that

$$1 = d\ddot{a}_{\overline{n}|} + v^n \quad \text{and} \quad (1 + i)^n = d\ddot{s}_{\overline{n}|} + 1,$$

and the formulas (2.5) and (2.6) follow immediately.

As a final example, we consider perpetuities, which are annuities continuing perpetually. Consols, which are a kind of British government bonds, and certain preferred stock can be modelled as perpetuities.

Definition 2.2.2. A *perpetuity immediate* is an annuity immediate continuing indefinitely. Its present value (one period before the first payment) is denoted $a_{\overline{\infty}|}$. A *perpetuity due* is an annuity due continuing indefinitely. Its present value (at the time of the first payment) is denoted $\ddot{a}_{\overline{\infty}|}$.

There is no symbol for the accumulated value of a perpetuity, because it would be infinite. It is not immediately obvious that the present value is finite, because it is the present value of an infinite sequence of payments. However, using the formula for the sum of an infinite geometric sequence ($\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$), we find that

$$\ddot{a}_{\overline{\infty}|} = \sum_{k=0}^{\infty} v^k = \frac{1}{1 - v} = \frac{1}{d}$$

and

$$a_{\overline{\infty}|} = \sum_{k=1}^{\infty} v^k = v \sum_{k=0}^{\infty} v^k = \frac{v}{1 - v} = \frac{1}{i}.$$

Alternatively, we can use $a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|}$ and $\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|}$ in combination with the formulas for $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$. This method gives the same result.

Example 2.2.3. You want to endow a fund which pays out a scholarship of £1000 every year in perpetuity. The first scholarship will be paid out in five years' time. Assuming an interest rate of 7%, how much do you need to pay into the fund?

Answer. The fund makes payments of £1000 at $t = 5, 6, 7, \ldots$, and we wish to compute the present value of these payments at $t = 0$. These payments form a perpetuity, so the value at $t = 5$ is $\ddot{a}_{\overline{\infty}|}$. We need to discount by five years to find the value at $t = 0$:

$$v^5 \ddot{a}_{\overline{\infty}|} = \frac{v^5}{d} = \frac{0.934579^5}{0.0654206} = 10.89850.$$

Thus, the fund should be set up with a contribution of £10898.50.

Alternatively, imagine that the fund would be making annual payments starting immediately. Then the present value at $t = 0$ would be $1000\ddot{a}_{\overline{\infty}|}$. However, we added imaginary payments at $t = 0, 1, 2, 3, 4$; the value at $t = 0$ of these imaginary payments is $1000\ddot{a}_{\overline{5}|}$. Thus, the value at $t = 0$ of the payments at $t = 5, 6, 7, \ldots$ is

$$\begin{aligned} 1000\ddot{a}_{\overline{\infty}|} - 1000\ddot{a}_{\overline{5}|} &= 1000 \cdot \frac{1}{d} - 1000 \cdot \frac{1 - v^5}{d} \\ &= 15285.71 - 4387.21 = 10898.50, \end{aligned}$$

as we found before. This alternative method is not faster in this example, but it illustrates a reasoning which is useful in many situations.

An annuity which starts paying in the future is called a *deferred annuity.* The perpetuity in the above example has its first payment in five years' time, so it can be considered as a perpetuity due deferred by five years. The actuarial symbol for the present value of such a perpetuity is ${}_{5|}\ddot{a}_{\overline{\infty}|}$. Alternatively, we can consider the example as a perpetuity immediate deferred by four years, whose present value is denoted by ${}_{4|}a_{\overline{\infty}|}$. Generally, the present value of an annuities over n years deferred by m years is given

$${}_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} \quad \text{and} \quad {}_{m|}\ddot{a}_{\overline{n}|} = v^m \ddot{a}_{\overline{n}|}.$$

2.3 Annuities over non-integer periods

It is fairly easy to compute the number n of payments of an annuity given its present or accumulated value, as the following example shows. However, the result is usually not an integer. This section explains how to interpret fractional periods.

Example 2.3.1. A loan of \$10000 is repaid by annual payments of \$1000 each at the end of the year. How long does it take to repay the loan on the basis of an interest rate of 6% p.a.?

Answer. The annual payments form an annuity immediate. Its present value is $1000a_{\overline{n}|}$. This should equal the amount borrowed, so we have to solve $a_{\overline{n}|} = 10$ for n. Using (2.3), we find:

$$\begin{aligned} a_{\overline{n}|} = 10 &\implies \frac{1 - v^n}{i} = 10 \\ &\implies v^n = 1 - 10i \\ &\implies n = \frac{\log(1 - 10i)}{\log(v)} = 15.725, \end{aligned}$$

so the solution is $n = 15.725$.

How should we to interpret the result $n = 15.725$? What does it mean to have an annuity over 15.725 years? One possibility is that the borrower should make 15 payments of \$1000 at $t = 1, \dots, 15$, and one final payment at time $t = 15.725$ (after 15 years and 265 days). The present value of the 15 payments of \$1000 is $1000a_{\overline{15}|} = 9712.25$ dollars, so after these payments \$287.75 of the loan is still outstanding. Thus, the final payment at $t = 15.725$ should be $(1+i)^{15.725} \cdot 287.75 = 718.38$ dollars. The loan is repaid by 15 payments of \$1000, paid annually in arrears, and a final payment of \$718.38, paid after 15 years and 265 days.

This interpretation is possible but not used very often in practice, because it is more convenient for both parties to have all payments at regular times. Usually, either the payment at $t = 15$ is increased so that the entire loan is repaid at that time, or a payment is added at $t = 16$ to repay the remainder of the loan. In the first case, the borrower has to pay $(1+i)^{15} \cdot 287.75 = 689.61$ dollars extra at $t = 15$, so the loan is repaid by 14 payments of \$1000 at $t = 1, \dots, 14$ and a final payment of \$1689.61. The payment of \$1689.61 at the end is called the *balloon payment*. In the second case, the borrower has to make an additional payment of $(1+i)^{16} \cdot 287.75 = 730.99$ dollars at $t = 16$, so the loan is repaid by 15 payments of \$1000 at $t = 1, \dots, 15$ and a final payment of \$730.99 at $t = 16$. The payment of \$730.99 at the end is called the *drop payment*. In conclusion, we found three possible interpretations of the annuity over 15.725 years, that resulted in the situation of Example 2.3.1:

- The borrower pays \$1000 at $t = 1, \dots, 14$ and \$1689.61 at $t = 15$.
- The borrower pays \$1000 at $t = 1, \dots, 15$ and \$718.38 at $t = 15.725$.
- The borrower pays \$1000 at $t = 1, \dots, 15$ and \$730.99 at $t = 16$.

In all these three cases, the present value of the payments at $t = 0$ is \$10000.

These notes use the convention that an annuity over 15.725 years is interpreted by one of the three sequences of payments listed above, whose present value is $1000a_{\overline{15.725}|}$. In other words, we define annuities over n years, where n is not an integer, such that the formulas (2.3) for the present value and (2.2) for the accumulated value remain valid. This definition is not generally accepted. Other writers define an annuity immediate of \$1000 over 15.725 years as 15 payments of \$1000 at $t = 1, \dots, 15$ and a final payment of \$725 at $t = 15.725$. This may seem more normal, but in that interpretation formulas (2.3) and (2.2) are no longer valid.

2.4 Unknown interest rate

In Sections 2.1 and 2.2 we derived the present and accumulated values of annuities with given period n and interest rate i. In Section 2.3, we studied how to find n. The topic of the current section is the determination of the rate i.

Example 2.4.1 (McCutcheon & Scott, p. 48)**.** A loan of £5000 is repaid by 15 annual payments of £500, with the first payment due in a year. What is the interest rate?

Answer. The repayments form an annuity. The value of this annuity at the time of the loan, which is one year before the first payment, is $500a_{\overline{15}|}$. This

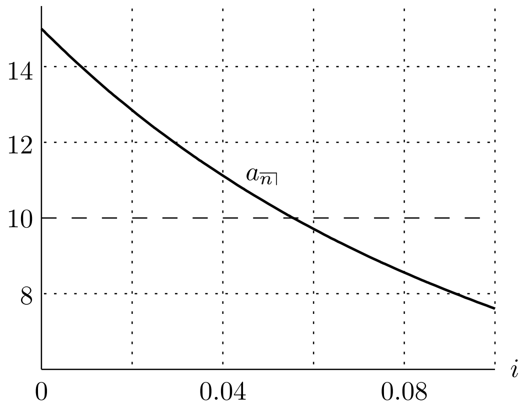

Figure 2.3: A plot of the present value of a 15-year annuity against the interest rate i (cf. Example 2.4.1). This shows that the solution of (2.9) lies between $i = 0.05$ and $i = 0.06$.

has to equal the principal, so we have to solve $500a_{\overline{15}|} = 5000$ or $a_{\overline{15}|} = 0.1$. Formula (2.3) for $a_{\overline{n}|}$ yields

$$a_{\overline{15}|} = \frac{1-v^{15}}{i} = \frac{1}{i}\left(1 - \left(\frac{1}{1+i}\right)^{15}\right),$$

so the equation that we have to solve is

$$\frac{1}{i}\left(1 - \left(\frac{1}{1+i}\right)^{15}\right) = 10. \qquad (2.9)$$

The solution of this equation is $i = 0.055565$, so the rate is 5.56%.

The above example is formulated in terms of a loan, but it can also be formulated from the view of the lender. The lender pays £5000 and gets 15 annual payments of £500 in return. The interest rate implied by the transaction is called the *yield* or the *(internal) rate of return* of the transaction. It is an important concept when analysing possible investments. Obviously, an investor wants to get high yield on his investment. We will return to this in Chapter 3.

Example 2.4.1 raises the question: how can we solve equations like (2.9)? It cannot be solved algebraically, so we have to use some numerical method to find an approximation to the solution. We present several methods here. Conceptually the simplest method is to consider a table like the following, perhaps by consulting a book of actuarial tables.

i	0	0.01	0.02	0.03	0.04	0.05
$a_{\overline{15}|}$	15.0000	13.8651	12.8493	11.9379	11.1184	10.3797

i	0.06	0.07	0.08	0.09	0.10	0.11
$a_{\overline{15}|}$	9.7122	9.1079	8.5595	8.0607	7.6061	7.1909

This shows that $a_{\overline{15}|} = 10$ for some value of i between 0.05 and 0.06, so the interest rate lies between 5% and 6%. The table also shows that the present

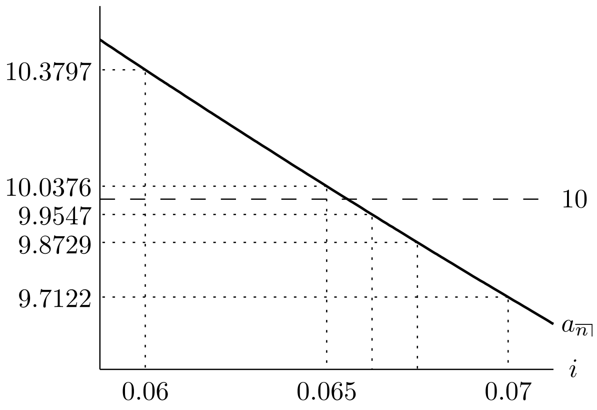

Figure 2.4: An illustration of the bisection method.

value $a_{\overline{15}|}$ decreases as the rate i increases (you should be able to understand this from first principles).

If we would like a more accurate approximation, we can apply the *bisection method*. This method takes the midpoint, which is here $5\frac{1}{2}\%$. We compute $a_{\overline{15}|}$ at $5\frac{1}{2}\%$, which turns out to be 10.0376. On the other hand, $a_{\overline{15}|}$ at 6% is 9.7122, so rate at which $a_{\overline{15}|} = 10$ lies between $5\frac{1}{2}\%$ and 6%. Another step of the bisection method takes $i = 5\frac{3}{4}\%$; at this rate $a_{\overline{15}|} = 9.8729$, so the rate we are looking for lies between $5\frac{1}{2}\%$ and $5\frac{3}{4}\%$. At the next step, we compute $a_{\overline{15}|}$ at $5\frac{5}{8}\%$, which turns out to be 9.9547, so we know that i should be between $5\frac{1}{2}\%$ and $5\frac{5}{8}\%$. As illustrated in Figure 2.4, the bisection method allows us to slowly zoom in on the solution.

Another possibility is to use *linear interpolation*. Again, we use that $a_{\overline{15}|}$ at 5% equals 10.3797, and that $a_{\overline{15}|}$ at 6% equals 9.7122. In other words, we know two points on the graph depicted in Figure 2.3, namely $(x_1, y_1) = (0.05, 10.3797)$ and $(x_2, y_2) = (0.06, 9.7122)$. The method of linear interpolation approximates the graph by a straight line between (x_1, y_1) and (x_2, y_2), as illustrated in Figure 2.5. The equation of this line is

$$y - y_1 = (x - x_1)\frac{y_2 - y_1}{x_2 - x_1}.$$

In the current example, we wish to find the value of x which corresponds to $y = 10$. If we denote the given value of y by y_*, then the unknown value of x_* is given by

$$x_* = x_1 + (y_* - y_1)\frac{x_2 - x_1}{y_2 - y_1}. \tag{2.10}$$

In the situation considered here, this evaluates to

$$0.05 + (10 - 10.3797) \cdot \frac{0.06 - 0.05}{9.7122 - 10.3797} = 0.055689.$$

This brings us in one step close to the solution. As with the bisection method, we can repeat this process to get more accurate approximations of the solution.

Some people may know *Newton's method*, also known as the *Newton–Raphson method*. This method is usually given for equations of the form $f(x) = 0$. We

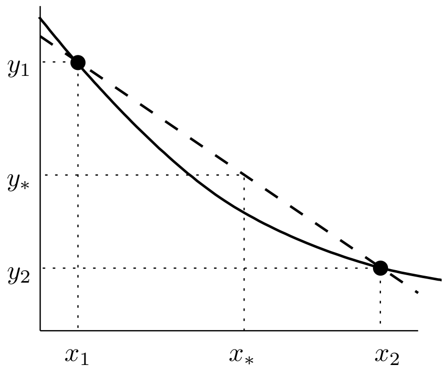

Figure 2.5: The method of linear interpolation takes two known points (x_1, y_1) and (x_2, y_2) on the graph and considers the line between them (the dashed line in the figure). This line approximates the graph and is used to find an approximation x_* to the x-value corresponding to y_*.

can write (2.9) in this form by taking

$$f(x) = \frac{1}{i}\left(1 - \left(\frac{1}{1+i}\right)^{15}\right) - 10.$$

Newton's method start from only one value of x, say x_*. It states that if x_* is a good approximation to the solution, then

$$x_{**} = x_* - \frac{f(x_*)}{f'(x_*)}$$

is an even better one. The disadvantage of Newton's method is that you have to differentiate the function in the equation. We will not consider this method any further.

All these methods are quite cumbersome to use by hand, so people commonly use some kind of machine to solve equations like these. Some graphical calculators allow you to solve equations numerically. Financial calculators generally have an option to find the interest rate of an annuity, given the number of payments, the size of every payment, and the present or accumulated value. There are also computer programs that can assist you with these computations. For example, in Excel the command `RATE(15,500,-5000)` computes the unknown rate in Example 2.4.1.

2.5 Annuities payable monthly, etc.

Up to now all annuities involved annual payments. However, other frequencies commonly arise in practice. The same theory as developed above apply to annuities with other frequencies.

