Chapter 4

Payment streams and variable interest rates

In this chapter we consider two extensions of the theory. Firstly, we look at payment streams. A payment stream is a payment that occurs continuously, just like the stream of water out of a tap. We model this as the limit of the following sequence: annual payments, monthly payments, weekly payments, daily payments, hourly payments, payments every minute, payments every second, et cetera. Mathematically speaking, we are taking the limit $p \to \infty$.

Secondly, we look at variable interest rates. In combination with continuous payment streams, this gives us the most general framework for cash flows. This is used in second- and third-year finance modules to price complicated financial instruments like options.

4.1 The force of interest

A long time ago, in Example 1.5.3, we computed that a nominal rate of 8% p.a. payable quarterly corresponds to an effective rate of 8.243% p.a. As shown in Figure 1.4, this means that a series of four payments of 2 each at the end of every quarter is equivalent to a single payment of 8.243 at the end of the year. The other results in Example 1.5.3 show that a series of twelve payments of $\frac{8}{12} = \frac{2}{3}$ each at the end of every month is equivalent to a single payment of 8.300 at the end of the year, that a series of 52 payments of $\frac{8}{52}$ each at the end of every week is equivalent to a single payment of 8.322 at the end of the year, and that a series of 365 payments of $\frac{8}{365}$ each at the end of every day equivalent to a single payment of 8.328 at the end of the year. More generally, a series of $p$ payments of $\frac{i(p)}{p}$ each at the end of every period of $1/p$ year equivalent to a single payment of

$$i = \left(1 + \frac{i(p)}{p}\right)^p - 1.$$ 

at the end of the year. As the conversion period gets shorter, $p$ gets larger, and it looks like the effective rate tends to a limiting value. In the limit, the interest is paid continuously. Although this is essentially a theoretical concept,
it is important both for conceptual reasons and as an approximation for when the conversion period is short.

To study this limit, we fix the effective interest rate $i$ and let $p$ go to infinity. Remember that the nominal rate $i(p)$ is given by

$$i(p) = p \left( (1 + i)^{1/p} - 1 \right). \quad (4.1)$$

We want to compute the limit of this quantity as $p \to \infty$. To this end, we introduce $h = 1/p$, the length of the period between two interest payments ($p$ is the number of interest payments per year). Substituting this in (4.1), we find

$$i^{(1/h)} = \frac{1}{h} \left( (1 + i)^{1/h} - 1 \right).$$

If $p$ goes to infinity, then $h = 1/p$ goes to zero, so the limit we want to compute is

$$\lim_{p \to \infty} i(p) = \lim_{h \to 0} i^{(1/h)} = \lim_{h \to 0} \frac{(1 + i)^{1/h} - 1}{h}.$$  

Now comes the smart bit. Recall the definition of the derivative of a function $f$ at a point $x_0$:

$$\frac{df}{dx}(x_0) = \lim_{h \to 0} f(x_0 + h) - f(x_0)/h.$$  

If we take $f(x) = (1 + i)^x$ and $x_0 = 0$, then we get precisely the limit we want to compute! Turning this around, we find that

$$\lim_{p \to \infty} i(p) = \lim_{h \to 0} i^{(1/h)} = \lim_{h \to 0} \frac{(1 + i)^{1/h} - 1}{h} = \frac{d}{dx} (1 + i)^x \big|_{x=0}.$$  

You can get the same result with l’Hôpital’s rule for limits, if you know that. However the result is arrived at, we now have to find the derivative of $f(x) = (1 + i)^x$. To differentiate this function, rewrite it as $f(x) = e^{x \log(1 + i)}$, so $f'(x) = \log(1 + i) \cdot e^{x \log(1 + i)} = \log(1 + i)(1 + i)^x$ and $f'(0) = \log(1 + i)$. Thus,

$$\lim_{p \to \infty} i(p) = \frac{d}{dx} (1 + i)^x \big|_{x=0} = \log(1 + i).$$  

We may refer to $\lim_{p \to \infty} i(p)$ as the nominal rate payable continuously, but in practice another term is used to refer to this quantity.

**Definition 4.1.1.** The *force of interest*, denoted $\delta$, is the limit of the nominal interest rate $i(p)$ as $p \to \infty$.

