Chapter 2

Annuities and loans

An annuity is a sequence of payments with fixed frequency. The term “annuity” originally referred to annual payments (hence the name), but it is now also used for payments with any frequency. Annuities appear in many situations; for instance, interest payments on an investment can be considered as an annuity. An important application is the schedule of payments to pay off a loan.

The word “annuity” refers in everyday language usually to a life annuity. A life annuity pays out an income at regular intervals until you die. Thus, the number of payments that a life annuity makes is not known. An annuity with a fixed number of payments is called an annuity certain, while an annuity whose number of payments depend on some other event (such as a life annuity) is a contingent annuity. Valuing contingent annuities requires the use of probabilities and this will not be covered in this module. These notes only looks at annuities certain, which will be called “annuity” for short.

2.1 Annuities immediate

The analysis of annuities relies on the formula for geometric sums:

\[ 1 + r + r^2 + \ldots + r^n = \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}. \quad (2.1) \]

This formula appeared already in Section 1.5, where it was used to relate nominal interest rates to effective interest rates. In fact, the basic computations for annuities are similar to the one we did in Section 1.5. It is illustrated in the following example.

**Example 2.1.1.** At the end of every year, you put £100 in a savings account which pays 5% interest. You do this for eight years. How much do you have at the end (just after your last payment)?

**Answer.** The first payment is done at the end of the first year and the last payment is done at the end of the eighth year. Thus, the first payment accumulates interest for seven years, so it grows to \((1 + 0.05)^7 \cdot 100 = 140.71\) pounds. The second payment accumulates interest for six years, so it grows to \(1.05^6 \cdot 100 = 134.01\) pounds. And so on, until the last payment which does not
Figure 2.1: The present and accumulated value of an annuity immediate.

accumulate any interest. The accumulated value of the eight payments is

\[ 1.05^7 \cdot 100 + 1.05^6 \cdot 100 + \cdots + 100 \]

\[ = 100 \left( 1 + \cdots + 1.05^6 + 1.05^7 \right) = 100 \sum_{k=0}^{7} 1.05^k. \]

This sum can be evaluated with the formula for a geometric sum. Substitute \( r = 1.05 \) and \( n = 7 \) in (2.1) to get

\[ \sum_{k=0}^{7} 1.05^k = \frac{1.05^8 - 1}{1.05 - 1} = 9.5491. \]

Thus, the accumulated value of the eight payments is £954.91.

In the above example, we computed the accumulated value of an annuity. More precisely, we considered an annuity with payments made at the end of every year. Such an annuity is called an annuity immediate (the term is unfortunate because it does not seem to be related to its meaning).

**Definition 2.1.2.** An annuity immediate is a regular series of payments at the end of every period. Consider an annuity immediate paying one unit of capital at the end of every period for \( n \) periods. The accumulated value of this annuity at the end of the \( n \)th period is denoted \( s_{\text{immediate}} \).

The accumulated value depends on the interest rate \( i \), but the rate is usually only implicit in the symbol \( s_{\text{immediate}} \). If it is necessary to mention the rate explicitly, the symbol \( s_{\text{immediate}} \) is used.

Let us derive a formula for \( s_{\text{immediate}} \). The situation is depicted in Figure 2.1. The annuity consists of payments of 1 at \( t = 1, 2, \ldots, n \) and we wish to compute the accumulated value at \( t = n \). The accumulated value of the first payment is \( (1 + i)^{n-1} \), the accumulated value of the second payment is \( (1 + i)^{n-2} \), and so on till the last payment which has accumulated value 1. Thus, the accumulated values of all payments together is

\[ (1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + 1 = \sum_{k=0}^{n-1} (1 + i)^k. \]

The formula for a geometric sum, cf. (2.1), yields

\[ \sum_{k=0}^{n-1} (1 + i)^k = \frac{(1 + i)^n - 1}{(1 + i) - 1} = \frac{(1 + i)^n - 1}{i}. \]
We arrive at the following formula for the accumulated value of an annuity immediate:

\[ s_n = \frac{(1 + i)^n - 1}{i}. \]  

(2.2)

This formula is not valid if \( i = 0 \). In that case, there is no interest, so the accumulated value of the annuities is just the sum of the payments: \( s_n = n \).

The accumulated value is the value of the annuity at \( t = n \). We may also be interested in the value at \( t = 0 \), the present value of the annuity. This is denoted by \( a_n \), as shown in Figure 2.1.

**Definition 2.1.3.** Consider an annuity immediate paying one unit of capital at the end of every period for \( n \) periods. The value of this annuity at the start of the first period is denoted \( a_n \).

A formula for \( a_n \) can be derived as above. The first payment is made after a year, so its present value is the discount factor \( v = \frac{1}{1+i} \). The present value of the second value is \( v^2 \), and so on till the last payment which has a present value of \( v^n \). Thus, the present value of all payments together is

\[ v + v^2 + \cdots + v^n = v(1 + v + \cdots + v^{n-1}) = v \sum_{k=0}^{n-1} v^k. \]

Now, use the formula for a geometric sum:

\[ v \sum_{k=0}^{n-1} v^k = v \frac{v^n - 1}{v - 1} = \frac{v}{1-v} (1 - v^n). \]

The fraction \( \frac{v}{1-v} \) can be simplified if we use the relation \( v = \frac{1}{1+i} \):

\[ \frac{v}{1-v} = \frac{1}{1 - \frac{1}{1+i}} = \frac{1}{(1+i) - 1} = \frac{1}{i}. \]

By combining these results, we arrive at the following formula for the present value of an annuity immediate:

\[ a_n = \frac{1 - v^n}{i}. \]  

(2.3)

Similar to equation (2.2) for \( s_n \), the equation for \( a_n \) is not valid for \( i = 0 \), in which case \( a_n = n \).

There is a simple relation between the present value \( a_n \) and the accumulated value \( s_n \). They are value of the same sequence of payments, but evaluated at different times: \( a_n \) is the value at \( t = 0 \) and \( s_n \) is the value at \( t = n \) (see Figure 2.1). Thus, \( a_n \) equals \( s_n \) discounted by \( n \) years:

\[ a_n = v^n s_n. \]  

(2.4)

This relation is easily checked. According to (2.2), the right-hand side evaluates to

\[ v^n s_n = v^n \left( \frac{(1+i)^n - 1}{i} \right) = \left( \frac{1+i}{i} \right)^n - v^n = \frac{1 - v^n}{i} = a_n. \]
where the last-but-one equality follows from $v = \frac{1}{1+i}$ and the last equality from (2.3). This proves (2.4).

One important application of annuities is the repayment of loans. This is illustrated in the following example.

**Example 2.1.4.** A loan of €2500 at a rate of $6\frac{1}{2}\%$ is paid off in ten years, by paying ten equal installments at the end of every year. How much is each installment?

**Answer.** Suppose that each installment is $x$ euros. Then the loan is paid off by a 10-year annuity immediate. The present value of this annuity is $x \alpha_{\overline{10}|}$ at $6\frac{1}{2}\%$.