Example 2.1.1 shows that the accumulated value of a sequence of eight annual payments of £100 at the time of the last payment is £954.91, if the rate of interest is 5% per annum. The result remains valid if we change the time unit.

The same computation shows that the accumulated value of a sequence of eight *monthly* payments of £100 at the time of the last payment is £954.91, if the rate of interest is 5% per *month*.

Interest rates per month are not used very often. As explained in Section 1.5, a rate of 5% per month corresponds to a nominal rate $i^{(12)}$ of 60% per year payable monthly (computed as $60 = 5 \times 12$). It also corresponds to an effective rate i of 79.59% per year, because $(1.05)^{12} = 1.7959$. Thus, the accumulated value of a sequence of eight monthly payments of £100 at the time of the last payment is £954.91, if the (effective) rate of interest is 79.59% p.a.

The preceding two paragraphs illustrate the basic idea of this section. The remainder elaborates on this and gives some definitions.

Definition 2.5.1. An *annuity immediate payable pthly* is a regular series of payments at the end of every period of $1/p$ time unit. Consider such an annuity lasting for n time units (so there are np payments), where every payment is $1/p$ unit of capital (so the total payment is n units). The present value of this annuity at the start of the first period is denoted $a^{(p)}_{\overline{n}|}$, and the accumulated value at the end of the npth period is denoted $s^{(p)}_{\overline{n}|}$.

The present and accumulated value of an annuity immediate payable $p$thly on the basis of an interest rate i per time unit can be calculated using several methods. Three methods will be presented here. All these methods use the nominal interest rate $i^{(p)}$ payable $p$thly, which is related to i by (1.5):

$$1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p.$$

The first method is the one used at the start of the section, in which a new time unit is introduced which equals the time between two payments (i.e., $1/p$ old time units). The rate i per old time unit corresponds to a rate of $j = i^{(p)}/p$ per new time unit, and the annuity payable $p$thly becomes a standard annuity over np (new) time units, with one payment of $1/p$ per (new) time unit. The future value of this annuity is $\frac{1}{p} s_{\overline{np}|j}$, which can be evaluated using (2.2):

$$s^{(p)}_{\overline{n}|i} = \tfrac{1}{p} s_{\overline{np}|j} = \frac{(1+j)^{np} - 1}{jp} = \frac{\left(1 + \frac{i^{(p)}}{p}\right)^{np} - 1}{i^{(p)}} = \frac{(1+i)^n - 1}{i^{(p)}}.$$

To compute the present value of the annuity over np time units, with one payment of $1/p$ per time unit, use (2.3) while bearing in mind that the discount factor in new time units is $1/(1+j)$:

$$a^{(p)}_{\overline{n}|i} = \tfrac{1}{p} a_{\overline{np}|j} = \frac{1 - \left(\frac{1}{1+j}\right)^{np}}{jp} = \frac{1 - \left(1 + \frac{i^{(p)}}{p}\right)^{-np}}{i^{(p)}} = \frac{1 - (1+i)^{-n}}{i^{(p)}} = \frac{1 - v^n}{i^{(p)}}.$$

The second method computes the present and accumulated value of annuities payable $p$thly from first principles using formula (2.1) for the sum of a geometric sequence. This is the same method used to derive formulas (2.2) and (2.3). The symbol $a^{(p)}_{\overline{n}|}$ denotes the present value at $t = 0$ of np payments of $1/p$ each. The first payment is at time $t = 1/p$, so its present value is $(1/p) \cdot v^{1/p}$; the second payment is at time $t = 2/p$, so its present value is $(1/p) \cdot v^{2/p}$; and so on till the last payment which is at time $t = n$, so its present value is $(1/p) \cdot v^n$. The sum of the present values is:

$$
\begin{aligned}
a_{\overline{n}|}^{(p)} &= \frac{1}{p}\left(v^{1/p} + v^{2/p} + \cdots + v^{n-\frac{1}{p}} + v^n\right) = \frac{1}{p}\sum_{k=1}^{np} v^{k/p} \\
&= \frac{v^{1/p}}{p} \cdot \frac{1-(v^{1/p})^{np}}{1-v^{1/p}} = \frac{1}{p} \cdot \frac{1-v^n}{(1+i)^{1/p}-1} = \frac{1-v^n}{i^{(p)}}.
\end{aligned}
$$

Similarly, the accumulated value is computed as

$$
\begin{aligned}
s_{\overline{n}|}^{(p)} &= \frac{1}{p}\left((1+i)^{n-\frac{1}{p}} + (1+i)^{n-\frac{2}{p}} + \cdots + (1+i)^{1/p} + 1\right) \\
&= \frac{1}{p}\sum_{k=0}^{np-1}(1+i)^{k/p} = \frac{1}{p} \cdot \frac{((1+i)^{1/p})^{np}-1}{(1+i)-1} = \frac{(1+i)^n-1}{i^{(p)}}.
\end{aligned}
$$

The third method compares an annuity payable $p$thly to an annuity payable annually over the same period. In one year, an annuity payable $p$thly consists of p payments (at the end of every period of $1/p$ year) and an annuity payable annually consists of one payment at the end of the year. If p payments are $i^{(p)}/p$ each and the annual payment is i, then these payments are equivalent, as was found in Section 1.5 (see Figure 4.1). Thus, an annuity with $p$thly payments of $i^{(p)}/p$ is equivalent to an annuity with annual payments of i, so their present and accumulated values are the same:

$$
i a_{\overline{n}|} = i^{(p)} a_{\overline{n}|}^{(p)} \quad \text{and} \quad i s_{\overline{n}|} = i^{(p)} s_{\overline{n}|}^{(p)}.
$$

All three methods leads to the same conclusion:

$$
a_{\overline{n}|}^{(p)} = \frac{1-v^n}{i^{(p)}} \quad \text{and} \quad s_{\overline{n}|}^{(p)} = \frac{(1+i)^n-1}{i^{(p)}}. \tag{2.11}
$$

The formulas for annuities payable $p$thly are the same as the formulas for standard annuities (that is, annuities payable annually), except that the formulas for annuities payable $p$thly have the nominal interest rate $i^{(p)}$ in the denominator instead of i.

A similar story holds for annuities due. An annuity due payable $p$thly is a sequence of payments at $t = 0, 1/p, \ldots, n-(1/p)$, whereas an annuity immediate payable $p$thly is a sequence of payments at $t = 1/p, 2/p, \ldots, n$. Thus, they represent the same sequence of payments, but shifted by one period of $1/p$ time unit. The present value of an annuity due payable $p$thly at $t = 0$ is denoted by $\ddot{a}_{\overline{n}|}^{(p)}$, and the accumulated value at $t = n$ is denoted by $\ddot{s}_{\overline{n}|}^{(p)}$. The corresponding expressions are

$$
\ddot{a}_{\overline{n}|}^{(p)} = \frac{1-v^n}{d^{(p)}} \quad \text{and} \quad \ddot{s}_{\overline{n}|}^{(p)} = \frac{(1+i)^n-1}{d^{(p)}}. \tag{2.12}
$$

The difference with the formulas (2.11) for annuities immediate is again only in the denominator: $i^{(p)}$ is replaced by $d^{(p)}$.

The above discussion tacitly assumed that p is an integer, but in fact the results are also valid for fractional values of p. This is illustrated in the following example.

Example 2.5.2. Consider an annuity of payments of £1000 at the end of every second year. What is the present value of this annuity if it runs for ten years and the interest rate is 7%?

Answer. The present value can be found from first principles by summing a geometric sequence. We have $i = 0.07$ so $v = 1/1.07 = 0.934579$, so the present value is

$$\begin{aligned}1000v^2 + 1000v^4 + 1000v^6 + 1000v^8 + 1000v^{10} \\ = 1000\sum_{k=1}^{5} v^{2k} = 1000v^2 \cdot \frac{1-(v^2)^5}{1-v^2} = 3393.03 \text{ pounds.}\end{aligned}$$

Alternatively, we can use (2.11) with $p = 1/2$, because there is one payment per two years. We compute $i^{(1/2)}$ from (1.5),

$$1 + i = \left(1 + \frac{i^{(1/2)}}{1/2}\right)^{1/2} \implies i^{(1/2)} = \frac{1}{2}\Big((1+i)^2 - 1\Big) = 0.07245,$$

and thus

$$a_{\overline{10}|}^{(1/2)} = \frac{1-v^{10}}{i^{(1/2)}} = 6.786069.$$

Remember that $a_{\overline{n}|}^{(p)}$ is the present value of an annuity paying $1/p$ units of capital every $1/p$ years for a period of n years, so $a_{\overline{10}|}^{(1/2)} = 6.786069$ is the present value of an annuity paying two units of capital every two years for a period of 10 years. Thus, the present value of the annuity in the question is $500 \cdot 6.786069 = 3393.03$ pounds. This is the same as we found from first principles.

2.6 Varying annuities

The annuities studied in the preceding sections are all level annuities, meaning that all payments are equal. This section studies annuities in which the size of the payments changes. In simple cases, these can be studied by splitting the varying annuities in a sum of level annuities, as the following example shows.

Example 2.6.1. An annuity pays €50 at the end of every month for two years, and €60 at the end of every month for the next three years. Compute the present value of this annuity on the basis of an interest rate of 7% p.a.

Answer. This annuity can be considered as the sum of two annuities: one of €50 per month running for the first two years, and one of €60 per month running for the next three years. The present value of the first annuity is $600a_{\overline{2}|}^{(12)}$ euros (remember that $a_{\overline{n}|}^{(12)}$ is the present value of an annuity paying 1/12 at the end of every month). The value of the second annuity one month before its first payment is $720a_{\overline{3}|}^{(12)}$, which we need to discount by two years. Thus, the present value of the annuity in the question is

$$600a_{\overline{2}|}^{(12)} + 720v^2 a_{\overline{3}|}^{(12)} = 600 \cdot \frac{1-v^2}{i^{(12)}} + 720v^2 \cdot \frac{1-v^3}{i^{(12)}}.$$

The interest rate is $i = 0.07$, so the discount factor is $v = 1/1.07 = 0.934579$ and the nominal interest rate is $i^{(12)} = 12(1.07^{1/12} - 1) = 0.0678497$, so

$$600 \cdot \frac{1-v^2}{i^{(12)}} + 720v^2 \cdot \frac{1-v^3}{i^{(12)}} = 1119.19 + 1702.67 = 2821.86.$$

Thus, the present value of the annuity in the question is €2821.86.

Alternatively, the annuity can be considered as the difference between an annuity of €60 per month running for five years and an annuity of €10 per month running for the first two years. This argument shows that the present value of the annuity in the question is

$$720a^{(12)}_{\overline{5}|} - 120a^{(12)}_{\overline{2}|} = 3045.70 - 223.84 = 2821.86.$$

Unsurprisingly, this is the same answer as we found before.

More complicated examples of varying annuities require a return to first principles. Let us consider a varying annuity immediate running over n time units, and denote the amount paid at the end of the kth time unit by P_k. The present value of this annuity, one time unit before the first payment, is given by

$$\sum_{k=1}^{n} P_k v^k,$$

and its accumulated value at the time of the last payment is given by

$$\sum_{k=1}^{n} P_k (1+i)^{n-i}.$$

For a level annuity, all the P_k are equal, and we arrive at the formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$. The next example considers an annuity whose payments increase geometrically.

Example 2.6.2. An annuity immediate pays £1000 at the end of the first year. The payment increases by 3% per year to compensate for inflation. What is the present value of this annuity on the basis of a rate of 7%, if it runs for 20 years?

Answer. The annuity pays £1000 at the end of the first year, £1030 at the end of the second year, and so on. The payment at the end of year k is given by $P_k = 1000 \cdot (1.03)^{k-1}$. Thus, the present value is

$$\begin{aligned}\sum_{k=1}^{20} 1000 \cdot (1.03)^{k-1} \cdot v^k &= \frac{1000}{1.03} \sum_{k=1}^{20} (1.03v)^k \\ &= \frac{1000}{1.03} \left(\sum_{k=0}^{20} (1.03v)^k - 1 \right) \\ &= \frac{1000}{1.03} \left(\frac{1-(1.03v)^{21}}{1-1.03v} - 1 \right) \\ &= \frac{1000}{1.03} (14.731613 - 1) = 13331.66.\end{aligned}$$

So the present value of the annuity is £13,331.66.

The case of an annuity whose payments increase in an arithmetic progression is important enough to have its own symbol.

Definition 2.6.3. The present value of an increasing annuity immediate which pays 1 at $t = 1$, 2 at $t = 2$, and so on until a final payment of n at $t = n$, is denoted by $(Ia)_{\overline{n}|}$. The present value of an increasing annuity due which pays 1 at $t = 0$, 2 at $t = 1$, and so on until a final payment of n at $t = n - 1$, is denoted by $(I\ddot{a})_{\overline{n}|}$.

Expressions for $(Ia)_{\overline{n}|}$ and $(I\ddot{a})_{\overline{n}|}$ can be derived as follows. Consider first the increasing annuity immediate. The first payment of 1 at $t = 1$ has present value v, the second payment of 2 at $t = 2$ has present value $2v^2$, et cetera, so

$$(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \cdots + nv^n = \sum_{k=1}^{n} kv^k. \tag{2.13}$$

For the increasing annuity due, the first payment of 1 at $t = 0$ has present value 1, the second payment of 2 at $t = 2$ has present value $2v$, et cetera, so

$$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \sum_{k=1}^{n} kv^{k-1}. \tag{2.14}$$

The annuity due is the same sequence of payments as the annuity immediate, but one year earlier. Thus, we have $(Ia)_{\overline{n}|} = v(I\ddot{a})_{\overline{n}|}$, as is already obvious from (2.13) and (2.14) above.

Now consider the difference between the annuity due and the annuity immediate. At time $t = 0$, the annuity due pays out 1 and the annuity immediate pays out nothing. At time $t = 1$, the annuity due pays out 2 and the annuity immediate pays out 1. The annuity due pays out 1 more than the annuity immediate at $t = 0, 1, \ldots, n - 1$. At the other hand, the annuity immediate pays out n at $t = n$ and the annuity due pays out nothing. Thus,

$$(I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n.$$

This can also be found by subtracting (2.13) from (2.14). Now use that $(Ia)_{\overline{n}|} = v(I\ddot{a})_{\overline{n}|}$, as we found above:

$$\frac{1}{v}(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n \implies (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\frac{1}{v} - 1} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.$$

This can be written as $\ddot{a}_{\overline{n}|} = i(Ia)_{\overline{n}|} + nv^n$, an equation with an interesting (but perhaps challenging) interpretation. Consider a transaction, in which one unit of capital is lent every year. The interest is i in the first year, $2i$ in the second year, and so on. At the end of n years, the amount borrowed is n, which is then paid back. The equation $\ddot{a}_{\overline{n}|} = i(Ia)_{\overline{n}|} + nv^n$ expresses that the payments done by the lender are equivalent to the payments by the borrower.

The formula for $(Ia)_{\overline{n}|}$ can be used to find the value of annuities with payments in an arithmetic progression. For instance, consider an annuity paying £1000 at the end of the first year, £950 at the end of the second year, £900 at the end of the third year, and so on, with the payment decreasing by £50 ever year. The last payment is £500 at the end of the eleventh year. The present value of this annuity is $1050a_{\overline{11}|} - 50(Ia)_{\overline{11}|}$.

2.7 Loans

Example 2.1.4 considered the repayment of a loan by a level annuity. The repayment of loans is an important application of annuities, which is studied further in this section.