The computation above gives us the relation between the force of interest $\delta$ and effective interest rate $i$:

$$\delta = \log(1 + i). \quad (4.2)$$

We can also write this relation as $i = e^\delta - 1$. If we put $\delta = 0.08$ in here, as in Example 1.5.3, we get $i = e^{0.08} - 1 = 0.083287$. Indeed, the AERs in the answer tend to a limiting rate of 8.2387%.

As a side note, if we take the limit $p \to \infty$ in (1.5), we get

$$1 + i = \lim_{p \to \infty} \left( 1 + \frac{\delta}{p} \right)^p,$$
and above we found that $1 + i = e^\delta$. We thus arrive at
\[ \lim_{p \to \infty} \left( 1 + \frac{\delta}{p} \right)^p = e^\delta. \]
This is one of the standard limits.

We have now defined five standard symbols: $i$, $d$, $i^{(p)}$, $d^{(p)}$ and $\delta$. A concise summary of their definitions is given in Figure 4.1. The figure shows five equivalent ways to pay interest on a principal of 1. The first four rows have already been explained in the previous section. The last row shows a continuous payment stream, lasting one time unit, with rate $\delta$ per time unit. The equivalence of these payments lead to the following algebraic relations, which follow from (1.3), (1.5), (1.7) and (4.2):

\begin{align*}
    i &= e^\delta - 1, & \quad i^{(p)} &= p(e^\delta/p - 1), \quad \text{(4.3)} \\
    d &= 1 - e^{-\delta}, & \quad d^{(p)} &= p(1 - e^{-\delta/p}). \quad \text{(4.4)}
\end{align*}
Assuming that the interest rate is positive, the rates are ordered as
\[ d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i. \]

### Exercises

1. Given that \( \delta = 0.08 \), find the values of \( i, d, \) and \( v \).
2. Given that \( d = 0.08 \), find the values of \( v, i, \) and \( \delta \).
3. Given that \( i = 0.08 \), find the values of \( v, d, \) and \( \delta \).
4. Given that \( v = 0.95 \), find the values of \( d, i, \) and \( \delta \).

### 4.2 Annuities paid continuously

The concept of a continuous payment stream can also be applied to annuities. An annuity paid continuously is a payment stream with a constant rate that lasts for a specified period of time. As with the other annuities, we are interested in the present and accumulated value of annuities paid continuously.

**Definition 4.2.1.** Consider a payment stream lasting for \( n \) time units with a rate of one unit of capital per time unit. The present value of this stream at the start \((t = 0)\) is denoted by \( \bar{a}_n \) and the future value at the end \((t = n)\) is denoted by \( \bar{s}_n \).

Formulas for \( \bar{a}_n \) and \( \bar{s}_n \) are derived by taking the limits of \( a(p)_n \) and \( s(p)_n \) as \( p \to \infty \):

\[
\bar{a}_n = \lim_{p \to \infty} a(p)_n = \lim_{p \to \infty} \frac{1 - v^n}{i(p)} = \frac{1 - v^n}{\delta},
\]

\[
\bar{s}_n = \lim_{p \to \infty} s(p)_n = \lim_{p \to \infty} \frac{(1 + i)^n - 1}{i(p)} = \frac{(1 + i)^n - 1}{\delta},
\]

where the final equality follows from \( \delta = \lim_{p \to \infty} i(p) \), see Definition 4.1.1.

The formulas for \( \bar{a}_n \) and \( \bar{s}_n \) differ from those for \( a_n \) and \( s_n \) only in that the formulas for annuities payable continuously have the force of interest \( \delta \) in the denominator, whereas the formulas for standard annuities have the interest rate \( i \) in the denominator. The reason for this is that in one year, an annuity payable continuously consists of a payment stream over the entire year and a standard annuity consists of a single payment at the end of the year. If the payment stream has rate \( \delta \) and the payment at the end of the year is \( i \), then these payments are equivalent, as was found in the previous section (see Figure 4.1). Thus, an annuity immediate with annual payments of \( i \) is equivalent to an annuity payable continuously with rate \( \delta \), so \( i\bar{a}_n = \delta \bar{a}_n \) and \( i\bar{s}_n = \delta \bar{s}_n \).

There is no need to distinguish between annuities immediate and annuities due for annuities payable continuously. Indeed, an annuity due payable \( p \)thly is the same as an annuity immediate payable \( p \)thly moved forward in time by \( 1/p \) time units. As \( p \to \infty \), the annuity payable \( p \)thly becomes an annuity payable continuously and the time shift tends to zero. Algebraically, this follows from \( \lim_{p \to \infty} i(p) = \lim_{p \to \infty} d(p) = \delta \).