We compute $v = \frac{1}{1+i} = 0.938967$ and

$$\alpha_{\overline{10}|} = \frac{1 - v^{10}}{i} = \frac{1 - 0.938967^{10}}{0.065} = 7.188830.$$ 

The present value should be equal to €2500, so the size of each installment is $x = 2500/\alpha_{\overline{10}|} = 347.7617$ euros. Rounded to the nearest cent, this is €347.76.

Every installment in the above example is used to both pay interest and pay back a part of the loan. This is studied in more detail in Section 2.6. Another possibility is to only pay interest every year, and to pay back the principal at the end. If the principal is one unit of capital which is borrowed for $n$ years, then the borrower pays $i$ at the end of every year and 1 at the end of the $n$ years. The payments of $i$ form an annuity with present value $ia_{\overline{n}|}$. The present value of the payment of 1 at the end of $n$ years is $v^n$. These payments are equivalent to the payment of the one unit of capital borrowed at the start. Thus, we find

$$1 = ia_{\overline{n}|} + v^n.$$ 

This gives another way to derive formula (2.3). Similarly, if we compare the payments at $t = n$, we find

$$(1 + i)^n = is_{\overline{n}|} + 1,$$

and (2.2) follows.

**Exercises**

1. On 15 November in each of the years 1964 to 1979 inclusive an investor deposited £500 in a special bank savings account. On 15 November 1983 the investor withdrew his savings. Given that over the entire period the bank used an annual interest rate of 7% for its special savings accounts, find the sum withdrawn by the investor.

2. A savings plan provides that in return for $n$ annual premiums of £$X$ (payable annually in advance), an investor will receive $m$ annual payments of £$Y$, the first such payments being made one payments after payment of the last premium.

(a) Show that the equation of value can be written as either

$$Ys_{\overline{m}|} - (X + Y)a_{\overline{n}|} = 0,$$

or as

$$(X + Y)s_{\overline{m}|} - Xs_{\overline{m}|} = 0.$$
Figure 2.2: The present and accumulated value of an annuity due.

(b) Suppose that \( X = 1000, Y = 2000, n = 10 \) and \( m = 10 \). Find the yield per annum on this transaction.

(c) Suppose that \( X = 1000, Y = 2000, \) and \( n = 10 \). For what values of \( m \) is the annual yield on the transaction between 8% and 10%?

(d) Suppose that \( X = 1000, Y = 2000, \) and \( m = 20 \). For what values of \( n \) is the annual yield on the transaction between 8% and 10%?

### 2.2 Annuities due and perpetuities

The previous section considered annuities immediate, in which the payments are made in arrears (that is, at the end of the year). Another possibility is to make the payments at advance. Annuities that pay at the start of each year are called annuities due.

**Definition 2.2.1.** An annuity due is a regular series of payments at the beginning of every period. Consider an annuity immediate paying one unit of capital at the beginning of every period for \( n \) periods. The value of this annuity at the start of the first period is denoted \( \overline{a}_n \), and the accumulated value at the end of the \( n \)th period is denoted \( \overline{s}_n \).

The situation is illustrated in Figure 2.2, which should be compared to the corresponding figure for annuities immediate. Both \( a_n \) and \( \overline{a}_n \) are measured at \( t = 0 \), while \( s_n \) and \( \overline{s}_n \) are both measured at \( t = n \). The present value of an annuity immediate \( (a_n) \) is measured one period before the first payment, while the present value of an annuity due \( (\overline{a}_n) \) is measured at the first payment. On the other hand, the accumulated value of an annuity immediate \( (s_n) \) is at the last payment, while the accumulated value of an annuity due \( (\overline{s}_n) \) is measured one period after the last payment.

We can easily derive formulas for \( \overline{a}_n \) and \( \overline{s}_n \). One method is to sum a geometric series. An annuity due consists of payments at \( t = 0, t = 1, \ldots, t = n - 1 \), so its value at \( t = 0 \) is

\[
\overline{a}_n = 1 + v + \cdots + v^{n-1} = \sum_{k=0}^{n-1} v^k = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}. \tag{2.5}
\]
The value at \( t = n \) is
\[
\bar{s}_n = (1 + i)^n + (1 + i)^{n-1} + \cdots + (1 + i) = \sum_{k=1}^{n} (1 + i)^k
\]
\[
= (1 + i) \frac{(1 + i)^n - 1}{(1 + i) - 1} = \frac{1 + i}{i} \left( (1 + i)^n - 1 \right) = \frac{(1 + i)^n - 1}{d}.
\]
(2.6)
If we compare these formulas with the formulas for \( a_n \) and \( s_n \) given in (2.3) and (2.2), we see that they are identical except that the denominator is \( d \) instead of \( i \). In other words,
\[
\bar{a}_n = \frac{i}{d} a_n = (1 + i) a_n \quad \text{and} \quad \bar{s}_n = \frac{i}{d} s_n = (1 + i) s_n.
\]
There is a simple explanation for this. An annuity due is an annuity immediate with all payments shifted one time period in the past (compare Figures 2.1 and 2.2). Thus, the value of an annuity due at \( t = 0 \) equals the value of an annuity immediate at \( t = 1 \). We know that an annuity immediate is worth \( a_n \) at \( t = 0 \), so its value at \( t = 1 \) is \( (1 + i) a_n \) and this has to equal \( \bar{a}_n \). Similarly, \( \bar{s}_n \) is not only the value of an annuity due at \( t = n \) but also the value of an annuity immediate at \( t = n + 1 \). Annuities immediate and annuities due refer to the same sequence of payments evaluated at different times.

There is another relationship between annuities immediate and annuities due. An annuity immediate over \( n \) years has payments at \( t = 1, \ldots, t = n \) and an annuity due over \( n + 1 \) years has payments at \( t = 0, t = 1, \ldots, t = n \). Thus, the difference is a single payment at \( t = 0 \). It follows that
\[
\bar{a}_{n+1} = a_n + 1.
\]
(2.7)
Similarly, \( \bar{s}_{n+1} \) is the value at \( t = n + 1 \) of a series of \( n + 1 \) payments at times \( t = 1, \ldots, n + 1 \), which is the same as the value at \( t = n \) of a series of \( n + 1 \) payments at \( t = 0, \ldots, n \). On the other hand, \( \bar{s}_n \) is the value at \( t = n \) of a series of \( n \) payments at \( t = 0, \ldots, n - 1 \). The difference is a single payment at \( t = n \), so
\[
\bar{s}_{n+1} = \bar{s}_n + 1.
\]
(2.8)
The relations (2.7) and (2.8) can be checked algebraically by substituting (2.2), (2.3), (2.5) and (2.6) in them.