Example 2.7.1 (Continuation of Example 2.1.4)**.** We computed that a loan of €2500 at $6\frac{1}{2}\%$ interest can be repaid by ten installments of €347.76, each being paid at the end of the year. What is the remaining balance of the loan after six years?

Answer. There are two methods to handle questions like this. The first method considers the payments in the first six years. This is called the *retrospective method*, because it looks back to payments already made. The second method considers the payments in the last four years. This is called the *prospective method*, because it looks forward to payments that have not been made yet. Obviously, both methods should give the same answer, and you should pick the method that seems more convenient.

The retrospective method uses that the remaining balance is the value of the original loan after six years minus the accumulated value of the payments that have already been made. The borrower has made six payments of €347.76 each at the end of the year. The accumulated value of these payments is

$$347.76 \cdot s_{\overline{6}|} = 347.76 \cdot 7.063728 = 2456.48.$$

The value of the loan after six years is

$$2500 \cdot (1+i)^6 = 2500 \cdot 1.459142 = 3647.86,$$

so the remaining balance of the loan is $3647.86 - 2456.48 = 1191.38$ euros.

The prospective method uses that the remaining balance equals the present value of the remaining payments. The borrower still has to make four payments of €347.76. We need the present value of these payments six years after the start of the loan. This is one year before the first of the four remaining payments is due, so the present value of the four remaining payments is

$$347.76 \cdot a_{\overline{4}|} = 347.76 \cdot 3.425799 = 1191.36$$

euros. Thus, the remaining balance of the loan after six years is €1191.36.

The results found by the retrospective and prospective methods differ by two cents. The difference is caused because at the end of Example 2.1.4, the value of 347.7617... was rounded to 347.76. The prospective and retrospective method would have given the same result if we had used the exact value.

The outstanding balance at the end of every year can be computed with either the retrospective method or the prospective method. The results are summarized in Table 2.1. This table is read as follows. At the end of the first year, the borrower pays €347.76. Part of this payment is interest on the loan. The interest part is $0.065 \cdot 2500 = 162.50$ euros. The remainder, $347.76 - 162.50 = 185.26$ euros, is used to repay the loan, so the remaining balance is $2500 - 185.26 = 2314.74$ euros. In the second year, the payment of €347.76 consist of €150.46 interest on the outstanding balance of €2314.74 and €197.30 repayment of the loan, and so on.

Year	Payment	Interest paid	Principal repaid	Outstanding balance
0	—	—	—	2500.00
1	347.76	162.50	185.26	2314.74
2	347.76	150.46	197.30	2117.44
3	347.76	137.63	210.13	1907.31
4	347.76	123.98	223.78	1683.53
5	347.76	109.43	238.33	1445.20
6	347.76	93.94	253.82	1191.38
7	347.76	77.44	270.32	921.06
8	347.76	59.87	287.89	633.17
9	347.76	41.16	306.60	326.57
10	347.76	21.23	326.53	0.04

Table 2.1: Loan schedule for a loan of €2500 which is repaid by 10 annual payments of €347.76.

According to the table, the loan is not fully repaid after 10 years, but there is an outstanding balance of four cents. The reason for this is that the table assumes that the annual payment is 347.76 euros, instead of the exact value of 347.7617... euros that was computed in Example 2.1.4. Had we used the exact value, then the loan would be exactly repaid after ten years. In practice, the final payment is often adjusted so that the loan is fully repaid. In our example, that would mean that the final payment is €347.80 instead of €347.76.

A table such as Table 2.1, is known as a *loan schedule* or *amortization schedule*. It shows the interest component of every payment, which is useful if the interest component is taxed differently from the repayment component. It also shows the outstanding balance after every payment. The outstanding balance is used if the terms of the loan changes, as shown in the forthcoming Example 2.7.2.

Loan schedules can be computed quite easily. Consider a loan over n years with an annual payment of P, so amount borrowed is $Pa_{\overline{n}|}$. At the end of year k, there are $n-k$ further payments to be done, so the outstanding balance is $Pa_{\overline{n-k}|}$, by the prospective method. Thus, the next payment (at the end of year $k+1$) has an interest component of $iPa_{\overline{n-k}|} = (1-v^{n-k})P$. The rest of the payment, namely $v^{n-k}P$, is used to repay the loan.

Example 2.7.2. A mortgage of £120,000 is repaid over 20 years by equal monthly payments. How much is every payment on the basis of an effective interest rate of 5.89% p.a.? Suppose that the rate increases by one percent point after eleven years. Compute the revised monthly payment.

Answer. The rate of $i = 0.0589$ corresponds to a nominal rate payable monthly of

$$i^{(12)} = 12\Big((1+i)^{1/12} - 1\Big) = 0.05736732.$$

If every monthly payment is 1/12, then the present value of all the monthly payments is

$$a^{(12)}_{\overline{20}|} = \frac{1-v^{20}}{i^{(12)}} = \frac{1-1.0589^{-20}}{0.05736732} = 11.882249.$$

Thus, the monthly repayment required to pay off the mortgage of £120,000 is

$$\frac{120000}{a^{(12)}_{\overline{20}|}} \cdot \frac{1}{12} = 841.59 \text{ pounds.}$$

Now consider the situation after eleven years. There are nine years of payments remaining. The present value of these payments is

$$841.59 \cdot 12 \cdot a^{(12)}_{\overline{9}|} = 841.49 \cdot 11 \cdot 7.016967 = 70864.91$$

pounds, so by the prospective method the outstanding balance is £70,864.91. It may seem surprising that less than half the loan is repaid in the first eleven years, but at the start of the loan most of the payments are used to pay the interest. The bulk of the loan is repaid in the second half.

To compute the revised monthly payment after the rate increase, we compute the revised nominal rate as

$$i^{(12)} = 12\Big((1.0689)^{1/12} - 1\Big) = 0.06681541.$$

There are nine years of monthly payments left, in which £70,864.91 needs to be repaid, so the revised monthly payment is

$$\frac{70864.91}{a^{(12)}_{\overline{9}|}} \cdot \frac{1}{12} = \frac{70864.91}{6.750054} \cdot \frac{1}{12} = 874.87.$$

Thus, the rate increase causes the monthly payment to be increased from £841.59 to £874.87.

Chapter 3

Cash flows

In Chapter 1 we looked at how the value of single payments changes in time. Chapter 2 discussed annuities, regular sequences of payments. The topic of this chapter is irregular sequences of payments, called cash flows.

The chapter starts with an analysis of cash flows and in particular with the computation of their rate of return or yield. This can be used to appraise different investment projects and to value bonds and other securities. The last sections discusses how to incorporate practical matters like inflation and taxation in the theoretical framework.

3.1 Cash flows and their value

Varying annuities were discussed in Section 2.6, where we considered a sequence of payments done at times $t = 1, 2, \ldots, n$, with the amount paid at time $t = k$ denoted by P_k. This chapter generalizes the situation even further by dropping the assumption that the payments are done at regular intervals. Instead, we consider at sequence of payments of size P_k done at time $t = t_k$. Such a sequence is called a *cash flow*.

Example 3.1.1. Consider a contractor building a house. Suppose that he buys the ground at a certain time for £15,000. A month later, he pays his workers in advance, for a total of £30,000. The building materials cost £25,000, to be paid one quarter after the purchase of the ground. These payments constitute a cash flow, with (measuring time in years) $P_1 = 15000$, $t_1 = 0$, $P_2 = 30000$, $t_2 = \frac{1}{12}$, $P_3 = 25000$ and $t_3 = \frac{1}{4}$.

The payments could also be considered as a varying annuity payable monthly, with a payment of zero after two months. However, the pattern of payments here is so irregular that it is more natural to consider it as a cash flow.

The example only considers money flowing out. We can consider money flowing in as a separate cash flow. It is easier however to combine money flowing in and money flowing out in one cash flow with positive and negative payments. The convention is to take money flowing in as positive and money flowing out as negative. Such a cash flow is called a *net cash flow*.

Example 3.1.2 (Continuation of Example 3.1.1)**.** Suppose that it takes the contractor nine months to build the house. When it is finished, he sells it for £75,000. The net cash flow is as given in the following table.

k	t_k	P_k
1	0	−15000
2	$\frac{1}{12}$	−30000
3	$\frac{1}{4}$	−25000
4	$\frac{3}{4}$	+75000

The value of a cash flow at a time t_* is computed as followed. Payment k is done at time t_k, so it accumulates interest for $t_* - t_k$ time units (this number may be negative, but that does not matter). Thus, the value of payment k at time t_* is $(1+i)^{t_*-t_k} P_k$. Summing over all payments yields that the value of the entire cash flow at time t_* is given by

$$\sum_k (1+i)^{t_*-t_k} P_k. \tag{3.1}$$

Of particular importance is the case where t_* is zero. The value of the net cash flow at $t = 0$ is called the *net present value (NPV)*.

Example 3.1.3 (Continuation of Example 3.1.2)**.** The net present value of the cash flow in Example 3.1.2 is

$$(-15000) \cdot (1+i)^0 + (-30000) \cdot (1+i)^{-1/12} \\ + (-25000) \cdot (1+i)^{-1/4} + 75000 \cdot (1+i)^{-3/4}. \tag{3.2}$$

If the interest rate is 6%, then this is

$$-15000 - 29854.68 - 24638.46 + 71792.96 = 2299.82$$

so the net present value is £2299.82. The fact that it is positive means the project will be profitable for the contractor, if he can borrow the money at 6%.

3.2 The internal rate of return

The net present value of a cash flow is given by (3.1). It clearly depends on the interest rate i. For instance, the net present value for the construction project in Example 3.1.3 is given by (3.2). The net present value for different interest rates is in the following table.

$$\begin{aligned}
i = 0: &\quad -15000.00 - 30000.00 - 25000.00 + 75000.00 = 5000.00 \\
i = 0.05: &\quad -15000.00 - 29878.27 - 24696.91 + 72305.16 = 2729.97 \\
i = 0.10: &\quad -15000.00 - 29762.67 - 24411.35 + 69825.93 = 651.91 \\
i = 0.15: &\quad -15000.00 - 29652.62 - 24141.57 + 67536.40 = -1257.80 \\
i = 0.20: &\quad -15000.00 - 29547.64 - 23886.07 + 65414.70 = -3019.01
\end{aligned}$$

Figure 3.1 shows how the net present value depends on the interest rate. We see that the net present value is the highest when $i = 0$ and decreases as the

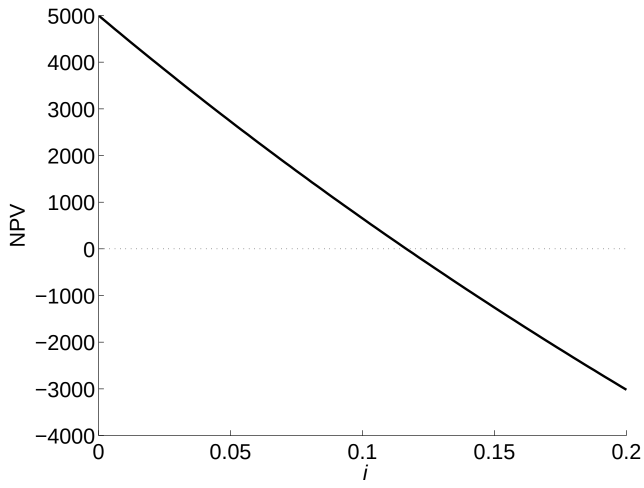

Figure 3.1: The net present value for the project in Example 3.1.3 plotted against the interest rate.

interest rate increases. Of particular interest is the rate at which the net present value is zero, that is, the value of i_* for which

$$-15000 - 40000(1+i_*)^{-1/12} - 25000(1+i_*)^{-1/4} + 75000(1+i_*)^{-3/4} = 0.$$

This equation is called the *equation of value* (for the interest rate), and the rate i_* is called the *yield* of the transaction. It is also known as the *internal rate of return (IRR)*; these terms will be used interchangeably here. If the contractor can borrow money for a rate lower than the internal rate of return, he will make a profit. If not, he will make a loss. The figure suggests that for this project, the yield is around 11% or 12% p.a.

Definition 3.2.1. Let NPV(i) denote the net present value of a cash flow as a function of the interest rate i. The equation "NPV(i) = 0" is the *equation of value* for this transaction. The *yield* or *internal rate of return* is the value of i which solves this equation.

If the total expenses equal the total income, the investor receives no return, and the yield is zero. If the expenses exceed the income, the yield is negative. We will assume that the yield is always larger than -1, as it is hard to give a meaning to yields smaller than -1.

The above definition uses the net value of the cash flow at time $t = 0$. In fact, the time t_* at which the cash flow is valued does not make a difference: if the present value at $t = 0$ is zero for some value of i, then the value at any time will be zero for that value of i. A smart choice of the time t_* at which the cash flow is valued may however simplify the computations.

In Section 2.4, we studied how to compute the interest rate for an annuity. The internal rate of return as defined here is a generalization to cash flows. As for annuities, the yield of a transaction can in general not be computed analytically, but the iterative methods discussed in Section 2.4 can be applied. Linear interpolation is often a fast way to compute the yield approximately.

Example 3.2.2 (Continuation of Example 3.1.3)**.** In Example 3.1.3 we computed that the net present value for the construction project is £2299.82 if the interest rate is 6%. If $i = 0$ then the net present value is £5000. The formula for linear interpolation is given in (2.10), which we reproduce here:

$$x_* = x_1 + (y_* - y_1)\frac{x_2 - x_1}{y_2 - y_1}.$$

Substitution of $x_1 = 0$, $y_1 = 5000$, $x_2 = 0.06$, $y_2 = 2299.82$ and $y_* = 0$ results in

$$x_* = -5000\frac{0.06}{2299.82 - 5000} = 0.1111.$$

Thus, linear interpolation yields an approximate yield of 11.11%. The net present value at this rate is given by (3.2):

$$-15000.00 - 29737.78 - 24350.15 + 69302.10 = 214.17.$$

For another iteration of linear interpolation, we substitute the values $x_1 = 0.06$, $y_1 = 2299.82$, $x_2 = 0.1111$, $y_2 = 214.17$ and $y_* = 0$ in the formula for linear interpolation, yielding

$$x_* = 0.06 - 2299.82 \cdot \frac{0.1111 - 0.06}{214.17 - 2299.82} = 0.1163$$

The yield is thus approximately 11.63%. This is already very close to the exact value for the yield, which is 11.66% (rounded to the nearest basis point).

The internal rate of return is often used to compare different transactions. For an investor or lender, who puts up some money at the start and receives money later, a higher yield indicates a more favorable transaction. For a borrower, who receives some money at the start of the transaction and has to pay money back later, lower yields are better. However, the internal rate of return is not always the correct measure. The next section, which is about the appraisal of investments, discusses some issues with use of the internal rate of return for this purpose.

A more fundamental issue is that a transaction may not necessarily have a rate of return, or it may have multiple rate of returns. For example, consider a project in which you receive £1000 at time $t = 0$ and £1155 at $t = 2$, in return for which you have to pay £2150 at $t = 1$. As noted just under Definition 3.2.1, we may choose any time t_* to value the cash flow. We choose $t_* = 2$; the value of the cash flow at that time is

$$1000(1 + i)^2 - 2150(1 + i) + 1155. \tag{3.3}$$

To find the yield, we need to solve $1000(1 + i)^2 - 2150(1 + i) + 1155 = 0$. This is a quadratic equation in $1 + i$ and its solution is

$$1 + i = \frac{2150 \pm \sqrt{2150^2 - 4 \cdot 1000 \cdot 1155}}{2000} = \frac{2150 \pm 50}{2000},$$

so $i = 0.10$ and $i = 0.05$. Thus, the transaction has a yield of 5% and also of 10%. The concept of a yield is not useful in this situation.