As an illustration, let us continue Example 3.1.3 in which we looked at a constructor that is building a house.
Example 4.2.2 (Continuation of Example 3.1.3). As before, the contractor has to pay £15,000 at $t = 0$ to purchase the land and £25,000 at $t = \frac{1}{4}$ to purchase the materials. The house is also still sold for £75,000 at $t = \frac{3}{4}$. The difference is that the £30,000 salary for the workers is not paid at once (at $t = \frac{1}{12}$), but continuously from the time the ground is purchased (at $t = 0$) until the house is sold (at $t = \frac{3}{4}$).

This continuous payment forms an annuity paid continuously lasting for three quarters of a year. The salary is £30,000 over three quarters, so the rate is 40,000 pounds per year. Thus the present value of the workers’ salary is $40000a_{3/4}$. Using equation (4.5), we find that

$$a_{3/4} = \frac{1 - v^{3/4}}{\delta} = \frac{1 - (1 + i)^{-3/4}}{\log(1 + i)}.$$

Thus, the net present value of the whole project is

$$-15000 - 40000 \cdot \frac{1 - (1 + i)^{-3/4}}{\log(1 + i)} = 25000(1 + i)^{-1/4} + 75000(1 + i)^{-3/4}.$$

If the interest rate is 6%, then this is

$$-15000 - 29353.92 - 24638.46 + 71792.96 = 2800.58.$$

This is higher than the net present value for the situation before in Example 3.1.3, where the workers were paid at $t = \frac{1}{12}$. There, we found that the net present value was £2299.82. Loosely speaking, the payments to the workers are later on average when they are paid continuously from $t = 0$ to $t = \frac{3}{4}$ than when they are paid at once at $t = \frac{1}{12}$. Thus, these payments are discounted by more when the workers are paid continuously, and this causes the net present value to increase from £2299.82 to £2800.58.

Exercises

1. (From the CT1 exam, April ’08; quite difficult) An investor is considering investing in a capital project.

   The project requires an outlay of £500,000 at outset and further payments at the end of each of the first 5 years, the first payment being £100,000 and each successive payment increasing by £10,000.

   The project is expected to provide a continuous income at a rate of £80,000 in the first year, £83,200 in the second year and so on, with income increasing each year by 4 per annum compound. The income is received for 25 years.

   It is assumed that, at the end of 15 years, a further investment of £300,000 will be required and that the project can be sold to another investor for £700,000 at the end of 25 years.

   (a) Calculate the net present value of the project at a rate of interest of 11 per annum effective.

   (b) Without doing any further calculations, explain how the net present value would alter if the interest rate had been greater than 11 per annum effective.
4.3 Variable interest rates

All the discussion up to now assumes that the interest rate stays constant. However, this is not a realistic assumption. In this section we consider situations in which the interest rate varies.

We start with a simple example.

Example 4.3.1. Suppose you invest 2000 dollars in an account that pays 4% interest in the first year, 5% in the second year, and 6% in the third year. How much do you have at the end of the third year?

Answer. You have \((1 + 0.04) \cdot 2000 = 2080\) dollars after the first year, \((1 + 0.05) \cdot 2080 = 2184\) dollars after the second year, and \((1 + 0.06) \cdot 2184 = 2315.04\) dollars after the third year.

The following computation is not valid: the average rate is 5%, and $2000 at 5% over three years accumulates to \((1 + 0.05)^3 \cdot 2000 = 2315.25\) dollars. Indeed, though the result is close to the correct answer, it is not the same.

We now consider a general model which covers the above example, but also simple and compound interest with constant rate. The central quantity in this model is the accumulation factor.

Definition 4.3.2. The accumulation factor \(A(t_1, t_2)\) is the amount that one unit of principal at time \(t_1\) accumulates to at time \(t_2\).

In Example 4.3.1, a principal of $2000 at time \(t = 0\) accumulates to $2080 at time \(t = 1\), so the accumulation factor over this period is \(A(0, 1) = \frac{2080}{2000} = 1.04\), corresponding to an interest rate of 4%. Similarly, \(A(0, 3) = \frac{2315.04}{2000} = 1.15752\).