There is an alternative method to derive the formulas for \( \bar{a}_n \) and \( \bar{s}_n \), analogous to the discussion at the end of the previous section. Consider a loan of one unit of capital over \( n \) years, and suppose that the borrower pays interest in advance and repays the principal after \( n \) years. As discussed in Section 1.4, the interest over one unit of capital is \( i \) if paid in advance, so the borrower pays an annuity due of size \( d \) over \( n \) years and a single payment of 1 after \( n \) years. These payments should be equivalent to the one unit of capital borrowed at the start. By evaluating this equivalence at \( t = 0 \) and \( t = n \), respectively, we find that
\[
1 = d\bar{a}_n + v^n \quad \text{and} \quad (1 + i)^n = d\bar{s}_n + 1,
\]
and the formulas (2.5) and (2.6) follow immediately.

As a final example, we consider perpetuities, which are annuities continuing perpetually. Consols, which are a kind of British government bonds, and certain preferred stock can be modelled as perpetuities.
Definition 2.2.2. A perpetuity immediate is an annuity immediate continuing indefinitely. Its present value (one period before the first payment) is denoted $a_\infty$. A perpetuity due is an annuity due continuing indefinitely. Its present value (at the time of the first payment) is denoted $\ddot{a}_\infty$.

There is no symbol for the accumulated value of a perpetuity, because it would be infinite. It is not immediately obvious that the present value is finite, because it is the present value of an infinite sequence of payments. However, using the formula for the sum of an infinite geometric sequence ($\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$), we find that

$$\ddot{a}_\infty = \sum_{k=0}^{\infty} v^k = \frac{1}{1-v} = \frac{1}{d}$$

and

$$a_\infty = \sum_{k=1}^{\infty} v^k = v \sum_{k=0}^{\infty} v^k = \frac{v}{1-v} = \frac{1}{i}.$$ 

Alternatively, we can use $a_\infty = \lim_{n \to \infty} a_n$ and $\ddot{a}_\infty = \lim_{n \to \infty} \ddot{a}_n$ in combination with the formulas for $a_n$ and $\ddot{a}_n$. This method gives the same result.

Example 2.2.3. You want to endow a fund which pays out a scholarship of £1000 every year in perpetuity. The first scholarship will be paid out in five years’ time. Assuming an interest rate of 7%, how much do you need to pay into the fund?

Answer. The fund makes payments of £1000 at $t = 5, 6, 7, \ldots$, and we wish to compute the present value of these payments at $t = 0$. These payments form a perpetuity, so the value at $t = 5$ is $\ddot{a}_\infty$. We need to discount by five years to find the value at $t = 0$:

$$v^5 \ddot{a}_\infty = \frac{v^5}{d} = \frac{0.934579^5}{0.0654206} = 10.89850.$$ 

Thus, the fund should be set up with a contribution of £10898.50.

Alternatively, imagine that the fund would be making annual payments starting immediately. Then the present value at $t = 0$ would be $1000\ddot{a}_\infty$. However, we added imaginary payments at $t = 0, 1, 2, 3, 4$; the value at $t = 0$ of these imaginary payments is $1000\ddot{a}_\infty$. Thus, the value at $t = 0$ of the payments at $t = 5, 6, 7, \ldots$ is

$$1000\ddot{a}_\infty - 1000\ddot{a}_\infty = 1000 \cdot \frac{1}{d} - 1000 \cdot \frac{1 - v^5}{d} = 15285.71 - 4387.21 = 10898.50,$$ 

as we found before. This alternative method is not faster in this example, but it illustrates a reasoning which is useful in many situations.

An annuity which starts paying in the future is called a deferred annuity. The perpetuity in the above example has its first payment in five years’ time, so it can be considered as a perpetuity due deferred by five years. The actuarial symbol for the present value of such a perpetuity is $5|\ddot{a}_\infty$. Alternatively, we can consider the example as a perpetuity immediate deferred by four years, whose present value is denoted by $4|a_{\infty}$. Generally, the present value of an annuities over $n$ years deferred by $m$ years is given

$$m|a_{\infty} = v^m a_{\infty} \quad \text{and} \quad m|\ddot{a}_{\infty} = v^m \ddot{a}_{\infty}.$$
Exercises

1. A loan of £2400 is to be repaid by 20 equal annual instalments. The rate of interest for the transaction is 10% per annum. Find the amount of each annual repayment, assuming that payments are made (a) in arrear and (b) in advance.

2.3 Unknown interest rate

In Sections 2.1 and 2.2 we derived the present and accumulated values of annuities with given period \( n \) and interest rate \( i \). In Section 2.7, we studied how to find \( n \). The topic of the current section is the determination of the rate \( i \).

Example 2.3.1 (McCutcheon & Scott, p. 48). A loan of £5000 is repaid by 15 annual payments of £500, with the first payment due in a year. What is the interest rate?

Answer. The repayments form an annuity. The value of this annuity at the time of the loan, which is one year before the first payment, is \( 500a_{15} \). This has to equal the principal, so we have to solve \( 500a_{15} = 5000 \) or \( a_{15} = 10 \).

Formula (2.3) for \( a_n \) yields

\[
a_{15} = \frac{1 - v^i}{i} = \frac{1}{i} \left( 1 - \left( \frac{1}{1+i} \right)^{15} \right),
\]

so the equation that we have to solve is

\[
\frac{1}{i} \left( 1 - \left( \frac{1}{1+i} \right)^{15} \right) = 10.
\]

The solution of this equation is \( i = 0.055565 \), so the rate is 5.56%.

The above example is formulated in terms of a loan, but it can also be formulated from the view of the lender. The lender pays £5000 and gets 15 annual payments of £500 in return. The interest rate implied by the transaction is called the yield or the (internal) rate of return of the transaction. It is an important concept when analysing possible investments. Obviously, an investor wants to get high yield on his investment. We will return to this in Chapter 3.

Example 2.3.1 raises the question: how can we solve equations like (2.9)? It cannot be solved algebraically, so we have to use some numerical method to find an approximation to the solution. We present several methods here. Conceptually the simplest method is to consider a table like the following, perhaps by consulting a book of actuarial tables.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( 0 )</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{15} )</td>
<td>15.0000</td>
<td>13.8651</td>
<td>12.8493</td>
<td>11.9379</td>
<td>11.1184</td>
<td>10.3797</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( i )</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
<th>0.10</th>
<th>0.11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{15} )</td>
<td>9.7122</td>
<td>9.1079</td>
<td>8.5595</td>
<td>8.0607</td>
<td>7.6061</td>
<td>7.1909</td>
</tr>
</tbody>
</table>

This shows that \( a_{15} = 10 \) for some value of \( i \) between 0.05 and 0.06, so the interest rate lies between 5% and 6%. The table also shows that the present
value $a_{15}$ decreases as the rate $i$ increases (you should be able to understand this from first principles).