The problem is that the net value of the cash flow, given by (3.3), is a quadratic function which crosses the x-axis twice. In contrast, the net value (3.2)

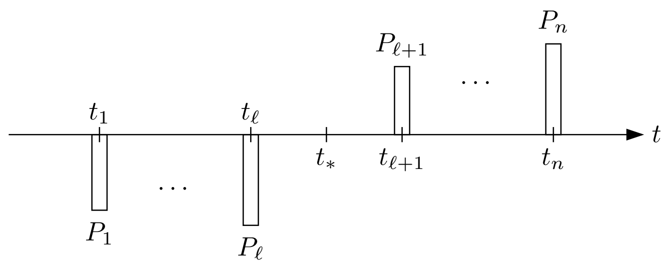

Figure 3.2: If all expenses are incurred before the earnings are due, then the internal rate of return is uniquely defined.

of the construction project, plotted in Figure 3.1, is a decreasing function and crosses the x-axis only once, and thus the yield is well defined. The reason is that in the construction project, the income (positive payments) is after the expenses (negative payments). If the interest rate increases, the effect of discounting increases. However, the income is affected more by this than the expenses, because the income is due after the expenses. Thus, the value of the income decreases more rapidly than the value of the expenses if the interest rate increases. This explains why the net value decreases as the interest rate increases.

More formally, suppose that there are n payments. The payments are in chronological order, meaning that $t_1 < \ldots < t_n$. Assume that the first ℓ payments (with $1 \leq \ell < n$) are negative and the rest are positive, as illustrated in Figure 3.2. The value of the cash flow at time t_* is

$$V(i) = P_1(1+i)^{t_*-t_1} + \cdots + P_n(1+i)^{t_*-t_n} = \sum_{k=1}^{n} P_k(1+i)^{t_*-t_k}. \qquad (3.4)$$

We wish to prove that this is a decreasing function, so we compute the derivative with respect to i:

$$V'(i) = \sum_{k=1}^{n} P_k(t_* - t_k)(1+i)^{t_*-t_k-1}. \qquad (3.5)$$

We now make a smart choice for t_*: we choose it so that all expenses are due before t_* and all earnings are due after t_*. In other words, t_* is between t_ℓ (the last expense) and $t_{\ell+1}$ (the first income). Now, if $k \leq \ell$ then $t_k < t_*$ and thus $t_* - t_k > 0$. Furthermore, the first ℓ payments are negative, so if $k < \ell$ then $P_k < 0$. In combination with our assumption that $i > -1$, we conclude that

$$\text{if } k \leq \ell \text{ then } P_k(t_* - t_k)(1+i)^{t_*-t_k-1} < 0.$$

On the other hand, if $k \geq \ell + 1$, then $t_* - t_k < 0$ and $P_k > 0$, so

$$\text{if } k \geq \ell + 1 \text{ then } P_k(t_* - t_k)(1+i)^{t_*-t_k-1} < 0.$$

This mean that all terms in the sum in (3.5) are negative, Thus, $V'(i) < 0$; the net value is a decreasing function with respect to the interest rate. This means that the graph of $V(i)$ can cross the x-axis at most once, so the internal rate of return is unique.

This does not show that the graph of $V(i)$ does in fact cross the x-axis. For that, we compute the value of the cash flow in the limits $i \downarrow -1$ and $i \to +\infty$. In the limit $i \downarrow -1$ (that is, i approaches -1 from above), $1 + i$ is small but positive. We have

$$\lim_{i \downarrow -1} (1+i)^a = \begin{cases} +\infty, & \text{if } a < 0; \\ 0, & \text{if } a > 0. \end{cases}$$

The kth term in the sum (3.4) for $V(i)$ is $P_k(1+i)^{t_*-t_k}$. First consider the case $k \le \ell$. The exponent $t_* - t_k$ is positive if $k \le \ell$, so $P_k(1+i)^{t_*-t_k} \to 0$ as $i \downarrow -1$. In the case $k \ge \ell + 1$, the exponent $t_* - t_k$ is negative, and P_k is positive, so $P_k(1+i)^{t_*-t_k} \to +\infty$ as $i \downarrow -1$. Adding all terms, we find that $V(i) \to +\infty$ in the limit $i \downarrow -1$.

Next, consider the limit $i \to \infty$. Now we have

$$\lim_{i \to \infty} (1+i)^a = \begin{cases} 0, & \text{if } a < 0; \\ +\infty, & \text{if } a > 0. \end{cases}$$

Thus, $P_k(1+i)^{t_*-t_k} \to -\infty$ if $k \le \ell$ and $P_k(1+i)^{t_*-t_k} \to 0$ if $k \ge \ell + 1$. We find that

$$\lim_{i \downarrow -1} V(i) = +\infty \quad \text{and} \quad \lim_{i \to \infty} V(i) = -\infty.$$

Since $V(i)$ goes from $+\infty$ to $-\infty$, there must be an $i \in (-1, \infty)$ for which $V(i) = 0$. Thus, the internal rate of return exists and is unique.

The above discussion shows that a cash flow where all expenses are incurred before the earnings are due has a well-defined yield. The same is true in the reverse situation, where all earnings are due before the expenses are incurred, as happens for instance when borrowing money. The proof is very similar.

There is another situation in which the yield is well-defined, namely, if all *cumulative* payments before a certain time t_* are negative and all cumulative payments afterwards are positive (or the other way around, with all cumulative payments before t_* positive and all after t_* negative). The kth cumulative payment C_k is the sum of the first k payments:

$$C_k = \sum_{j=1}^{k} P_j. \tag{3.6}$$

Thus, C_k is the total amount the investor has received at the time of the kth payment. If there is one and only one time at which the cumulative cash flow changes sign, then there is only one positive value of i for which the net value of the cash flow is zero. We will not prove this result in these lecture notes.

Example 3.2.3. A cash flow with payments -5, 3, -1, 7, -1 and 3 after 0, 2, 3, 7, 8 and 10 years, respectively, has cumulative payments as shown in the table below:

k	t_k	P_k	C_k
1	0	-5	-5
2	2	3	-2
3	3	-1	-3
4	7	7	4
5	8	-1	3
6	10	3	6

Even though the negative and positive payments are interleaved, all negative C_k are before the positive C_k. After seven years, at the time of the fourth payment, the cumulative cash flow changes from negative to positive (as explained in the next section, this time is called the *payback period*). Thus the yield of this cash flow is well defined. A computation as in Example 3.2.2 shows that the yield is approximately 13.55% p.a.

The two situations discussed above (payments change sign only once, and cumulative payments change sign only once) cover many cash flows occurring in practice. The internal rate of return of a investment project is usually well defined.

3.3 Investment project appraisal

Section 3.1 defined the net present value of a cash flow, while Section 3.2 discussed the internal rate of return. Both quantities are often used to evaluate and compare investments and other projects. Project with higher net present values and yields are more profitable to invest in.

If, as in many investments, there is a period of outlays followed by a period of profits, then the net present value is a decreasing function of the interest rate. The net present value is positive, and thus the project profitable, if the investor can borrow and lend money at a rate smaller than the internal rate of return. Thus, a project with a higher yield has more chance to be profitable.

This does not mean, however, that a project with a higher yield will always be more profitable than a project with a lower yield. It is the project with the higher net value for a given interest rate that is the most profitable.

Example 3.3.1 (McCutcheon & Scott, p. 92)**.** An investor is contemplating two investment projects. Project A requires an initial payment of £10000, in return for which the investor will receive £250 at the end of every quarter for 15 years. Project B requires an initial payment of £11000. In return, the investor will be paid £605 at the end of every year for 18 years and the initial payment of £11000 will be repaid at the end.

Both projects have one outlay at the start, and payments to the investor afterwards, so they both have a well-defined internal rate of return. The net present value for Project A is given by

$$\mathrm{NPV}_A(i) = -10000 + 1000 a^{(4)}_{\overline{15}|}.$$

The yield is found by solving $\mathrm{NPV}_A(i) = 0$, or $a^{(4)}_{\overline{15}|} = 10$, resulting in a yield of (approximately) 5.88% p.a. On the other hand, the net present value for Project B is

$$\mathrm{NPV}_B(i) = -11000 + 605 a_{\overline{18}|} + 11000 v^{18}.$$

The internal rate of return for Project B is 5.5% p.a.

The net present values of both projects are plotted in Figure 3.3. This shows that if the interest rate is low enough, Project B is more profitable than Project A, even though it has a lower yield. The rate at which the graph cross, that is, the rate i at which $\mathrm{NPV}_A(i) = \mathrm{NPV}_B(i)$, is called the *cross-over rate*. In this example, the cross-over rate is approximately 5.11%.

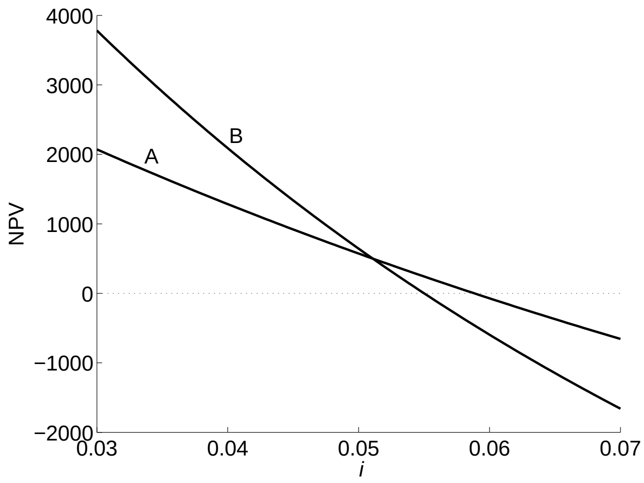

Figure 3.3: The net present values of the two projects in Example 3.3.1.

If the investor can borrow money for a lower rate than the cross-over rate, he will make a larger profit on Project B than on Project A. For instance, if the investor may lend or borrow money at 4%, then the profit on Project A is $\mathrm{NPV}_A(0.04) = 1283.81$, while the profit on Project B is $\mathrm{NPV}_B(0.04) = 2088.82$.

Another quantity that is sometimes used to evaluate investment projects is the *payback period*. This is the answer to the simple but compelling question: "When do I get my money back?" The payback period is the first time t_*, such that all payments up to that time added together are positive. In terms of the cumulative payments C_k defined in (3.6), if j is the smallest integer such that $C_j > 0$, then the payback period is t_j.

A disadvantage of the payback period is that it does not take the time value of money into account. This leads to a more sophisticated measure, called the *discounted payback period (DPP)*. Given an interest rate i, the discounted payback period is the first time t_*, such that the value of all payments up to that time is positive. In other words, if the cumulative discounted payments are defined by

$$\tilde{C}_k = \sum_{j=1}^{k} v^{t_j} P_j, \tag{3.7}$$

and j is the smallest integer such that $\tilde{C}_j > 0$, then the payback period is t_j.

The discounted payback period is the period that investors have to wait to get their money back if they borrow the money that they invest in the project against a rate i.

Example 3.3.2 (Continuation of Example 3.2.3)**.** Consider the same project as before, with payments −5, 3, −1, 7, −1 and 3 at times 0, 2, 3, 7, 8 and 10, respectively. We compute the discounted payments using a rate of 8% and 12%.

				$i = 0.08$		$i = 0.12$	
k	t_k	P_k	C_k	$v^{t_k} P_k$	C'_k	$v^{t_k} P_k$	C'_k
1	0	-5	-5	-5.00	-5.00	-5.00	-5.00
2	2	$+3$	-2	$+2.57$	-2.43	$+2.39$	-2.61
3	3	-1	-3	-0.79	-3.22	-0.71	-3.32
4	7	$+7$	$+4$	$+4.08$	$+0.86$	$+3.17$	-0.15
5	8	-1	$+3$	-0.54	$+0.32$	-0.40	-0.56
6	10	$+3$	$+6$	$+1.39$	$+1.71$	$+0.97$	$+0.41$

As explained in Example 3.2.3, the fourth cumulative payment is the first one that is positive, thus the payback period is the time of the fourth payment, which is $t = 7$. This does not change if all payments are discounted at a rate of 8%; the discounted payback period is still seven years. However, if a rate of 12% is used, then the discounted payback period is 10 years.

The table also shows that the net present value a rate of 8% is 1.71. This is the profit the investor makes (valued at $t = 0$) assuming that the investor borrows the money he invests in the project, and that he can both borrow and deposit money at an interest rate of 8%. However, this assumption is not very realistic: banks generally pay less interest on a deposit than they charge on a loan. This leads us to consider variable interest rates later in the module, in Chapter 4.

As the example shows, the discounted payback period computed on the basis of an interest rate of 8% measures how long it takes before the investor recoups his investment, if he borrows the money against a rate of 8%. The accumulated value of the project does not only depend on the rate against which the investor can borrow money, but also on the rate at which the profits can be reinvested (the *reinvestment rate*). The definition of the internal rate of return (IRR) assumes that these rates are equal. This assumption is not always valid. When it is not valid, it is dangerous to compare different investment projects using the internal rate of return, especially when the interest rate fluctuate widely or the investment periods are very different.

We end this section on the appraisal of investment projects by acknowledging that nothing is certain in life except death and taxes. Investment may not make the returns that we expect them to made. A borrower may default on a loan. These risks are difficult to estimate and handle, but taking them properly into account is of great importance. An easy way to compensate for the risk of default is to demand a higher interest rate or a higher yield on a transaction (a *risk premium*), but then the question becomes how high the risk premium should be. More sophisticated methods use the theory of probability. This will be treated in *Financial Mathematics II* and other modules.

3.4 Investment funds

An investment fund is a collective scheme which pools and invests money from multiple parties. The pool of money administered by the fund changes over time. This complicates the assessment of the performance of an investment fund. The value of the fund changes not only due to the performance of its investments, but also due to additional investments or withdrawals by its members.

The following example shows that the internal rate of return can be used to assess the performance of an investment fund.

Example 3.4.1. Consider an investment fund that starts out with £100,000. After one year, this has increased to £105,000. The investors deposit an additional £50,000 to the funds, so the total sum administered is now £155,000. The second years turns out to be very profitable: the value of the funds increases to £195,000. At this time, the investors withdraw £60,000, leaving £135,000 in the funds. Unfortunately, the investments go sour in the third year, and the value of the funds decreases to £125,000. What is the yield achieved by the investment fund over these three years?

Answer. If the fund would be wound up, the fund would have to pay £125,000 back to the investors. Thus, the cash flow is +100 (measured in thousands of pounds) at $t = 0$ (measured in years), +50 at $t = 1$, −60 at $t = 2$ and −125 at $t = 3$. The equation of value at $t = 0$ is

$$100 + 50(1+i)^{-1} - 60(1+i)^{-2} - 125(1+i)^{-3} = 0. \tag{3.8}$$

The solution of this equation is $i = 0.0937$, so the yield rate is 9.37%.