Two examples are particularly important. The accumulation factor for simple interest with constant rate \(i\) is given by

\[ A(t_1, t_2) = 1 + i(t_2 - t_1). \]  (4.7)

The accumulation factor for compound interest with constant rate \(i\) is given by

\[ A(t_1, t_2) = (1 + i)^{t_2 - t_1}. \]  (4.8)

As we discussed at the end of Section 1.1, simple interest is not consistent in that you receive less interest if you put money in an account for two years than if you put money in one account for a year and then take it out with interest and put it in another account for a year. On the other hand, you get the same amount if compound interest is used. We say that the principle of consistency holds with compound interest.

Definition 4.3.3. An accumulation function \(A\) satisfies the principle of consistency if the equation \(A(t_1, t_2) A(t_2, t_3) = A(t_1, t_3)\) holds for all values of \(t_1\), \(t_2\), and \(t_3\).

For instance, with compound interest we have:

\[ A(t_1, t_2) A(t_2, t_3) = (1 + i)^{t_2 - t_1} (1 + i)^{t_3 - t_2} = (1 + i)^{t_3 - t_1}, \]

and

\[ A(t_1, t_3) = (1 + i)^{t_3 - t_1}, \]
so \( A(t_1, t_2)A(t_2, t_3) = A(t_1, t_3) \) for all \( t_1, t_2, \) and \( t_3 \). On the other hand, with simple interest we have:

\[
A(t_1, t_2)A(t_2, t_3) = (1 + (t_2 - t_1)i)(1 + (t_3 - t_2)i) = 1 + (t_3 - t_1)i + (t_3 - t_2)(t_2 - t_1)i^2,
\]

and

\[
A(t_1, t_3) = 1 + (t_3 - t_1)i,
\]

which are not equal in general, so simple interest does not satisfy the principle of consistency. Unless stated otherwise, we assume that \( A \) satisfies the principle of consistency.

We proceed to generalize the definitions of the quantities \( i, \ i^{(p)}, \ \delta, \ v, \ d \) and \( d^{(p)} \) from the previous sections to arbitrary accumulation functions. One unit of capital at time \( t \) accumulates to \( A(t, t+1) \) in one time period. Thus, the effective interest rate per unit time under the accumulation function \( A \) is given by

\[
i(t) = A(t, t+1) - 1.
\]

In general, the effective interest rate depends on the time \( t \). In Example 4.3.1, we have \( i(0) = 0.04, \ i(1) = 0.05 \) and \( i(2) = 0.06 \). Given the principle of consistency, the accumulation function over a whole number \( n \) of time periods can be expressed in terms of the interest rates:

\[
A(0, n) = A(0, 1)A(1, 2)A(2, 3)\ldots A(n-1, n) = (1 + i(0))(1 + i(1))(1 + i(2))\ldots (1 + i(n-1))
\]

(4.9)

The nominal interest rate per unit time for a term \( h \) is the effective interest rate over the term \( h \) divided by the length of the term:

\[
i_h(t) = \frac{A(t, t+h) - 1}{h}.
\]

This is related to nominal rate payable \( p \)thly, \( i^{(p)} \), as defined in Definition 1.5.2, by \( i^{(p)} = i\frac{p}{p} \). Furthermore, we have \( i = i_1 = i^{(1)} \).

The force of interest is found by taking the limit as \( h \to 0 \):

\[
\delta(t) = \lim_{h \to 0} i_h(t) = \lim_{h \to 0} \frac{A(t, t+h) - 1}{h}.
\]

The accumulation function associated to compound interest is given in (4.8),

\[
A(t_1, t_2) = (1 + i)^{t_2 - t_1},
\]

so

\[
\delta(t) = \lim_{h \to 0} \frac{(1+i)^h - 1}{h} = \frac{d}{dt}((1+i)^t) \bigg|_{t=0} = \frac{d}{dt}e^{tlog(1+i)} \bigg|_{t=0} = log(1+i),
\]

where we used the definition of derivatives to evaluate the limit. This is another derivation for the relation \( \delta = log(1+i) \) which we found before in Equation (4.2). The relations (4.3) and (4.4) at the end of Section 4.1 may be understood more easily in this context.
We can now model the situation of Example 4.3.1, where the interest rate was 4% in the first year, 5% in the second year, and 6% in the third year, more rigorously as:

\[
\delta(t) = \begin{cases} 
\log(1.04), & \text{if } t \in [0, 1), \\
\log(1.05), & \text{if } t \in [1, 2), \\
\log(1.06), & \text{if } t \in [2, 3). 
\end{cases}
\]