If we would like a more accurate approximation, we can apply the bisection method. This method takes the midpoint, which is here $5\frac{1}{2}$%. We compute $a_{15}$ at $5\frac{1}{2}$%, which turns out to be 10.0376. On the other hand, $a_{15}$ at 6% is 9.7122, so rate at which $a_{15}$ = 10 lies between $5\frac{1}{2}$% and 6%. Another step of the bisection method takes $i = 5\frac{3}{4}$%; at this rate $a_{15}$ = 9.8729, so the rate we are looking for lies between $5\frac{3}{4}$% and $5\frac{1}{2}$%. At the next step, we compute $a_{15}$ at $5\frac{5}{8}$%, which turns out to be 9.9547, so we know that $i$ should be between $5\frac{1}{2}$% and $5\frac{5}{8}$%. As illustrated in Figure 2.4, the bisection method allows us to slowly zoom in on the solution.

Another possibility is to use linear interpolation. Again, we use that $a_{15}$ at 5% equals 10.3797, and that $a_{15}$ at 6% equals 9.7122. In other words, we know two points on the graph depicted in Figure 2.3, namely $(x_1, y_1) = (0.05, 10.3797)$
Figure 2.5: The method of linear interpolation takes two known points \((x_1, y_1)\) and \((x_2, y_2)\) on the graph and considers the line between them (the dashed line in the figure). This line approximates the graph and is used to find an approximation \(x_*\) to the \(x\)-value corresponding to \(y_*\).

and \((x_2, y_2) = (0.06, 9.7122)\). The method of linear interpolation approximates the graph by a straight line between \((x_1, y_1)\) and \((x_2, y_2)\), as illustrated in Figure 2.5. The equation of this line is

\[
y - y_1 = \frac{x - x_1}{x_2 - x_1} (y_2 - y_1)
\]

In the current example, we wish to find the value of \(x\) which corresponds to \(y = 10\). If we denote the given value of \(y\) by \(y_*\), then the unknown value of \(x_*\) is given by

\[
x_* = x_1 + \frac{y_* - y_1}{y_2 - y_1} (x_2 - x_1).
\]

(2.10)

In the situation considered here, this evaluates to

\[
0.05 + (10 - 10.3797) \cdot \frac{0.06 - 0.05}{9.7122 - 10.3797} = 0.055689.
\]

This brings us in one step close to the solution. As with the bisection method, we can repeat this process to get more accurate approximations of the solution.

Some people may know Newton’s method, also known as the Newton–Raphson method. This method is usually given for equations of the form \(f(x) = 0\). We can write (2.9) in this form by taking

\[
f(x) = \frac{1}{i} \left(1 - \left(\frac{1}{1 + i}\right)^{15}\right) - 10.
\]

Newton’s method start from only one value of \(x\), say \(x_*\). It states that if \(x_*\) is a good approximation to the solution, then

\[
x_{**} = x_* - \frac{f(x_*)}{f'(x_*)}
\]

is an even better one. The disadvantage of Newton’s method is that you have to differentiate the function in the equation. We will not consider this method any further.
All these methods are quite cumbersome to use by hand, so people commonly use some kind of machine to solve equations like these. Some graphical calculators allow you to solve equations numerically. Financial calculators generally have an option to find the interest rate of an annuity, given the number of payments, the size of every payment, and the present or accumulated value. There are also computer programs that can assist you with these computations. For example, in Excel the command $\text{RATE}(15,500,-5000)$ computes the unknown rate in Example 2.3.1.

Exercises

1. A borrower agrees to repay a loan of £3000 by 15 annual repayments of £500, the first repayment being due after five years. Find the annual yield for this transaction.

2. (From the sample exam) A loan of £50,000 is repaid by annual payments of £4000 in arrear over a period of 20 years. Write down the equation of value and use linear interpolation with trial values of $i = 0.04$ and $i = 0.07$ to approximate the effective rate of interest per annum.

2.4 Annuities payable monthly, etc.

Up to now all annuities involved annual payments. However, other frequencies commonly arise in practice. The same theory as developed above apply to annuities with other frequencies.

Example 2.1.1 shows that the accumulated value of a sequence of eight annual payments of £100 at the time of the last payment is £954.91, if the rate of interest is 5% per annum. The result remains valid if we change the time unit. The same computation shows that the accumulated value of a sequence of eight monthly payments of £100 at the time of the last payment is £954.91, if the rate of interest is 5% per month.

Interest rates per month are not used very often. As explained in Section 1.5, a rate of 5% per month corresponds to a nominal rate $i^{(12)}$ of 60% per year payable monthly (computed as $60 = 5 \times 12$). It also corresponds to an effective rate $i$ of 79.59% per year, because $(1.05)^{12} = 1.7959$. Thus, the accumulated value of a sequence of eight monthly payments of £100 at the time of the last payment is £954.91, if the (effective) rate of interest is 79.59% p.a.

The preceding two paragraphs illustrate the basic idea of this section. The remainder elaborates on this and gives some definitions.

Definition 2.4.1. An annuity immediate payable $p$thly is a regular series of payments at the end of every period of $1/p$ time unit. Consider such an annuity lasting for $n$ time units (so there are $np$ payments), where every payment is $1/p$ unit of capital (so the total payment is $n$ units). The present value of this annuity at the start of the first period is denoted $a^{(p)}_n$, and the accumulated value at the end of the $np$th period is denoted $s^{(p)}_n$.

The present and accumulated value of an annuity immediate payable $p$thly on the basis of an interest rate $i$ per time unit can be calculated using several
methods. Three methods will be presented here. All these methods use the nominal interest rate \( i^{(p)} \) payable \( p \)thly, which is related to \( i \) by (1.5):

\[
1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p.
\]

The first method is the one used at the start of the section, in which a new time unit is introduced which equals the time between two payments (i.e., \( 1/p \) old time units). The rate \( i \) per old time unit corresponds to a rate of \( j = i^{(p)}/p \) per new time unit, and the annuity payable \( p \)thly becomes a standard annuity over \( np \) (new) time units, with one payment of \( 1/p \) per (new) time unit. The future value of this annuity is \( \frac{1}{p} \ddot{s}_{\overline{np}|i^{(p)}} \), which can be evaluated using (2.2):

\[
\ddot{s}_{\overline{np}} = \frac{1}{p} \ddot{s}_{\overline{np|i^{(p)}}} = \frac{(1 + j)^{np} - 1}{jp} = \frac{(1 + \frac{i^{(p)}}{p})^{np} - 1}{i^{(p)}} = \frac{(1 + i)^n - 1}{i^{(p)}}.
\]

To compute the present value of the annuity over \( np \) time units, with one payment of \( 1/p \) per time unit, use (2.3) while bearing in mind that the discount factor in new time units is \( 1/(1 + j) \):

\[
a^{(p)} = \frac{1}{p} a^{(p)} = \frac{1 - (\frac{1}{1+j})^{np}}{jp} = \frac{1 - (1 + \frac{i^{(p)}}{p})^{-np}}{i^{(p)}} = \frac{1 - (1 + i)^{-n}}{i^{(p)}} = \frac{1 - v^n}{i^{(p)}}.
\]