The fund manager in the example achieved a return of $\frac{5}{100} = 5\%$ in the first year, $\frac{40}{155} = 25.81\%$ in the second year, and $-\frac{10}{135} = -7.41\%$ in the third year. If you invest money against these rates, the yield over three years is

$$\left(\left(1+\frac{5}{100}\right)\left(1+\frac{40}{155}\right)\left(1-\frac{10}{135}\right)\right)^{1/3} - 1 = (1.223118)^{1/3} - 1 = 0.0694,$$

or 6.94%. The yield rate of 9.37% that was computed in the example is considerably higher because that computation gives more weight to periods in which more money is invested in the funds. In the example, the second year has the most money in the funds, so that year has more influence on the yield rate than the first and last year. Coincidentally, the funds performed considerably better in the second year than in the first and last year. This explains why the yield rate is so high.

Because the internal rate of return gives more weight to periods in which more money is invested in the funds, it is often called the *money-weighted rate of return* in this context. Yet another name is the *dollar-weighted rate of return*, though this is obviously not appropriate for British funds. In contrast, the rate of 6.94% is called the *time-weighted rate of return*. The time-weighted rate of return is a fairer measure of the performance of a fund manager, because it eliminates the effect of investors depositing or withdrawing money from the funds. However, the money-weighted computation is a valid measure of the actual investment results achieved.

The formal definitions of the money-weighted rate of return and the time-weighted rate of return are as follows.

Definition 3.4.2. Let V_k with $k = 0, 1, \ldots, n$ denote the value of an investment fund at time t_k, where the times are ordered such that $t_0 < t_1 < \ldots < t_n$. Let c_k with $k = 1, 2, \ldots, n_1$ denote the new cash deposited into the fund at time t_k (if $c_k < 0$ then cash is withdrawn). The value V_k is measured just before the new cash is deposited or withdrawn. The *money-weighted rate of return* is the rate i which satisfies

$$V_0(1+i)^{t_n-t_0} + c_1(1+i)^{t_n-t_1} + c_2(1+i)^{t_n-t_2} + \cdots + c_{n-1}(1+i)^{t_n-t_{n-1}} = V_n. \quad (3.9)$$

The *time-weighted rate of return* is the rate i which satisfies

$$(1+i)^{t_n-t_0} = \frac{V_1}{V_0} \cdot \frac{V_2}{V_1+c_1} \cdot \frac{V_3}{V_2+c_2} \cdots \frac{V_n}{V_{n-1}+c_{n-1}}. \quad (3.10)$$

Example 3.4.3 (Continuation of Example 3.4.1)**.** In the situation of Example 3.4.1, the value of the fund is $V_0 = 100000$, $V_1 = 105000$, $V_2 = 155000$ and $V_3 = 125000$ at times $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ and $t_3 = 3$, respectively. The flow of new money is given by $c_1 = 50000$ and $c_2 = -60000$. Thus, equation (3.9) for the money-weighted rate of return gives

$$100000(1+i)^3 + 50000(1+i)^2 - 60000(1+i) = 125000.$$

This is equivalent to equation (3.10) which we found before; the money-weighted rate of return is 9.37%.

Equation (3.10) for the time-weighted rate of return gives

$$(1+i)^3 = \frac{105000}{100000} \cdot \frac{195000}{105000+50000} \cdot \frac{125000}{195000-60000}.$$

The solution of this equation is $i = 0.0694$, so the time-weighted rate of return is 6.94%, as we found before.

The *linked internal rate of return* is another measure that is sometimes used. It is a variant of the time-weighted rate of return which approximates the flow of new money into and out of the funds. We will not be considering the linked internal rate of return in this module.

3.5 Bonds

A *bond* is a certificate of indebtedness issued by a borrower. It promises to pay a certain amount of money at a specified date (or several amounts at several dates). If a company or government agency wants to borrow money, it can go to a bank or it can issue bonds and sell these to investors. Bonds are usually issued for a round value such as £100 in the U.K. and \$1000 in the U.S.; this is called the *face value* of the bond.

Most bonds specify when the loan is paid back. This is called the *maturity date*, and the period between the issue of the bond and its maturity is the *term* of the bond. The amount paid to the investor at the maturity date is called the *redemption value* of the bond. The redemption value usually equals the face value; in this case the bond is said to be redeemed *at par*. We will assume in this module that all bonds are redeemed at par.

Many bonds also pay interest on the loan in addition to repaying it at maturity. These interest payments are called *coupons*. The coupons are computed by multiplying an interest rate specified on the bond (the *coupon rate*) by the face value. Thus, a bond with a coupon rate of 8% and a face value of £100 would pay coupons of £8 if the coupons are paid annually. Typically, coupons are payable semi-annually and the nominal rate is quoted, so a bond with coupons of 8% payable semi-annually pays coupons of £4 every half year.

Governments usually have to raise large sums of money and they do this by issuing bonds. Bonds issued by the U.K. government are known as *gilts*. Bonds issued by the U.S. governments are called *Treasury notes* if the term is relatively long, and *Treasury bills* (or *T-bills*) if the term is relatively short.

A fundamental computation is to compute the price an investor has to pay for a bond to achieve a certain yield. The basic rule is that the price of the bond must be equal to the present value of the coupons plus the present value of the redemption value, where the present values are computed using the given yield rate.

Example 3.5.1. Find the price of a five-year bond with a face value of £100 and coupons at 8% p.a. payable semi-annually. The bond is redeemable at par and bought to yield 5%.

Answer. The investor will receive half-yearly payments of £4 for five years. We use time units of a year. The present value of these payments is $8a^{(2)}_{\overline{5}|}$ (remember that $a^{(2)}_{\overline{n}|}$ is the present value of semi-annual payments of *a half*). Additionally, the investor will receive £100 at maturity; the present value of this payment is $100v^5$. Here, the yield rate is given as $i = 0.05$, so $v = 0.952381$ and $i^{(2)} = 0.0493901$. We compute the present value of both the coupons and the redemption value using $i = 0.05$ (the given yield rate) as

$$\begin{aligned} 8a^{(2)}_{\overline{5}|} + 100v^5 &= 8 \cdot \frac{1-v^5}{i^{(2)}} + 100v^5 = 8 \cdot 4.382935 + 100 \cdot 0.783526 \\ &= 35.0635 + 78.3526 = 113.4161. \end{aligned}$$

So, the price of the bond is £113.42.

We can also use time units of half a year. The yield rate of 5% p.a. is equivalent to a rate of 2.4695% per half-year, so the discount factor is $v = 1/1.024695 = 0.975900$. The present value of the coupons and the payment at maturity is now computed as

$$\begin{aligned} 4a_{\overline{10}|} + 100v^{10} &= 4 \cdot \frac{1-v^{10}}{i} + 100v^{10} = 4 \cdot 8.765870 + 100 \cdot 0.783526 \\ &= 35.0635 + 78.3526 = 113.4161, \end{aligned}$$

which is of course the same as we found using time units of a year.

The same reasoning shows that the price to be paid to achieve a yield of i for a bond with a face value of F and coupons payable half-yearly at a rate of D per annum, redeemable at par in n years, is

$$FDa^{(2)}_{\overline{n}|} + Fv^n \quad \text{at a rate of } i. \tag{3.11}$$

The coupon rate of D p.a. payable semi-annually corresponds to an effective annual rate of $(1 + \frac{1}{2}D)^2 - 1$; this is formula (1.5) relating nominal and effective interest rates. For the bond in Example 3.5.1, the coupon rate is 8% p.a. payable semi-annually, which corresponds to an effective rate of 8.16% p.a. If the yield rate equals the effective coupon rate, meaning that $i = (1 + \frac{1}{2}D)^2 - 1$, then $i^{(2)} = D$ and thus the price of the bond is

$$FDa_{\overline{n}|}^{(2)} + Fv^n = F(1 - v^n) + Fv^n = F.$$

Thus, if the yield rate equals the coupon rate, the price of a bond equals the face value. In this case, the bond is said to sell *at par*. If the yield rate i decreases, the present value of the coupons and the redemption value increase. Thus, the price of the bond is higher than the face value if the coupon rate is higher than the yield rate. This is the case in Example 3.5.1. We say that the bond sells *at a premium* (the buyer of the bond has to pay a premium above the face value and gets higher coupons in return). In the opposite situation, where the coupon rate is lower than the yield rate and the price of the bond is lower than the face value, the bond is said to sell *at a discount*. This is yet another meaning of the word "discount" in financial mathematics.

The relationship between the price and the yield of a bond can also be understood algebraically, starting from equation (3.11) for the price of a bond. The theory of annuities treated in Chapter 2, specifically equation (2.11), implies that

$$v^n = 1 - i^{(2)} a_{\overline{n}|}^{(2)},$$

so we can rewrite the formula for the price as

$$\begin{aligned} \text{price} = FDa_{\overline{n}|}^{(2)} + Fv^n &= FDa_{\overline{n}|}^{(2)} + F\Big(1 - i^{(2)} a_{\overline{n}|}^{(2)}\Big) \\ &= F + \underbrace{F\Big(D - i^{(2)}\Big) a_{\overline{n}|}^{(2)}}_{\text{premium/discount}} \end{aligned} \tag{3.12}$$

This is called the *premium/discount formula*, because it gives the price as a sum of the face value and the premium or discount. In the situation of Example 3.5.1, we have

$$\text{premium} = F\Big(D - i^{(2)}\Big) a_{\overline{n}|}^{(2)} = 100 \cdot (0.08 - 0.0493901) \cdot 4.382935 = 13.416,$$

which corresponds to the price of £113.42 that we found before.

The premium/discount formula shows immediately that if the yield equals the coupon rate (both expressed as nominal rates), the price of the bond equals the face value. If the coupon rate is higher than the yield ($D > i^{(2)}$), then the bond trades at a premium, and if the coupon rate is lower than the yield ($D < i^{(2)}$), then the bond trades as a premium. If the yield increases, then $i^{(2)}$ increases and thus the price of the bond decreases.

A consequence of the inverse relationship between the price and the yield of a bond is that the bonds become cheaper if the interest rate (or more precisely, the expected interest rate in the future) increases. This is the main source of fluctuations in the bond market.

Example 3.5.1 above shows how to compute the price of a bond given the yield rate. The opposite computation, where the yield rate is to be found given the price, can be performed as explained in Section 3.2: write down the equation of value and solve it for the rate i. Usually, we can find only an approximate solution, using techniques like linear interpolation.

Example 3.5.2. A bond with a face value of £100 and coupons at 8% p.a. payable semi-annually is redeemable in five years at par. Find the yield if the bond is bought for £119.25.

Answer. This is the same bond as in Example 3.5.1. The price of the bond should equal the present value of the semi-annual coupons of £4 each plus the present value of the £100 paid at maturity:

$$8a^{(2)}_{\overline{5}|} + 100v^5 = 119.25.$$

Writing everything in terms of the rate i, this becomes

$$8 \cdot \frac{1-(1+i)^{-5}}{2((1+i)^{1/2}-1)} + 100(1+i)^{-5} = 119.25.$$

The solution of this equation is $i = 0.0377774$ so the yield is 3.78%.

Unfortunately, this solution cannot be found analytically. We can find find an approximate solution for the yield rate using linear interpolation if we know the bond price for two yield rates. In Example 3.5.1, we found that if the yield is 5% p.a., the bond costs £113.42. Furthermore, we know that the price of the bond equals the face value if the yield rate equals the coupon rates. Here, the coupon rate is 8% p.a. payable semi-annually, which corresponds to an effective rate of $(1 + 0.08/2)^2 - 1 = 0.0816$ or 8.16% p.a. Thus, we use the formula for linear interpolation

$$x_* = x_1 + (y_* - y_1)\frac{x_2 - x_1}{y_2 - y_1}.$$

with $x_1 = 0.05$, $y_1 = 113.42$, $x_2 = 0.0816$, $y_2 = 100$ and $y_* = 119.25$. We find $x_* = 0.03627$, so the approximate yield of the bond is 3.63%. This is fairly close to the exact solution of 3.78%. Another round of interpolation would allow us to come even closer to the exact solution.

As illustrated in the example, the yield of a bond that is trading at a premium (that is, the price is higher than the face value) is lower than the coupon rate. The relation between yield and price is shown in Figure 3.4.

A source of confusion is the difference between effective and nominal rates. The coupon rate is always quoted as a nominal rate, convertible with the same frequency as the coupon payments. The yield rate of 3.78% computed in the above example is an effective rate. The corresponding nominal rate is

$$2\big((1 + 0.0378)^{1/2} - 1\big) = 0.0374 \text{ or } 3.74\%.$$

When comparing two rates, you should make sure they both are both effective rates or that they both are nominal rates convertible with the same frequency. Thus, in the premium/discount formula, equation (3.12), the coupon rate and yield are both nominal rates. Bond yields in the financial press are usually quoted using nominal rates. This is appropriate when comparing different bonds because (almost) all bonds pay coupons with the same frequency. However, when comparing bond yields with yields on other investments, the bond yield has to be converted to an effective annual rate.

Another source of confusion is that people sometimes consider another yield rate called the *interest yield*. This is the yield computed on the basis of only the interest payments, while ignoring the redemption value. The interest yield of the

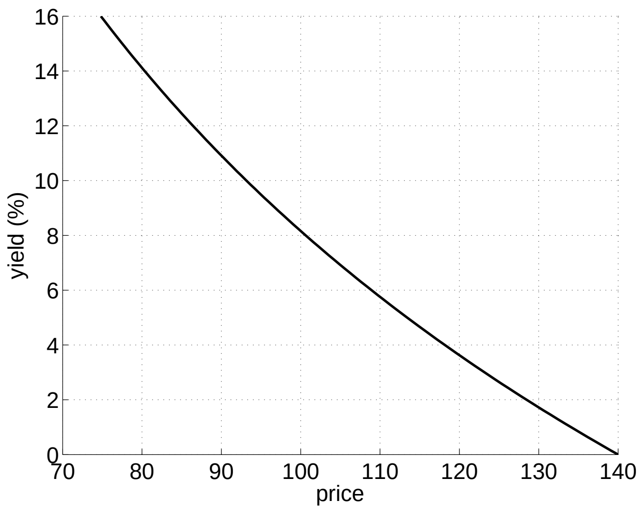

Figure 3.4: The relation between the price and the yield of a five-year bond with a face value of £100 and coupons at 8% p.a. payable semi-annually (as treated in the examples in this section). If the price increases, the yield decreases, and vice versa.

bond in Example 3.5.2, which has a coupon rate of 8% and trades at £119.25, is $8/119.25 = 0.0671$ or 6.71%. This rate is the nominal rate p.a. convertible semi-annually; the effective rate is 6.82%. The interest yield is also known as the *flat yield*. The yield that we considered in the rest of this section, which does take the redemption value into account, is sometimes called the *redemption yield* or the *yield to maturity* if it is necessary to distinguish it from the interest yield.

If a bond is trading at a premium then the investor will receive less money at maturity than he paid for the bond, so he will suffer a loss at maturity. Thus, the redemption yield is lower than the interest yield for bonds trading at a premium. On the other hand, the coupon rate is higher than the interest yield for bonds trading at a premium: the interest yield is the coupon payments divided by the price of the bond, while the coupon rate is the coupon payments divided by the face value, and the price is higher than the face value because the bond trades at a premium. This is the case in Example 3.5.2, where the bond has a redemption yield of 3.74%, an interest yield of 6.71%, and a coupon rate of 8% (all quoted as nominal rates). Conversely, the redemption yield is higher than the interest yield which is higher than the coupon rate when the bond is trading at a discount.

The difference between the redemption yield and the interest yield depends on the term to redemption. If it takes very long before the bond matures, then the present value of the profit or loss for the investor at maturity is small, because it is discounted for a very long time. Thus, the difference between the redemption yield and the interest yield decreases as the redemption term increases. In the limit $n \to \infty$, the bond never matures (so the coupons form a perpetuity), and the redemption and interest yields are the same.