It can be argued that the force of interest should be considered as the fundamental quantity. Suppose that the principle of consistency holds and that \( A \) is a differentiable function. Define \( f(t) = A(0, t) \). Then,

\[
\delta(t) = \lim_{h \to 0} \frac{A(t, t+h) - A(t, t)}{h} = \lim_{h \to 0} \frac{A(0, t+h) - A(0, t)}{h} = \frac{f(t+h) - f(t)}{h} = f'(t)
\]

where we used the principle of consistency on the third line. Thus, \( f \) satisfies the differential equation \( f'(t) = \delta(f) f(t) \). Furthermore, \( f(0) = A(0, 0) = 1 \) by the consistency principle. The solution of this differential equation is

\[
f(t) = \exp \left( \int_0^t \delta(\tau) \, d\tau \right).
\]

We can find \( A(t_1, t_2) \) by noting that \( A(t_1, t_2) = A(t_1, 0) A(0, t_2) = \frac{A(0, t_2)}{A(0, t_1)} \) by the consistency principle, and thus

\[
A(t_1, t_2) = \frac{f(t_2)}{f(t_1)} = \exp \left( \int_0^{t_2} \delta(\tau) \, d\tau - \int_0^{t_1} \delta(\tau) \, d\tau \right) = \exp \left( \int_{t_1}^{t_2} \delta(\tau) \, d\tau \right).
\]

Thus, the accumulation function can be found in terms of the force of interest:

\[
A(t_1, t_2) = \exp \left( \int_{t_1}^{t_2} \delta(\tau) \, d\tau \right), \quad (4.10)
\]

This can be seen as a continuous version of (4.9). The effective rate of interest and the nominal rate of interest are both defined in terms of \( A \). In a sense, the force of interest can be seen as the most fundamental quantity.
Example 4.3.4. Stoodley’s formula for the force of interest is

$$\delta(t) = p + \frac{s}{1 + re^st},$$

where $p$, $r$ and $s$ are constants. What is the accumulation factor $A(0, t)$?

Answer. Applying (4.10) yields

$$A(0, t) = \exp \left( \int_0^t p + \frac{s}{1 + re^st} \, d\tau \right)$$

$$= \exp \left( \int_0^t p + s - \frac{rse^{st}}{1 + re^st} \, d\tau \right)$$

$$= \exp \left( \left[ (p + s)\tau - \log(1 + re^st) \right]_0^1 \right)$$

$$= \exp \left( (p + s)t - \log(1 + re^st) + \log(1 + r) \right)$$

$$= e^{(p+s)t} \frac{1}{1 + re^st}.$$

Stoodley’s formula is sometimes used to model gradually increasing or decreasing interest rates, because the integral can be evaluated exactly.

Exercises

1. The effective rate of interest per annum on a certain building society account is at present 7% but in two years’ time it will be reduced to 6%. Find the accumulation in five years’ time of £4000 in this account.

2. (From the 2010 exam)

   (a) A capital of £100 at time $t = 0$ grows to £110 at $t = 2$, while a capital of £100 at $t = 2$ grows to £120 at $t = 5$. Assume that the principle of consistency holds. Compute the accumulated value at $t = 5$ of a payment of £50 at $t = 0$, showing clearly where the principle of consistency is used.

   (b) Assume in addition to the data in question (a) that the force of interest varies according to $\delta(t) = a + bt$. Show that $a \approx 0.0424076$ and that $b \approx 0.0052475$.

   (c) Compute the present value at $t = 0$ of a payment of £500 due at $t = 3$, assuming that the force of interest varies as in question (b).

   (d) An annuity pays £75 at $t = 1$, $t = 2$ and $t = 3$. Compute its present value at $t = 0$, assuming that the force of interest varies as in question (b).

3. (From the CT1 exam, September ’08) The force of interest, $\delta(t)$, is a function of time and at any time $t$, measured in years, is given by the formula:

   $$\delta(t) = \begin{cases} 
   0.05 + 0.02t, & \text{for } 0 \leq t \leq 5; \\
   0.15, & \text{for } t > 5.
   \end{cases}$$

   (a) Calculate the present value of 1,000 due at the end of 12 years.