The second method computes the present and accumulated value of annuities payable \( p \)thly from first principles using formula (2.1) for the sum of a geometric sequence. This is the same method used to derive formulas (2.2) and (2.3). The symbol \( a^{(p)}_{\overline{np|}} \) denotes the present value at \( t = 0 \) of \( np \) payments of \( 1/p \) each. The first payment is at time \( t = 1/p \), so its present value is \( (1/p) \cdot v^{1/p} \); the second payment is at time \( t = 2/p \), so its present value is \( (1/p) \cdot v^{2/p} \); and so on till the last payment which is at time \( t = n \), so its present value is \( (1/p) \cdot v^n \). The sum of the present values is:

\[
a^{(p)}_{\overline{np|}} = \frac{1}{p} \left(v^{1/p} + v^{2/p} + \ldots + v^{n-\frac{1}{p}} + v^n\right) = \frac{1}{p} \sum_{k=1}^{np} v^{k/p} = v^{1/p} \cdot \frac{1 - (v^{1/p})^{np}}{1 - v^{1/p}} = \frac{1}{p} \cdot \frac{1 - v^n}{1 + i^{1/p} - 1} = \frac{1 - v^n}{i^{(p)}}.
\]

Similarly, the accumulated value is computed as

\[
s^{(p)} = \frac{1}{p} \left((1 + i)^{n-\frac{1}{p}} + (1 + i)^{n-\frac{2}{p}} + \ldots + (1 + i)^{1/p} + 1\right)
\]

\[
= \frac{1}{p} \sum_{k=0}^{np-1} (1 + i)^{k/p} = \frac{1}{p} \cdot \frac{((1 + i)^{1/p})^{np} - 1}{(1 + i) - 1} = \frac{(1 + i)^n - 1}{\frac{p}{i^{(p)}}}.
\]

The third method compares an annuity payable \( p \)thly to an annuity payable annually over the same period. In one year, an annuity payable \( p \)thly consists of \( p \) payments (at the end of every period of \( 1/p \) year) and an annuity payable annually consists of one payment at the end of the year. If \( p \) payments are \( i^{(p)}/p \) each and the annual payment is \( i \), then these payments are equivalent, as was
found in Section 1.5 (see Figure 4.1). Thus, an annuity with \( p \)thly payments of \( i^{(p)} / p \) is equivalent to an annuity with annual payments of \( i \), so their present and accumulated values are the same:

\[
\ddot{a}_n^{(p)} = i^{(p)} a_n^{(p)} \quad \text{and} \quad \ddot{s}_n^{(p)} = i^{(p)} s_n^{(p)}.
\]

All three methods leads to the same conclusion:

\[
\ddot{a}_n^{(p)} = \frac{1 - v^n}{d^{(p)}} \quad \text{and} \quad \ddot{s}_n^{(p)} = \frac{(1 + i)^n - 1}{d^{(p)}}. \quad (2.11)
\]

The formulas for annuities payable \( p \)thly are the same as the formulas for standard annuities (that is, annuities payable annually), except that the formulas for annuities payable \( p \)thly have the nominal interest rate \( i^{(p)} \) in the denominator instead of \( i \).

A similar story holds for annuities due. An annuity due payable \( p \)thly is a sequence of payments at \( t = 0, 1/p, \ldots, n - (1/p) \), whereas an annuity immediate payable \( p \)thly is a sequence of payments at \( t = 1/p, 2/p, \ldots, n \). Thus, they represent the same sequence of payments, but shifted by one period of \( 1/p \) time unit. The present value of an annuity due payable \( p \)thly at \( t = 0 \) is denoted by \( \ddot{a}_n^{(p)} \), and the accumulated value at \( t = n \) is denoted by \( \ddot{s}_n^{(p)} \). The corresponding expressions are

\[
\ddot{a}_n^{(p)} = \frac{1 - v^n}{d^{(p)}} \quad \text{and} \quad \ddot{s}_n^{(p)} = \frac{(1 + i)^n - 1}{d^{(p)}}. \quad (2.12)
\]

The difference with the formulas (2.11) for annuities immediate is again only in the denominator: \( i^{(p)} \) is replaced by \( d^{(p)} \).

The above discussion tacitly assumed that \( p \) is an integer, but in fact the results are also valid for fractional values of \( p \). This is illustrated in the following example.

**Example 2.4.2.** Consider an annuity of payments of £1000 at the end of every second year. What is the present value of this annuity if it runs for ten years and the interest rate is 7%?

**Answer.** The present value can be found from first principles by summing a geometric sequence. We have \( i = 0.07 \) so \( v = 1 / 1.07 = 0.934579 \), so the present value is

\[
1000v^2 + 1000v^4 + 1000v^6 + 1000v^8 + 1000v^{10}
\]

\[
= 1000 \sum_{k=1}^{5} v^{2k} = 1000v^2 \cdot \frac{1 - (v^2)^5}{1 - v^2} = 3393.03 \text{ pounds.}
\]

Alternatively, we can use (2.11) with \( p = 1/2 \), because there is one payment per two years. We compute \( i^{(1/2)} \) from (1.5),

\[
1 + i = \left(1 + \frac{i^{(1/2)}}{1/2}\right)^{1/2} \quad \Rightarrow \quad i^{(1/2)} = \frac{1}{2} \left(\frac{1 + i^2 - 1}{1 + i^2}\right) = 0.07245,
\]

and thus

\[
\ddot{a}_{10}^{(1/2)} = \frac{1 - v^{10}}{i^{(1/2)}} = 6.786069.
\]
Remember that $a^{(p)}_n$ is the present value of an annuity paying $1/p$ units of capital every $1/p$ years for a period of $n$ years, so $a^{(1/2)}_n = 6.786069$ is the present value of an annuity paying two units of capital every two years for a period of 10 years. Thus, the present value of the annuity in the question is $500 \cdot 6.786069 = 3393.03$ pounds. This is the same as we found from first principles.

**Exercises**

1. *From the 2010 exam*
   
   (a) A savings plan requires you to make payments of £250 each at the end of every month for a year. The bank will then make six equal monthly payments to you, with its first payment due one month after the last payment you make to the bank. Compute the size of each monthly payment made by the bank, assuming a nominal interest rate of 4% p.a. payable monthly.

   (b) The situation is the same as in question (a): you make payments of £250 each at the end of every month for a year, and the rate is 4% p.a. payable monthly. However, now the bank will make equal annual payments to you in perpetuity, with the first payment due three years after the last payment you make to the bank. Compute the size of the annual payments.

2. *From the sample exam* A 20-year loan of £50,000 is repaid as follows. The borrower pays only interest on the loan, annually in arrear at a rate of 5.5% per annum. The borrower will take out a separate savings policy which involves making monthly payments in advance such that the proceeds will be sufficient to repay the loan at the end of its term. The payments into the savings policy accumulate at a rate of interest of 4% per annum effective.

   Compute the monthly payments into the savings account which ensures that it contains £50,000 after 20 years, and write down the equation of value for the effective rate of interest on the loan if it is repaid using this arrangement.