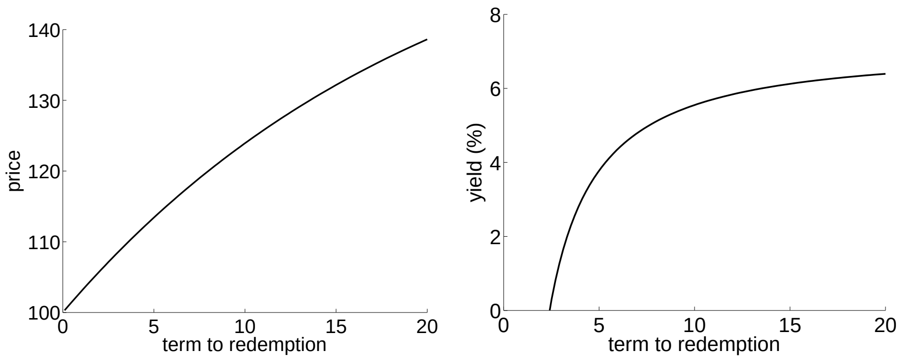

Figure 3.5: The relation between the redemption term and the price and yield of a bond. Consider a bound with a face value of £100 and a coupon rate of 8% payable semi-annually, whose redemption term varies. The left plot shows the price of the bond if it is bought to yield 5%, as in Example 3.5.1. The right plot shows the yield of the bond if it is bought for £119.25, as in Example 3.5.2.

If a bond is trading at a premium, then the redemption yield is lower than the interest yield. Since the redemption yield approaches the interest yield as the redemption term increases, and the interest yield stays constant, the redemption yield must increase. This is shown in the right plot of Figure 3.5, which illustrate the influence of the redemption term on the yield of the bond in Example 3.5.2, which is trading at a premium. In the opposite situation, the bond is trading at a discount, the redemption yield is higher than the interest yield, and the redemption yield decreases as the redemption term increases.

The effect of the term to redemption on the price can be seen from formula (3.11) for the price of a bond, which we rewrite as

$$\text{price} = FDa_{\overline{n}|}^{(2)} + Fv^n = FD\frac{1-v^n}{i^{(2)}} + Fv^n = \frac{FD}{i^{(2)}} + F\left(1 - \frac{D}{i^{(2)}}\right)v^n.$$

In the limit $n \to \infty$, the second term goes to zero, so the price of the bond approaches $FD/i^{(2)}$. We also know that the price equals the face value F if the bond matures immediately ($n = 0$). The price of the bond converges to the face value as the maturity date is approached; this is known as the *pull to par*.

If $i^{(2)} < D$, the yield is lower than the coupon rate so the bond is trading at a discount. In this case, the price is lower than the face value, and the above formula for the price shows that price increases monotonically as the redemption rate increases if the yield is held constant. This is illustrated in the left plot of Figure 3.5. In the opposite situation, where the bond is trading at a discount, the price decreases as the redemption rate increases if the yield is held constant.

Zero-coupon bonds, as the name indicates, are bonds that do not pay out any coupons. The bond holder does receives a payment (the redemption value) when the bond matures, but he does not receive any interest payments. Zero-coupon bonds are used when the term of the bond is short. For instance, U.S. Treasure bills are zero-coupon bonds. Zero-coupon bonds with longer terms are created by brokers, who take a bond with non-zero coupons and sell the redemption payment and the coupon payments separately.

Zero-coupon bond are easy to analyse, because the bond holder receives only one payment. The price of a zero-coupon bond is simply the redemption payment at maturity, discounted to the present time. A bond with face value F maturing in n years costs Fv^n. This agrees with (3.11) if considering a zero-coupon bond as a bond with coupon rate $D = 0$. Thus, zero-coupon bonds always trade at a discount and the price of a zero-coupon bond increases over time as the bond approaches maturity.

It is also easy to compute the yield of a zero-coupon bond. Consider for example a zero-coupon bond with a face value of £100 which matures in one year. If this bond is bought for £95, then the equation of value at the time of purchase is $95 = (1 + i)^{-1} \cdot 100$. The yield is found by solving this equation: $i = \frac{100}{95} - 1 = 0.05263$, so the yield is 5.26%.

Normal bonds, for which the coupon and redemption payments are fixed, are called *vanilla bonds*. There are also more exotic bonds, where the payments are not fixed beforehand. For example, the coupons and redemption value of an *index-linked bond* change according to the inflation index. These bonds are discussed in the next section. Another example are bonds with no fixed redemption date; instead, the terms of the bond specify that the borrower may redeem the bond when he wants, as long as it is redeemed between two specified date. In the past, the U.K. government issued many of these bonds with optional redemption rates. They are slowly disappearing from the market as they reach maturity. One of the remaining issues is 12% Exchequer stock 2013–2017, which pays 12% interest and has to be redeemed by the government between 2013 and 2017. Undated gilts, which may be redeemed by the government in any year of its choosing, also exist. The only issue that is actively traded is the $3\frac{1}{2}$% War Loan, which was issued by the U.K. government to finance World War I. Since $3\frac{1}{2}$% is historically a very low rate, it is considered unlikely that the government will redeem this gilt, instead choosing to make coupon payments forever. These bonds are therefore usually modelled as a perpetuity (see Section 2.2).

Bond are *fixed-income securities*: the payments of the bonds are specified beforehand (at least, for vanilla bonds). However, there is still a risk of *default*, which means that the borrower fails to make the promised payments. The risk of default is considered very low for bonds issued by the U.K. and U.S. governments. However, it does happen that national government default on their bonds. Argentina defaulted on parts of its debt in 2002. The risk of default is bigger for bonds issued by companies because the companies may go bankrupt. Investors want to be compensated for this risk and thus companies have to offer higher interest rates when issuing bonds. Bonds with significant risk of default are called *junk bonds* or *high-yield bonds*. The valuation of bonds in the presence of a default risk requires probability theory and is thus outside the scope of this module.

The techniques in this module can also be used for the valuation of other investment like shares and real-estate property. The buyer of an *equity share* in a company becomes owner of a part of the company. This allows the shareholder to share in the profits in the company in the form of *dividends* paid by the company to its shareholder. Usually shareholders also have a say in the running of the company. If a company goes bankrupt, creditors such as bondholders are paid before owners such as shareholders. The owner of real-estate property derives a regular income from it in the form of *rent*. Both these cash flows can be valued with the techniques discussed in this module. However, the income derived from equity shares and real-estate property is subject to many uncertainties, and it may be best to use a more sophisticated model reflecting these risks.

3.6 Inflation

Inflation is a general increase in prices over time. Inflation decreases the purchasing power of money: you can buy less for £1000 in 2010 than you could buy for £1000 in 2000. It is one of the reasons that you receive interest when depositing money on the bank: you need to be compensated for the loss of purchasing power of the money that you deposit.

Example 3.6.1. The interest rate is 6% and the rate of inflation is 4%. This means that if you deposit £100, you will have £106 one year later. However, products that would cost £100 when you deposited the money, now cost £104. If we say that the 100 pounds you deposited had a purchasing power of 1, then the purchasing power of the 106 pounds you have after a year is $106/104 = 1.01923$. Thus, the interest rate after taking the effects of inflation into account is 1.923% p.a.

The rate of 1.923% in the example is called the *real rate of interest*. The actual interest rate of 6% is called the *nominal rate of interest*. The word "nominal" has a very different meaning in this context than in Section 1.5, when we discussed interest payable with different frequencies (as in the nominal rate payable monthly).

The nominal rate of interest is the rate of interest that we discussed in this module and that we denoted by i. In this section, we denote the real rate of interest by i' and the rate of inflation by r. These rates are related by

$$1 + i = (1 + i')(1 + r). \tag{3.13}$$

The nominal rate of interest combines the real rate of interest with the effect of inflation. In the example, we have $1.06 = 1.01923 \cdot 1.04$.

When computing the real rate of interest, as we did in Example 3.6.1, we need to solve (3.13) for i':

$$1 + i' = \frac{1 + i}{1 + r}, \quad \text{so} \quad i' = \frac{i - r}{1 + r}.$$

Substituting the values from the example in this formula, we get

$$i' = \frac{0.06 - 0.04}{1 + 0.04} = 0.01923.$$

If we multiply out the right-hand side of (3.13), we find $1 + i = 1 + i' + r + i'r$, which simplifies to

$$i = i' + r + i'r.$$

The term $i'r$ is often ignored in practice, because it is small and the rate of inflation is not accurately known. This leads to the approximation $i \approx i' + r$. In the example where the nominal rate of interest is 6% and the rate of inflation is 4%, this would yield a real rate of interest of 2% instead of 1.923%.

Inflation is usually measured with the help of a *price index*. A price index is a weighted average of prices of several goods and services. Usually the price index is normalized so that it equals 100 in a specific year. The National Office of Statistics in the U.K. computes the *Retail Price Index (RPI)*, which is normalized to be 100 in 1987. It reached 200 points in 2006. The U.S. equivalent is the *Consumer Price Index (CPI)*, computed by the Bureau of Labor Statistics. The inflation rate over a particular year equals the increase in the price index over that year divided by the price index at the start of the year.

Just as the real rate of interest is the nominal rate of interest adjusted for inflation, the *real rate of return* (or *real yield*) is the nominal rate of return adjusted for inflation. If the rate of inflation is constant, then we can use formula (3.13) to compute the real rate of return from the nominal rate of return. For instance, if the rate of inflation is constant and equal to 4% p.a., and the nominal rate of return is 6% p.a., then the real rate of return is 1.923% p.a.

The computation of the real rate of return of an investment is more complicated if the rate of inflation is not constant. Recall that the (nominal) value at time t_* of a payment P due at time t is $P(1+i)^{t_*-t}$. Thus, the equation of value for an investment consisting of payments P_k due at times t_k is

$$\sum_k P_k(1+i)^{t_*-t_k} = 0,$$

and the value of i that solves this equation is the (nominal) yield. However, the payment of P due at time t equals $P/Q(t)$ in real terms, where $Q(t)$ denotes the value of the price index at time t; the payment can buy $P/Q(t)$ units of the index. Thus, the equation of value in real terms is

$$\sum_k \frac{P_k}{Q(t_k)}(1+i')^{t_*-t_k} = 0, \tag{3.14}$$

and the real rate of return is the value of i' that solves this equation.

The procedure is illustrated in the following example.

Example 3.6.2 (After McCutcheon & Scott, p. 180)**.** A bank lends £1000 in January 2005 against 5% interest. The borrower pays £50 in interest for four years, and repays the loan in January 2009. The RPI for January 2005, 2006, 2007, 2008 and 2009 is 188.9, 193.4, 201.6, 209.8, and 210.1, respectively. Compute the real rate of return.

Answer. The equation of value in real terms is found by substituting the date into (3.14), yielding

$$-\frac{1000}{188.9}(1+i')^4 + \frac{50}{193.4}(1+i')^3 + \frac{50}{201.6}(1+i')^2 + \frac{50}{209.8}(1+i') + \frac{1050}{210.1} = 0.$$

Here, we chose the end of the loan (January 2009) for t_*. We can find the real yield approximately with linear interpolation.

The rate of inflation is approximately 3% p.a., so let's try $i' = 0.02$ as initial guess. The left-hand side of the equation of value in real terms evaluates to

$$-5.7302 + 0.2744 + 0.2580 + 0.2431 + 4.9976 = 0.0429.$$

This is slightly above zero. We then try $i' = 0.03$; the left-hand side is now

$$-5.9582 + 0.2825 + 0.2631 + 0.2455 + 4.9976 = -0.1695.$$

This shows that the real yield is between 2% and 3%. Linear interpolation yields the approximation

$$0.02 + (0.02 - 0.03) \cdot \frac{0.0429 - 0}{0.0429 - (-0.1695)} = 0.02202.$$

The real yield is thus approximately 2.2% p.a.

Index-linked bonds are bonds whose coupons and maturity value depends on a price index. For instance, the $1\frac{1}{8}\%$ *Index-linked Treasury Gilt 2037* pays coupons at a rate which lies $1\frac{1}{8}\%$ above inflation, as measured by the RPI. The payments at maturity is also corrected for inflation. This 30-year gilt was issued by the U.K. Treasury in 2007. An analogous index-linked bond in the U.S. market is the Treasury Inflation-Protected Security (abbreviated TIPS).

Index-linked bonds are attractive for investors, especially for parties who have future obligations which rise in line with inflation, because they guarantee a certain *real* rate of return, independent of the inflation rate. The investor is thus shielded against the effects of inflation.

The coupons of an index-linked bond are computed as follows. Let $Q(t)$ be the relevant price index at time t, and consider a £100 bond with coupon rate D paid semi-annually issued at time t_0. Then the coupon payment at time t is

$$100 \, \frac{D}{2} \, \frac{Q(t)}{Q(t_0)}. \tag{3.15}$$

This has the same purchasing power as a payment of $100\frac{D}{2}$ at time t_0 when the bond was issued. The maturity value of the bond is also linked to the price index. If the bond matures at time t_m, then it pays out $100Q(t_m)/Q(t_0)$ at maturity.

The meaning of "the relevant price index at time t" is a bit subtle in practice. It takes some time for the National Office of Statistics to compute the RPI, and thus the RPI for a given month, say March, is only known some time afterwards, say in the middle of April. Thus, it is not possible to compute and pay the coupons of an index-linked bond in March on the basis of the RPI of March. For this reason, the coupons and maturity value of an index-linked bond are usually linked to the value of some price index several months ago. The $1\frac{1}{8}\%$ Index-linked Treasury Gilt 2037 mentioned above is linked to the RPI with a time-lag of three months.

Example 3.6.3. An index-linked bond with an annual coupon rate of 1% payable semi-annually is issued in April 2005. The coupons are paid on April and October of every year. The bond has a face value of £100 and matures at par in April 2008. The bond is linked to the RPI with a time-lag of three months. The relevant values of the RPI are given in the following table:

Month	Jan '05	Jul '05	Jan '06	Jul '06	Jan '07	Jul '07	Jan '08
RPI	188.9	192.2	193.4	198.5	201.6	206.1	209.8

An investor buys the bond at the issue date for £101.20. What are the coupons and maturity payments that the investor receives? Compute the nominal rate of return.

Answer. The bond is issued in April 2005. The relevant price index at that time is the RPI three month before the issue date, which is January 2005, so $Q(t_0) = 188.9$. The first coupon date is October 2005. This coupon is linked to the RPI in July 2005: 192.2. Equation (3.15) for the coupon payments tells us that the first coupon is

$$100 \cdot \frac{0.01}{2} \cdot \frac{192.2}{188.9} = 0.5087.$$

Similarly, we compute that the second coupon in April 2006 is $0.5 \cdot 193.4/188.9 = 0.5119$. The other four coupons are 0.5254, 0.5336, 0.5455 and 0.5553. Finally, the payment at maturity in April 2008 is $100 \cdot 209.8/188.9 = 111.0641$.