   (b) Calculate the annual effective rate of discount implied by the transaction in (a).
4.4 Cash flows in full generality

In Section 3.1 we looked at cash flows, irregular sequences of payments. The main formula from that section is (3.1): the value at time \( t_* \) of a cash flow consisting of payments \( P_k \) at time \( t_k \) is

\[
\sum_k (1 + i)^{t_* - t_k} P_k.
\]

We may also incorporate continuous payment streams in the framework. Furthermore, the rate of the stream may vary. Say that the rate is given by \( \rho(t) \). As before, a positive \( \rho \) indicates income and a negative \( \rho \) indicates expenditure. Now consider the period between time \( t \) and \( t + \Delta t \), with \( \Delta t \) small. In this period, the payment stream pays approximately \((\Delta t) \cdot \rho(t)\), because the rate is the amount paid divided by the length of the period. The value of this payment at time \( t_* \) is approximately \((1 + i)^{t_* - t} \cdot (\Delta t) \cdot \rho(t)\). If we sum up the values of all these payments and then take the limit \( \Delta t \to 0 \), we get the integral of \((1 + i)^{t_* - t} \rho(t)\). Thus, we find that the value of varying payment stream at time \( t_* \) equals

\[
\int_{-\infty}^{\infty} (1 + i)^{t_* - t} \rho(t) \, dt.
\] (4.11)

We will not concern ourselves with convergence issues.

A special case is annuities payable continuously. Definition 4.2.1 states that \( \bar{a}_n \) denotes the value at time \( t = 0 \) of a payment stream with rate 1 lasting from \( t = 0 \) to \( t = n \). This can be brought in the framework considered here by defining the rate \( \rho(t) \) as

\[
\rho(t) = \begin{cases} 
1, & \text{if } 0 < t \leq n; \\
0, & \text{if } t \leq 0 \text{ or } t > n.
\end{cases}
\]

Then, \( \bar{a}_n \) is the present value at \( t_* = 0 \) of the payment stream with rate \( \rho(t) \). Substituting this in (4.11) yields

\[
\bar{a}_n = \int_{-\infty}^{\infty} (1 + i)^{t_* - t} \rho(t) \, dt = \int_0^n (1 + i)^{-t} \, dt = \int_0^n e^{-t \log(1+i)} \, dt
\]

\[
= \left[ \frac{1}{\log(1+i)} e^{-t \log(1+i)} \right]_0^n = \left[ \frac{(1 + i)^{-t}}{\log(1+i)} \right]_0^n
\]

\[
= \frac{(1 + i)^0}{\log(1+i)} - \frac{(1 + i)^{-n}}{\log(1+i)} = \frac{1 - (1 + i)^{-n}}{\log(1+i)} = \frac{1 - v^n}{\delta}.
\]

This is the same expression for \( \bar{a}_n \) as we found earlier in Section 4.2.1.

Some situations combine discrete and continuous cash flows. Suppose that there are (discrete) payments of \( P_k \) at time \( t_k \), and also a continuous payment stream with rate \( \rho(t) \). The value of the combined cash flow at time \( t = t_* \) is

\[
\sum_k (1 + i)^{t_* - t_k} P_k + \int_{-\infty}^{\infty} (1 + i)^{t_* - t} \rho(t) \, dt.
\] (4.12)

This is a combination of (3.1) and (4.11). A further generalization is that the interest rate may also vary. This was analyzed in the previous section, where
we introduced the accumulation function \( A(t_1, t_2) \). The value of the combined 
discrete/continuous cash flow in this setting is

\[
P_k A(t_k, t_*) + \int_{-\infty}^{\infty} \rho(t) A(t, t_*) \, dt.
\]

Compound interest with a constant interest rate \( i \) has accumulation function 
\( A(t_1, t_2) = (1 + i)^{t_2 - t_1} \), and substitution of this in the formula above re-
trieves (4.12).

### Exercises

1. (From the CT1 exam, April ’08) The force of interest, \( \delta(t) \), is a function 
of time and at any time \( t \), measured in years, is given by the formula:

\[
\delta(t) = \begin{cases} 
0.06, & 0 \leq t \leq 4; \\
0.10 - 0.01t, & 4 < t \leq 7; \\
0.01t - 0.04, & 7 < t.
\end{cases}
\]

(a) Calculate the value at time \( t = 5 \) of £1,000 due for payment at time 
\( t = 10 \).

(b) Calculate the constant rate of interest per annum convertible monthly 
which leads to the same result as in (a) being obtained.

(c) Calculate the accumulated amount at time \( t = 12 \) of a payment 
stream, paid continuously from time \( t = 0 \) to \( t = 4 \), under which the 
rate of payment at time \( t \) is \( \rho(t) = 100e^{0.02t} \).

Success with the exam!!!