3. *From the CT1 exam, Sept ’08* A bank offers two repayment alternatives for a loan that is to be repaid over ten years. The first requires the borrower to pay £1,200 per annum quarterly in advance and the second requires the borrower to make payments at an annual rate of £1,260 every second year in arrears. Determine which terms would provide the best deal for the borrower at a rate of interest of 4% per annum effective.

### 2.5 Varying annuities

The annuities studied in the preceding sections are all level annuities, meaning that all payments are equal. This section studies annuities in which the size of the payments changes. In simple cases, these can be studied by splitting the varying annuities in a sum of level annuities, as the following example shows.
Example 2.5.1. An annuity pays €50 at the end of every month for two years, and €60 at the end of every month for the next three years. Compute the present value of this annuity on the basis of an interest rate of 7% p.a.

Answer. This annuity can be considered as the sum of two annuities: one of €50 per month running for the first two years, and one of €60 per month running for the next three years. The present value of the first annuity is 600\(a^{(12)}_{\text{2}}\) euros (remember that \(a^{(12)}_{\text{n}}\) is the present value of an annuity paying 1/12 at the end of every month). The value of the second annuity one month before its first payment is 720\(a^{(12)}_{\text{3}}\), which we need to discount by two years. Thus, the present value of the annuity in the question is

\[
600a^{(12)}_{\text{2}} + 720v^2a^{(12)}_{\text{3}} = 600 \cdot \frac{1 - v^2}{i^{(12)}} + 720v^2 \cdot \frac{1 - v^3}{i^{(12)}}.
\]

The interest rate is \(i = 0.07\), so the discount factor is \(v = 1/1.07 = 0.934579\) and the nominal interest rate is \(i^{(12)} = 12(1.07^{1/12} - 1) = 0.0678497\), so

\[
600 \cdot \frac{1 - v^2}{i^{(12)}} + 720v^2 \cdot \frac{1 - v^3}{i^{(12)}} = 1119.19 + 1702.67 = 2821.86.
\]

Thus, the present value of the annuity in the question is €2821.86.

Alternatively, the annuity can be considered as the difference between an annuity of €60 per month running for five years and an annuity of €10 per month running for the first two years. This argument shows that the present value of the annuity in the question is

\[
720a^{(12)}_{\text{5}} - 120a^{(12)}_{\text{2}} = 3045.70 - 223.84 = 2821.86.
\]

Unsurprisingly, this is the same answer as we found before.

More complicated examples of varying annuities require a return to first principles. Let us consider a varying annuity immediate running over \(n\) time units, and denote the amount paid at the end of the \(k\)th time unit by \(P_k\). The present value of this annuity, one time unit before the first payment, is given by

\[
\sum_{k=1}^{n} P_k v^k,
\]

and its accumulated value at the time of the last payment is given by

\[
\sum_{k=1}^{n} P_k (1 + i)^{n-k}.
\]

For a level annuity, all the \(P_k\) are equal, and we arrive at the formulas for \(a^m\) and \(s^m\). The next example considers an annuity whose payments increase geometrically.

Example 2.5.2. An annuity immediate pays £1000 at the end of the first year. The payment increases by 3% per year to compensate for inflation. What is the present value of this annuity on the basis of a rate of 7%, if it runs for 20 years?
Answer. The annuity pays £1000 at the end of the first year, £1030 at the end of the second year, and so on. The payment at the end of year $k$ is given by $P_k = 1000 \cdot (1.03)^{k-1}$. Thus, the present value is

$$
\sum_{k=1}^{20} 1000 \cdot (1.03)^{k-1} \cdot v^k = \frac{1000}{1.03} \sum_{k=1}^{20} (1.03v)^k
$$

$$
= \frac{1000}{1.03} \left( \sum_{k=0}^{20} (1.03v)^k - 1 \right)
$$

$$
= \frac{1000}{1.03} \left( \frac{1 - (1.03v)^{21}}{1 - 1.03v} - 1 \right)
$$

$$
= \frac{1000}{1.03} \left( 14.731613 - 1 \right) = 13331.66.
$$

So the present value of the annuity is £13,331.66.

The case of an annuity whose payments increase in an arithmetic progression is important enough to have its own symbol.

**Definition 2.5.3.** The present value of an increasing annuity immediate which pays 1 at $t = 1$, 2 at $t = 2$, and so on until a final payment of $n$ at $t = n$, is denoted by $(Ia)\underline{n}$. The present value of an increasing annuity due which pays 1 at $t = 0$, 2 at $t = 1$, and so on until a final payment of $n$ at $t = n-1$, is denoted by $(I\ddot{a})\underline{n}$.

Expressions for $(Ia)\underline{n}$ and $(I\ddot{a})\underline{n}$ can be derived as follows. Consider first the increasing annuity immediate. The first payment of 1 at $t = 1$ has present value $v$, the second payment of 2 at $t = 2$ has present value $2v^2$, et cetera, so

$$
(Ia)\underline{n} = v + 2v^2 + 3v^3 + \cdots + nv^n = \sum_{k=1}^{n} kv^k.
$$

(2.13)

For the increasing annuity due, the first payment of 1 at $t = 0$ has present value 1, the second payment of 2 at $t = 2$ has present value $2v$, et cetera, so

$$
(I\ddot{a})\underline{n} = 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \sum_{k=1}^{n} kv^{k-1}.
$$

(2.14)

The annuity due is the same sequence of payments as the annuity immediate, but one year earlier. Thus, we have $(Ia)\underline{n} = v(I\ddot{a})\underline{n}$ as is already obvious from (2.13) and (2.14) above.

Now consider the difference between the annuity due and the annuity immediate. At time $t = 0$, the annuity due pays out 1 and the annuity immediate pays out nothing. At time $t = 1$, the annuity due pays out 2 and the annuity immediate pays out 1. The annuity due pays out 1 more than the annuity immediate at $t = 0, 1, \ldots, n-1$. At the other hand, the annuity immediate pays out $n$ at $t = n$ and the annuity due pays out nothing. Thus,

$$
(I\ddot{a})\underline{n} - (Ia)\underline{n} = \ddot{a} - nv^n.
$$
This can also be found by subtracting (2.13) from (2.14). Now use that \((Ia)_{\bar{m}} = v(I\bar{a})_{\bar{m}}\), as we found above:

\[
\frac{1}{v}(Ia)_{\bar{m}} - (Ia)_{\bar{m}} = \ddot{a}_{\bar{m}} - nv^n \implies (Ia)_{\bar{m}} = \frac{\ddot{a}_{\bar{m}} - nv^n}{\frac{1}{v} - 1} = \frac{\ddot{a}_{\bar{m}} - nv^n}{i}.
\]

This can be written as \(\ddot{a}_{\bar{m}} = i(Ia)_{\bar{m}} + nv^n\), an equation with an interesting (but perhaps challenging) interpretation. Consider a transaction, in which one unit of capital is lent every year. The interest is \(i\) in the first year, \(2i\) in the second year, and so on. At the end of \(n\) years, the amount borrowed is \(n\), which is then paid back. The equation \(\ddot{a}_{\bar{m}} = i(Ia)_{\bar{m}} + nv^n\) expresses that the payments done by the lender are equivalent to the payments by the borrower.