To compute the nominal rate of return, we write down the equation of value (in nominal terms, that is, in pounds). The investor pays 101.20 and receives six coupon payments and the maturity payment, leading to the following equation at value:

$$\begin{aligned} 101.20 &= 0.5087(1+i)^{-1/2} + 0.5119(1+i)^{-1} + 0.5254(1+i)^{-3/2} \\ &\quad + 0.5336(1+i)^{-2} + 0.5455(1+i)^{-5/2} + (0.5553 + 111.0641)(1+i)^{-3}. \end{aligned}$$

To approximate the nominal rate of return, we evaluate the right-hand side at the trial values of $i = 0$ and $i = 0.05$ and use linear interpolation to approximate the value of i for which the right-hand side equals 101.20. If $i = 0$, the right-hand side evaluates to 114.2446, and if $i = 0.05$, it evaluates to 98.8602. The formula for linear interpolation is

$$0 + (0.05 - 0) \cdot \frac{101.20 - 114.2446}{98.8602 - 114.2446} = 0.0424,$$

so the nominal rate of return is approximately 4.2% p.a.

If we want compute the real rate of return for the bond in the example, we write down the equation of value in real terms, as illustrated in (3.14). However, the coupon at time t is defined to have the same purchasing power as £0.50 in April 2005, when the bond was issued; see (3.15) (we are ignoring the three-month time lag here for the sake of illustration). Similarly, the maturity payment is defined to have the same purchasing power as £100 when the bond was issued. Thus, the equation of value in real terms is

$$\begin{aligned} 101.20 &= 0.5(1+i')^{-1/2} + 0.5(1+i')^{-1} + 0.5(1+i')^{-3/2} \\ &\qquad + 0.5(1+i')^{-2} + 0.5(1+i')^{-5/2} + 100.5(1+i')^{-3}. \end{aligned}$$

The solution of this is $i' = 0.0060$, so the real rate of return is 0.6% p.a. As expected, the price index does not appear in the equation of value in real terms; the real rate of return of an index-linked bond does not depend on the inflation rate (in practice, there is a small dependance because of the time lag). On the other hand, the nominal rate of return does depend on the inflation rate because the coupon and maturity payments depend on the price index. The situation is opposite with a normal bond, which is not linked to a price index. For a normal bond, the nominal rate of return is independent of the inflation, but the real rate of return is not.

3.7 Taxes

In the previous section we discussed the difference between nominal and real rates of return. The latter compensate for the effect of inflation. We end this module with a discussion of the effects of taxation.

In the presence of taxes, we need to distinguised between gross and net income. Gross income is the income before taxes, and net income is what is left after you have paid the taxes. The rate of return computed using gross income is known as the *gross rate of return* (or gross yield), while the rate of return computed using net income is the (surprise, surprise . . .) *net rate of return* (or net yield).

The rules and laws around taxes are extremely complicated and far beyond the scope of this module. We will satisfy ourselves with a simple example featuring *income tax* and *capital gains tax*. Income tax is levied on income such as wages and interest. Capital gains tax is a tax levied on the profit one makes because the value of some asset increases. In the context of bonds, income tax is levied on coupon payments, and capital gains tax is levied on the difference between the price at which you sell a bond and the price at which you buy a bond. If the bond is held until maturity, capital gains tax is levied on the difference between the maturity value and the purchase price.

Example 3.7.1. A bond with a face value of £100 and coupons at 4% p.a. payable semi-annually is redeemable in five years at par. Suppose the bought for £91.22 by an investor who is subject to income tax at 40% on the coupons and capital gains tax at 18% on the profit on redemption. Compute the gross and net yield on this transaction.

Answer. The investor receives ten coupons of £2 each on which he has to pay 40% income tax, so he pays £0.80 of every coupon in tax and receives a net payment of £1.20. At maturity, the investor makes a profit of $100-91.22 = 8.78$ pounds, which is taxed at 18%. Thus, he has to pay £1.58 capital gains tax. At maturity, the investor receives a net payment of $100 - 1.58 = 98.42$ pounds.

The equation of value for the gross yield is

$$\text{price} = FDa_{\overline{n}|}^{(2)} + Fv^n \quad \text{or} \quad 91.22 = 4a_{\overline{5}|}^{(2)} + 100v^5.$$

If $i^{(2)} = D = 0.04$, corresponding to an effective interest rate of $i = (1.02)^2-1 = 0.0404$, then the bond trades at par and costs £100. In fact, the bond sells for £91.22, so it is trading at a discount, which means that the gross yield is higher than that. We try $i = 0.1$; the price corresponding to this yield is

$$FDa_{\overline{n}|}^{(2)} + Fv^n \text{ (at 10\%)} = 15.5332 + 62.0921 = 77.6253.$$

Linear interpolation suggests that the gross yield is approximately

$$0.0404 + (0.1 - 0.0404) \cdot \frac{91.22 - 100}{77.6253 - 100} = 0.06379.$$

To compute the net yield we need to write down the equation of value with net payments. After tax, the investor receives coupons of £1.20 and £98.42 at maturity. Thus, the equation of value for the net yield is

$$91.22 = 1.20 \cdot 2a_{\overline{5}|}^{(2)} + 98.42\, v^5.$$

We guess trial values of $i = 0.03$ and $i = 0.06$ for the net yield:

$$\begin{aligned} \text{at } 3\%: \quad & 1.20 \cdot 2a_{\overline{5|}}^{(2)} + 98.42\, v^5 = 12.9186 + 84.8980 = 97.8166, \\ \text{at } 6\%: \quad & 1.20 \cdot 2a_{\overline{5|}}^{(2)} + 98.42\, v^5 = 11.9690 + 73.5451 = 85.5141. \end{aligned}$$

Another round of linear interpolation suggests that the net yield is approximately

$$0.03 + (0.06 - 0.03) \cdot \frac{91.22 - 97.8166}{85.5141 - 97.8166} = 0.04609.$$

Thus, we arrive at a net yield of 4.6% p.a. In contrast, we found that the investor makes a gross yield of 6.4% p.a.

Chapter 4

Payment streams and variable interest rates

In this chapter we consider two extensions of the theory. Firstly, we look at payment streams. A payment stream is a payment that occurs continuously, just like the stream of water out of a tap. We model this as the limit of the following sequence: annual payments, monthly payments, weekly payments, daily payments, hourly payments, payments every minute, payments every second, et cetera. Mathematically speaking, we are taking the limit $p \to \infty$.

Secondly, we look at variable interest rates. In combination with continuous payment streams, this give us the most general framework for cash flows. This is used in second- and third-year finance modules to price complicated financial instruments like options.

4.1 The force of interest

A long time ago, in Example 1.5.3, we computed that a nominal rate of 8% p.a. payable quarterly corresponds to an effective rate of 8.243% p.a. As shown in Figure 1.4, this means that a series of four payments of 2 each at the end of every quarter is equivalent to a single payment of 8.243 at the end of the year. The other results in Example 1.5.3 show that a series of twelve payments of $\frac{8}{12} = \frac{2}{3}$ each at the end of every month is equivalent to a single payment of 8.300 at the end of the year, that a series of 52 payments of $\frac{8}{52}$ each at the end of every week is equivalent to a single payment of 8.322 at the end of the year, and that a series of 365 payments of $\frac{8}{365}$ each at the end of every day equivalent to a single payment of 8.328 at the end of the year. More generally, a series of p payments of $i^{(p)}/p$ each at the end of every period of $1/p$ year equivalent to a single payment of

$$i = \left(1 + \frac{i^{(p)}}{p}\right)^p - 1.$$

at the end of the year. As the conversion period gets shorter, p gets larger, and it looks like the effective rate tends to a limiting value. In the limit, the interest is paid continuously. Although this is essentially a theoretical concept, it is important both for conceptual reasons and as an approximation for when the conversion period is short.

To study this limit, we fix the effective interest rate i and let p go to infinity. Remember that the nominal rate $i^{(p)}$ is given by

$$i^{(p)} = p\Big((1+i)^{1/p} - 1\Big). \tag{4.1}$$

We want to compute the limit of this quantity as $p \to \infty$. To this end, we introduce $h = 1/p$, the length of the period between two interest payments (p is the number of interest payments per year). Substituting this in (4.1), we find

$$i^{(1/h)} = \frac{1}{h}\Big((1+i)^{1/h} - 1\Big).$$

If p goes to infinity, then $h = 1/p$ goes to zero, so the limit we want to compute is

$$\lim_{p\to\infty} i^{(p)} = \lim_{h\to 0} i^{(1/h)} = \lim_{h\to 0} \frac{(1+i)^{1/h} - 1}{h}.$$

Now comes the smart bit. Recall the definition of the derivative of a function f at a point x_0:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) = \lim_{h\to 0} f(x_0+h) - f(x_0)h.$$

If we take $f(x) = (1+i)^x$ and $x_0 = 0$, then we get precisely the limit we want to compute! Turning this around, we find that

$$\lim_{p\to\infty} i^{(p)} = \lim_{h\to 0} i^{(1/h)} = \lim_{h\to 0} \frac{(1+i)^{1/h} - 1}{h} = \frac{\mathrm{d}}{\mathrm{d}x}\,(1+i)^x\big|_{x=0}\,.$$

You can get the same result with l'Hôpital's rule for limits, if you know that. However the result is arrived at, we now have to find the derivative of $f(x) = (1+i)^x$. To differentiate this function, rewrite it as $f(x) = \mathrm{e}^{x\log(1+i)}$, so $f'(x) = \log(1+i)\cdot \mathrm{e}^{x\log(1+i)} = \log(1+i)(1+i)^x$ and $f'(0) = \log(1+i)$. Thus,

$$\lim_{p\to\infty} i^{(p)} = \frac{\mathrm{d}}{\mathrm{d}x}\,(1+i)^x\big|_{x=0} = \log(1+i).$$

We may refer to $\lim_{p\to\infty} i^{(p)}$ as the nominal rate payable continuously, but in practice another term is used to refer to this quantity.

Definition 4.1.1. The *force of interest*, denoted δ, is the limit of the nominal interest rate $i^{(p)}$ as $p \to \infty$.

The computation above gives us the relation between the force of interest δ and effective interest rate i:

$$\delta = \log(1+i). \tag{4.2}$$

We can also write this relation as $i = \mathrm{e}^\delta - 1$. If we put $\delta = 0.08$ in here, as in Example 1.5.3, we get $i = \mathrm{e}^{0.08} - 1 = 0.083287$. Indeed, the AERs in the answer tend to a limiting rate of 8.2387%.

As a side note, if we take the limit $p \to \infty$ in (1.5), we get

$$1 + i = \lim_{p\to\infty}\left(1 + \frac{\delta}{p}\right)^p,$$

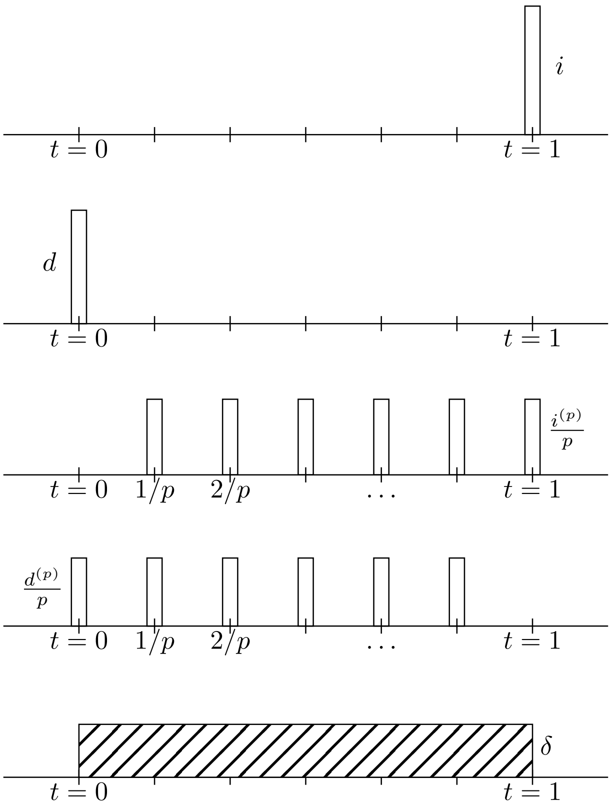

Figure 4.1: The following five situations are equivalent: A payment of i at the end of the year, a payment of d at the beginning of the year, a series of p payments of $i^{(p)}/p$ each at the end of every $1/p$ of a year, a series of p payments of $d^{(p)}/p$ each at the beginning of every $1/p$ of a year, and a continuous payment for one year at rate δ per year. This extends Figure 1.4, in which only the first four situations were given.

and above we found that $1 + i = \mathrm{e}^{\delta}$. We thus arrive at

$$\lim_{p\to\infty} \left(1 + \frac{\delta}{p}\right)^p = \mathrm{e}^{\delta}.$$

This is one of the standard limits.

We have now defined five standard symbols: i, d, $i^{(p)}$, $d^{(p)}$ and δ. A concise summary of their definitions is given in Figure 4.1. The figure shows five equivalent ways to pay interest on a principal of 1. The first four rows have already been explained in the previous section. The last row shows a continuous payment stream, lasting one time unit, with rate δ per time unit. The equivalence of these payments lead to the following algebraic relations, which follow from (1.3), (1.5), (1.7) and (4.2):

$$i = \mathrm{e}^{\delta} - 1, \qquad i^{(p)} = p\big(\mathrm{e}^{\delta/p} - 1\big), \tag{4.3}$$

$$d = 1 - \mathrm{e}^{-\delta}, \qquad d^{(p)} = p\big(1 - \mathrm{e}^{-\delta/p}\big). \tag{4.4}$$

Assuming that the interest rate is positive, the rates are ordered as

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i.$$

4.2 Annuities paid continuously

The concept of a continuous payment stream can also be applied to annuities. An annuity paid continuously is a payment stream with a constant rate that lasts for a specified period of time. As with the other annuities, we are interested in the present and accumulated value of annuities paid continuously.

Definition 4.2.1. Consider a payment stream lasting for n time units with a rate of one unit of capital per time unit. The present value of this stream at the start ($t = 0$) is denoted by $\bar{a}_{\overline{n}|}$ and the future value at the end ($t = n$) is denoted by $\bar{s}_{\overline{n}|}$.

Formulas for $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ are derived by taking the limits of $a^{(p)}_{\overline{n}|}$ and $s^{(p)}_{\overline{n}|}$ as $p \to \infty$:

$$\begin{aligned} \bar{a}_{\overline{n}|} &= \lim_{p\to\infty} a^{(p)}_{\overline{n}|} = \lim_{p\to\infty} \frac{1 - v^n}{i^{(p)}} = \frac{1 - v^n}{\delta}, \\ \bar{s}_{\overline{n}|} &= \lim_{p\to\infty} s^{(p)}_{\overline{n}|} = \lim_{p\to\infty} \frac{(1+i)^n - 1}{i^{(p)}} = \frac{(1+i)^n - 1}{\delta}, \end{aligned} \tag{4.5}$$

where the final equality follows from $\delta = \lim_{p\to\infty} i^{(p)}$, see Definition 4.1.1.

The formulas for $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ differ from those for $a_{\overline{n}|}$ and $s_{\overline{n}|}$ only in that the formulas for annuities payable continuously have the force of interest δ in the denominator, whereas the formulas for standard annuities have the interest rate i in the denominator. The reason for this is that in one year, an annuity payable continuously consists of a payment stream over the entire year and a standard annuity consists of a single payment at the end of the year. If the payment stream has rate δ and the payment at the end of the year is i, then these payments are equivalent, as was found in the previous section (see Figure 4.1). Thus, an annuity immediate with annual payments of i is equivalent to an annuity payable continuously with rate δ, so $ia_{\overline{n}|} = \delta\bar{a}_{\overline{n}|}$ and $is_{\overline{n}|} = \delta\bar{s}_{\overline{n}|}$.

There is no need to distinguish between annuities immediate and annuities due for annuities payable continuously. Indeed, an annuity due payable $p$thly is the same as an annuity immediate payable $p$thly moved forward in time by $1/p$ time units. As $p \to \infty$, the annuity payable $p$thly becomes an annuity payable continuously and the time shift tends to zero. Algebraically, this follows from $\lim_{p\to\infty} i^{(p)} = \lim_{p\to\infty} d^{(p)} = \delta$.

As an illustration, let us continue Example 3.1.3 in which we looked at a constructor that is building a house.

Example 4.2.2 (Continuation of Example 3.1.3)**.** As before, the contractor has to pay £15,000 at $t = 0$ to purchase the land and £25,000 at $t = \frac{1}{4}$ to purchase the materials. The house is also still sold for £75,000 at $t = \frac{3}{4}$. The difference is that the £30,000 salary for the workers is not paid at once (at $t = \frac{1}{12}$), but continously from the time the ground is purchased (at $t = 0$) until the house is sold (at $t = \frac{3}{4}$).

This continous payment forms an annuity paid continuously lasting for three quarters of a year. The salary is £30,000 over three quarters, so the rate is 40,000 pounds per year. Thus the present value of the workers' salary is $40000\bar{a}_{\overline{3/4}|}$. Using equation (4.5), we find that

$$\bar{a}_{\overline{3/4}|} = \frac{1 - v^{3/4}}{\delta} = \frac{1 - (1+i)^{-3/4}}{\log(1+i)}.$$

Thus, the net present value of the whole project is

$$-15000 - 40000 \cdot \frac{1 - (1+i)^{-3/4}}{\log(1+i)} - 25000(1+i)^{-1/4} + 75000(1+i)^{-3/4}. \quad (4.6)$$

If the interest rate is 6%, then this is

$$-15000 - 29353.92 - 24638.46 + 71792.96 = 2800.58.$$

This is higher than the net present value for the situation before in Example 3.1.3, where the workers were paid at $t = \frac{1}{12}$. There, we found that the net present value was £2299.82. Loosely speaking, the payments to the workers are later on average when they are paid continuously from $t = 0$ to $t = \frac{3}{4}$ than when they are paid at once at $t = \frac{1}{12}$. Thus, these payments are discounted by more when the workers are paid continuously, and this causes the net present value to increase from £2299.82 to £2800.58.

4.3 Variable interest rates

All the discussion up to now assumes that the interest rate stays constant. However, this is not a realistic assumption. In this section we consider situations in which the interest rate varies.

We start with a simple example.

Example 4.3.1. Suppose you invest 2000 dollars in an account that pays 4% interest in the first year, 5% in the second year, and 6% in the third year. How much do you have at the end of the third year?

Answer. You have $(1+0.04)\cdot 2000 = 2080$ dollars after the first year, $(1+0.05)\cdot 2080 = 2184$ dollars after the second year, and $(1+0.06)\cdot 2184 = 2315.04$ dollars after the third year.

The following computation is *not* valid: the average rate is 5%, and \$2000 at 5% over three years accumulates to $(1+0.05)^3 \cdot 2000 = 2315.25$ dollars. Indeed, though the result is close to the correct answer, it is not the same.

We now consider a general model which covers the above example, but also simple and compound interest with constant rate. The central quantity in this model is the accumulation factor.

Definition 4.3.2. The *accumulation factor* $A(t_1, t_2)$ is the amount that one unit of principal at time t_1 accumulates to at time t_2.

In Example 4.3.1, a principal of \$2000 at time $t = 0$ accumulates to \$2080 at time $t = 1$, so the accumulation factor over this period is $A(0,1) = \frac{2080}{2000} = 1.04$, corresponding to an interest rate of 4%. Similarly, $A(0,3) = \frac{2315.25}{2000} = 1.157625$.

Two examples are particularly important. The accumulation factor for simple interest with constant rate i is given by

$$A(t_1, t_2) = 1 + i(t_2 - t_1). \tag{4.7}$$

The accumulation factor for compound interest with constant rate i is given by

$$A(t_1, t_2) = (1 + i)^{t_2 - t_1}. \tag{4.8}$$

As we discussed at the end of Section 1.1, simple interest is not consistent in that you receive less interest if you put money in an account for two years than if you put money in one account for a year and then take it out with interest and put it in another account for a year. On the other hand, you get the same amount if compound interest is used. We say that the principle of consistency holds with compound interest.

Definition 4.3.3. An accumulation function A satisfies the *principle of consistency* if the equation $A(t_1, t_2)\, A(t_2, t_3) = A(t_1, t_3)$ holds for all values of t_1, t_2, and t_3.

For instance, with compound interest we have:

$$A(t_1, t_2)\, A(t_2, t_3) = (1 + i)^{t_2 - t_1}\, (1 + i)^{t_3 - t_2} = (1 + i)^{t_3 - t_1},$$

and

$$A(t_1, t_3) = (1 + i)^{t_3 - t_1},$$

so $A(t_1, t_2)A(t_2, t_3) = A(t_1, t_3)$ for all t_1, t_2, and t_3. On the other hand, with simple interest we have:

$$\begin{aligned} A(t_1, t_2)\, A(t_2, t_3) &= \big(1 + (t_2 - t_1)i\big)\big(1 + (t_3 - t_2)i\big) \\ &= 1 + (t_3 - t_1)i + (t_3 - t_2)(t_2 - t_1)i^2, \end{aligned}$$

and

$$A(t_1, t_3) = 1 + (t_3 - t_1)i,$$

which are not equal in general, so simple interest does not satisfy the principle of consistency. Unless stated otherwise, we assume that A satisfies the principle of consistency.

We proceed to generalize the definitions of the quantities i, $i^{(p)}$, δ, v, d and $d^{(p)}$ from the previous sections to arbitrary accumulation functions. One unit of capital at time t accumulates to $A(t, t + 1)$ in one time period. Thus, the effective interest rate per unit time under the accumulation function A is given by

$$i(t) = A(t, t + 1) - 1.$$

In general, the effective interest rate depends on the time t. In Example 4.3.1, we have $i(0) = 0.04$, $i(1) = 0.05$ and $i(2) = 0.06$. Given the principle of consistency, the accumulation function over a whole number n of time periods can be expressed in terms of the interest rates:

$$\begin{aligned} A(0, n) &= A(0, 1)\, A(1, 2)\, A(2, 3) \ldots A(n - 1, n) \\ &= \big(1 + i(0)\big)\big(1 + i(1)\big)\big(1 + i(2)\big) \cdots \big(1 + i(n - 1)\big) \end{aligned} \tag{4.9}$$

The nominal interest rate per unit time for a term h is the effective interest rate over the term h divided by the length of the term:

$$i_h(t) = \frac{A(t, t+h) - 1}{h}.$$

This is related to nominal rate payable $p$thly, $i^{(p)}$, as defined in Definition 1.5.2, by $i^{(p)} = i_{1/p}$. Furthermore, we have $i = i_1 = i^{(1)}$.

The force of interest is found by taking the limit as $h \to 0$:

$$\delta(t) = \lim_{h \to 0} i_h(t) = \lim_{h \to 0} \frac{A(t, t+h) - 1}{h}.$$

The accumulation function associated to compound interest is given in (4.8), $A(t_1, t_2) = (1+i)^{t_2 - t_1}$, so

$$\begin{aligned}
\delta(t) &= \lim_{h \to 0} \frac{(1+i)^h - 1}{h} = \frac{\mathrm{d}}{\mathrm{d}t}\big((1+i)^t\big)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{t \log(1+i)}\Big|_{t=0} \\
&= \mathrm{e}^{t \log(1+i)} \log(1+i)\Big|_{t=0} = \log(1+i),
\end{aligned}$$

where we used the definition of derivatives to evaluate the limit. This is another derivation for the relation $\delta = \log(1+i)$ which we found before in Equation (4.2). The relations (4.3) and (4.4) at the end of Section 4.1 may be understood more easily in this context.

We can now model the situation of Example 4.3.1, where the interest rate was 4% in the first year, 5% in the second year, and 6% in the third year, more rigorously as:

$$\delta(t) = \begin{cases} \log(1.04), & \text{if } t \in [0, 1), \\ \log(1.05), & \text{if } t \in [1, 2), \\ \log(1.06), & \text{if } t \in [2, 3). \end{cases}$$

It can be argued that the force of interest should be considered as the fundamental quantity. Suppose that the principle of consistency holds and that A is a differentiable function. Define $f(t) = A(0, t)$. Then,

$$\begin{aligned}
\delta(t) &= \lim_{h \to 0} \frac{A(t, t+h) - 1}{h} \\
&= \frac{1}{A(0,t)} \lim_{h \to 0} \frac{A(0,t)\, A(t, t+h) - A(0,t)}{h} \\
&= \frac{1}{A(0,t)} \lim_{h \to 0} \frac{A(0, t+h) - A(0,t)}{h} \\
&= \frac{1}{f(t)} \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\
&= \frac{f'(t)}{f(t)},
\end{aligned}$$

where we used the principle of consistency on the third line. Thus, f satisfies the differential equation $f'(t) = \delta(f) f(t)$. Furthermore, $f(0) = A(0,0) = 1$ by the consistency principle. The solution of this differential equation is

$$f(t) = \exp\left(\int_0^t \delta(\tau)\, \mathrm{d}\tau\right).$$

We can find $A(t_1, t_2)$ by noting that $A(t_1, t_2) = A(t_1, 0)\, A(0, t_2) = \frac{A(0,t_2)}{A(0,t_1)}$ by the consistency principle, and thus

$$\begin{aligned} A(t_1, t_2) &= \frac{f(t_2)}{f(t_1)} \\ &= \frac{\exp\left(\int_0^{t_2} \delta(\tau)\,\mathrm{d}\tau\right)}{\exp\left(\int_0^{t_1} \delta(\tau)\,\mathrm{d}\tau\right)} \\ &= \exp\left(\int_0^{t_2} \delta(\tau)\,\mathrm{d}\tau - \int_0^{t_1} \delta(\tau)\,\mathrm{d}\tau\right) \\ &= \exp\left(\int_{t_1}^{t_2} \delta(\tau)\,\mathrm{d}\tau\right). \end{aligned}$$

Thus, the accumulation function can be found in terms of the force of interest:

$$A(t_1, t_2) = \exp\left(\int_{t_1}^{t_2} \delta(\tau)\,\mathrm{d}\tau\right). \tag{4.10}$$

This can be seen as a continuous version of (4.9). The effective rate of interest and the nominal rate of interest are both defined in terms of A. In a sense, the force of interest can be seen as the most fundamental quantity.

Example 4.3.4. Stoodley's formula for the force of interest is

$$\delta(t) = p + \frac{s}{1 + r\mathrm{e}^{st}},$$

where p, r and s are constants. What is the accumulation factor $A(0, t)$?

Answer. Applying (4.10) yields

$$\begin{aligned} A(0, t) &= \exp\left(\int_0^t p + \frac{s}{1 + r\mathrm{e}^{s\tau}}\,\mathrm{d}\tau\right) \\ &= \exp\left(\int_0^t p + s - \frac{rs\mathrm{e}^{st}}{1 + r\mathrm{e}^{s\tau}}\,\mathrm{d}\tau\right) \\ &= \exp\left(\Big[(p + s)\tau - \log(1 + r\mathrm{e}^{s\tau})\Big]_0^t\right) \\ &= \exp\big((p + s)t - \log(1 + r\mathrm{e}^{st}) + \log(1 + r)\big) \\ &= \mathrm{e}^{(p+s)t}\frac{1 + r}{1 + r\mathrm{e}^{st}}. \end{aligned}$$

Stoodley's formula is sometimes used to model gradually increasing or decreasing interest rates, because the integral can be evaluated exactly.

4.4 Cash flows in full generality

In Section 3.1 we looked at cash flows, irregular sequences of payments. The main formula from that section is (3.1): the value at tim t_* of a cash flow consisting of payments P_k at time t_k is

$$\sum_k (1 + i)^{t_* - t_k} P_k.$$

We may also incorporate continuous payment streams in the framework. Furthermore, the rate of the stream may vary. Say that the rate is given by $\rho(t)$. As before, a positive ρ indicates income and a negative ρ indicates expenditure. Now consider the period between time t and $t + \Delta t$, with Δt small. In this period, the payment stream pays approximately $(\Delta t) \cdot \rho(t)$, because the rate is the amount paid divided by the length of the period. The value of this payment at time t_* is approximately $(1+i)^{t_*-t} \cdot (\Delta t) \cdot \rho(t)$. If we sum up the values of all these payments and then take the limit $\Delta t \to 0$, we get the integral of $(1+i)^{t_*-t}\rho(t)$. Thus, we find that the value of varying payment stream at time t_* equals

$$\int_{-\infty}^{\infty} (1+i)^{t_*-t} \rho(t) \, \mathrm{d}t. \tag{4.11}$$

We will not concern ourselves with convergence issues.

A special case is annuities payable continuously. Definition 4.2.1 states that $\bar{a}_{\overline{n|}}$ denotes the value at time $t = 0$ of a payment stream with rate 1 lasting from $t = 0$ to $t = n$. This can be brought in the framework considered here by defining the rate $\rho(t)$ as

$$\rho(t) = \begin{cases} 1, & \text{if } 0 < t \le n; \\ 0, & \text{if } t \le 0 \text{ or } t > n. \end{cases}$$

Then, $\bar{a}_{\overline{n|}}$ is the present value at $t_* = 0$ of the payment stream with rate $\rho(t)$. Substituting this in (4.11) yields

$$\begin{aligned} \bar{a}_{\overline{n|}} &= \int_{-\infty}^{\infty} (1+i)^{-t} \rho(t) \, \mathrm{d}t = \int_0^n (1+i)^{-t} \, \mathrm{d}t = \int_0^n \mathrm{e}^{-t \log(1+i)} \, \mathrm{d}t \\ &= \left[-\frac{1}{\log(1+i)} \mathrm{e}^{-t \log(1+i)} \right]_0^n = \left[-\frac{(1+i)^{-t}}{\log(1+i)} \right]_0^n \\ &= \frac{(1+i)^0}{\log(1+i)} - \frac{(1+i)^{-n}}{\log(1+i)} = \frac{1-(1+i)^{-n}}{\log(1+i)} = \frac{1-v^n}{\delta}. \end{aligned}$$

This is the same expression for $\bar{a}_{\overline{n|}}$ as we found earlier in Section 4.2.1.

Some situations combine discrete and continuous cash flows. Suppose that there are (discrete) payments of P_k at time t_k, and also a continuous payment stream with rate $\rho(t)$. The value of the combined cash flow at time $t = t_*$ is

$$\sum_k (1+i)^{t_*-t_k} P_k + \int_{-\infty}^{\infty} (1+i)^{t_*-t} \rho(t) \, \mathrm{d}t. \tag{4.12}$$

This is a combination of (3.1) and (4.11). A further generalization is that the interest rate may also vary. This was analyzed in the previous section, where we introduced the accumulation function $A(t_1, t_2)$. The value of the combined discrete/continuous cash flow in this setting is

$$P_k A(t_k, t_*) + \int_{-\infty}^{\infty} \rho(t) A(t, t_*) \, \mathrm{d}t.$$

Compound interest with a constant interest rate i has accumulation function $A(t_1, t_2) = (1+i)^{t_2-t_1}$, and substitution of this in the formula above retrieves (4.12).

Success with the exam!!!