The formula for \((Ia)_{\bar{m}}\) can be used to find the value of annuities with payments in an arithmetic progression. For instance, consider an annuity paying £1000 at the end of the first year, £950 at the end of the second year, £900 at the end of the third year, and so on, with the payment decreasing by £50 every year. The last payment is £500 at the end of the eleventh year. The present value of this annuity is \(1050\ddot{a}_{\bar{11}} - 50(Ia)_{\bar{11}}\).

**Exercises**

1. An annuity is payable in arrear for 20 years. The first payment is of amount £8000 and the amount of each subsequent payment decreases by £300 each year. Find the present value of the annuity on the basis of an interest rate of 5% per annum.
2. An annuity is payable half-yearly for six years, the first half-yearly payment of amount £1800 being due after two years. The amount of subsequent payments decreases by £30 every half-year. On the basis of an interest rate of 5% per half-year, find the present value of the annuity.
3. *(From the 2010 exam)* An annuity pays out on 1 January in every year, from 1 January 2011 up to (and including) 1 January 2030. The annuity pays £1000 in odd years (2011, 2013, 2015, etc.) and £2000 in even years (2012, 2014, 2016, etc.). Compute the present value of this annuity on 1 January 2011 on the basis on an interest rate of 6% p.a.
4. *(From the sample exam)* An individual wishes to receive an annuity which is payable monthly in arrears for 15 years. The annuity is to commence in exactly 10 years at an initial rate of £12,000 per annum. The payments increase at each anniversary by 3% per annum (so the first twelve payments are £1000, the next twelve payments are £1030, and so on). Compute the amount needed to purchase this annuity now assuming an interest rate of 6% per annum effective.
5. *(From the CT1 exam, Sept '09)* A member of a pensions savings scheme invests £1,200 per annum in monthly instalments, in advance, for 20 years from his 25th birthday. From the age of 45, the member increases his investment to £2,400 per annum. At each birthday thereafter the annual rate of investment is further increased by £100 per annum. The investments continue to be made monthly in advance for 20 years until the individuals 65th birthday.
(a) Calculate the accumulation of the investment at the age of 65 using a rate of interest of 6% per annum effective.

(b) At the age of 65, the scheme member uses his accumulated investment to purchase an annuity with a term of 20 years to be paid half-yearly in arrear. At this time the interest rate is 5% per annum convertible half-yearly. Calculate the annual rate of payment of the annuity.

2.6 Loans

Example 2.1.4 considered the repayment of a loan by a level annuity. The repayment of loans is an important application of annuities, which is studied further in this section.

Example 2.6.1 (Continuation of Example 2.1.4). We computed that a loan of €2500 at 6\% interest can be repaid by ten installments of €347.76, each being paid at the end of the year. What is the remaining balance of the loan after six years?

Answer. There are two methods to handle questions like this. The first method considers the payments in the first six years. This is called the retrospective method, because it looks back to payments already made. The second method considers the payments in the last four years. This is called the prospective method, because it looks forward to payments that have not been made yet. Obviously, both methods should give the same answer, and you should pick the method that seems more convenient.

The retrospective method uses that the remaining balance is the value of the original loan after six years minus the accumulated value of the payments that have already been made. The borrower has made six payments of €347.76 each at the end of the year. The accumulated value of these payments is

\[ 347.76 \cdot s_{6} = 347.76 \cdot 7.063728 = 2456.48. \]

The value of the loan after six years is

\[ 2500 \cdot (1 + i)^{6} = 2500 \cdot 1.459142 = 3647.86, \]

so the remaining balance of the loan is 3647.86 – 2456.48 = 1191.38 euros.

The prospective method uses that the remaining balance equals the present value of the remaining payments. The borrower still has to make four payments of €347.76. We need the present value of these payments six years after the start of the loan. This is one year before the first of the four remaining payments is due, so the present value of the four remaining payments is

\[ 347.76 \cdot a_{4} = 347.76 \cdot 3.425799 = 1191.36 \]

euros. Thus, the remaining balance of the loan after six years is €1191.36.

The results found by the retrospective and prospective methods differ by two cents. The difference is caused because at the end of Example 2.1.4, the value of 347.7617... was rounded to 347.76. The prospective and retrospective method would have given the same result if we had used the exact value.
Table 2.1: Loan schedule for a loan of €2500 which is repaid by 10 annual payments of €347.76.

The outstanding balance at the end of every year can be computed with either the retrospective method or the prospective method. The results are summarized in Table 2.1. This table is read as follows. At the end of the first year, the borrower pays €347.76. Part of this payment is interest on the loan. The interest part is \(0.065 \cdot 2500 = 162.50\) euros. The remainder, \(347.76 - 162.50 = 185.26\) euros, is used to repay the loan, so the remaining balance is \(2500 - 185.26 = 2314.74\) euros. In the second year, the payment of €347.76 consist of €150.46 interest on the outstanding balance of €2314.74 and €197.30 repayment of the loan, and so on.

According to the table, the loan is not fully repaid after 10 years, but there is an outstanding balance of four cents. The reason for this is that the table assumes that the annual payment is 347.76 euros, instead of the exact value of 347.7617... euros that was computed in Example 2.1.4. Had we used the exact value, then the loan would be exactly repaid after ten years. In practice, the final payment is often adjusted so that the loan is fully repaid. In our example, that would mean that the final payment is €347.80 instead of €347.76.

A table such as Table 2.1, is known as a loan schedule or amortization schedule. It shows the interest component of every payment, which is useful if the interest component is taxed differently from the repayment component. It also shows the outstanding balance after every payment. The outstanding balance is used if the terms of the loan changes, as shown in the forthcoming Example 2.6.2.

Loan schedules can be computed quite easily. Consider a loan over \(n\) years with an annual payment of \(P\), so amount borrowed is \(Pa_n\). At the end of year \(k\), there are \(n - k\) further payments to be done, so the outstanding balance is \(Pa_{n-k}\) by the prospective method. Thus, the next payment (at the end of year \(k + 1\)) has an interest component of \(iPa_{n-k} = (1 - v^{n-k})P\). The rest of the payment, namely \(v^{n-k} P\), is used to repay the loan.

**Example 2.6.2.** A mortgage of £120,000 is repaid over 20 years by equal monthly payments. How much is every payment on the basis of an effective interest rate of 5.89% p.a.? Suppose that the rate increases by one percent...
point after eleven years. Compute the revised monthly payment.

Answer. The rate of \( i = 0.0589 \) corresponds to a nominal rate payable monthly of

\[
 i^{(12)} = 12 \left( (1 + i)^{1/12} - 1 \right) = 0.05736732.
\]

If every monthly payment is 1/12, then the present value of all the monthly payments is

\[
a^{(12)}_{9/12} = \frac{1 - v^{20}}{i^{(12)}} = \frac{1 - 1.0589^{-20}}{0.05736732} = 11.882249.
\]

Thus, the monthly repayment required to pay off the mortgage of £120,000 is

\[
\frac{120000}{a^{(12)}_{9/12} \cdot \frac{1}{12}} = 841.59 \text{ pounds}.
\]

Now consider the situation after eleven years. There are nine years of payments remaining. The present value of these payments is

\[
841.59 \cdot 12 \cdot a^{(12)}_{7} = 841.49 \cdot 11 \cdot 7.016967 = 70864.91
\]

pounds, so by the prospective method the outstanding balance is £70,864.91. It may seem surprising that less than half the loan is repaid in the first eleven years, but at the start of the loan most of the payments are used to pay the interest. The bulk of the loan is repaid in the second half.

To compute the revised monthly payment after the rate increase, we compute the revised nominal rate as

\[
 i^{(12)} = 12 \left( (1.0689)^{1/12} - 1 \right) = 0.06681541.
\]

There are nine years of monthly payments left, in which £70,864.91 needs to be repaid, so the revised monthly payment is

\[
\frac{70864.91}{a^{(12)}_{7} \cdot \frac{1}{12}} = \frac{70864.91}{6.750054 \cdot \frac{1}{12}} = 874.87.
\]

Thus, the rate increase causes the monthly payment to be increased from £841.59 to £874.87.

Exercises

1. (From the 2010 exam) A four-year loan of £5000 is repaid by equal annual payments at the end of each year. Compute the annual payment on the basis of an interest rate of 6% p.a. and draw up a loan schedule, showing the interest component of every payment and the outstanding balance.

2. (From the CT1 exam, April ’08) A mortgage company offers the following two deals to customers for twenty-five year mortgages.

(a) A mortgage of £100,000 is offered with level repayments of £7,095.25 made annually in arrear. There are no arrangement or exit fees.
(b) A mortgage of £100,000 is offered whereby a monthly payment in advance is calculated such that the customer pays an effective rate of return of 4% per annum ignoring arrangement and exit fees. In addition the customer also has to pay an arrangement fee of £6,000 at the beginning of the mortgage and an exit fee of £5,000 at the end of the twenty-five year term of the mortgage.

Compare the annual effective rates of return paid by customers on the two products.

3. (From the CT1 exam, April ’09) A loan is to be repaid by an annuity payable annually in arrear. The annuity starts at a rate of £300 per annum and increases each year by £30 per annum. The annuity is to be paid for 20 years. Repayments are calculated using a rate of interest of 7% per annum effective.

Calculate:

(a) The amount of the loan.
(b) The capital outstanding immediately after the 5th payment has been made.
(c) The capital and interest components of the final payment.

### 2.7 Annuities over non-integer periods

It is fairly easy to compute the number $n$ of payments of an annuity given its present or accumulated value, as the following example shows. However, the result is usually not an integer. This section explains how to interpret fractional periods.

**Example 2.7.1.** A loan of $10000 is repaid by annual payments of $1000 each at the end of the year. How long does it take to repay the loan on the basis of an interest rate of 6% p.a.?

**Answer.** The annual payments form an annuity immediate. Its present value is $10000 \alpha_{1\overline{n}}$. This should equal the amount borrowed, so we have to solve $a_{1\overline{n}} = 10$ for $n$. Using (2.3), we find:

$$a_{1\overline{n}} = 10 \implies \frac{1 - v^n}{i} = 10$$

$$\implies v^n = 1 - 10i$$

$$\implies n = \frac{\log(1 - 10i)}{\log(v)} = 15.725,$$

so the solution is $n = 15.725$.

How should we to interpret the result $n = 15.725$? What does it mean to have an annuity over 15.725 years? One possibility is that the borrower should make 15 payments of $1000 at $t = 1, \ldots, 15$, and one final payment at time $t = 15.725$ (after 15 years and 265 days). The present value of the 15 payments of $1000 is $10000 \alpha_{1\overline{15}} = 9712.25$ dollars, so after these payments $287.75$ of the loan is still outstanding. Thus, the final payment at $t = 15.725$ should be
The loan is repaid by 15 payments of $1000, paid annually in arrears, and a final payment of $718.38, paid after 15 years and 265 days.

This interpretation is possible but not used very often in practice, because it is more convenient for both parties to have all payments at regular times. Usually, either the payment at \( t = 15 \) is increased so that the entire loan is repaid at that time, or a payment is added at \( t = 16 \) to repay the remainder of the loan. In the first case, the borrower has to pay \((1 + i)^{15} \cdot 287.75 = 689.61 \) dollars extra at \( t = 15 \), so the loan is repaid by 14 payments of $1000 at \( t = 1, \ldots, 14 \) and a final payment of $1689.61. The payment of $1689.61 at the end is called the balloon payment. In the second case, the borrower has to make an additional payment of \((1 + i)^{16} \cdot 287.75 = 730.99 \) dollars at \( t = 16 \), so the loan is repaid by 15 payments of $1000 at \( t = 1, \ldots, 15 \) and a final payment of $730.99 at \( t = 16 \). The payment of $730.99 at the end is called the drop payment. In conclusion, we found three possible interpretations of the annuity over 15.725 years, that resulted in the situation of Example 2.7.1:

- The borrower pays $1000 at \( t = 1, \ldots, 14 \) and $1689.61 at \( t = 15 \).
- The borrower pays $1000 at \( t = 1, \ldots, 15 \) and $718.38 at \( t = 15.725 \).
- The borrower pays $1000 at \( t = 1, \ldots, 15 \) and $730.99 at \( t = 16 \).

In all these three cases, the present value of the payments at \( t = 0 \) is $10000.

These notes use the convention that an annuity over 15.725 years is interpreted by one of the three sequences of payments listed above, whose present value is 10000(1+\(i\))\(^{15.725}\). In other words, we define annuities over \( n \) years, where \( n \) is not an integer, such that the formulas (2.3) for the present value and (2.2) for the accumulated value remain valid. This definition is not generally accepted. Other writers define an annuity immediate of $1000 over 15.725 years as 15 payments of $1000 at \( t = 1, \ldots, 15 \) and a final payment of $725 at \( t = 15.725 \). This may seem more normal, but in that interpretation formulas (2.3) and (2.2) are no longer valid.

**Exercises**

1. *(From the 2010 exam)* A loan of $1000 is repaid by payments of $100 at the end of each quarter, and a smaller final payment made one quarter after the last regular payment (a drop payment). The interest rate is 10% p.a. as long as the outstanding balance is greater than $500 and 8% p.a. afterwards.

   (a) Compute the outstanding balance after one year.
   
   (b) Which payment causes the outstanding balance to drop below $500?
   
   (c) How long does it take to pay off the loan?

   (d) Compute the final payment.