MATH1510
Financial Mathematics I

Jitse Niesen
University of Leeds

January – May 2012

Description of the module

This is the description of the module as it appears in the module catalogue.

Objectives

Introduction to mathematical modelling of financial and insurance markets with particular emphasis on the time-value of money and interest rates. Introduction to simple financial instruments. This module covers a major part of the Faculty and Institute of Actuaries CT1 syllabus (Financial Mathematics, core technical).

Learning outcomes

On completion of this module, students should be able to understand the time value of money and to calculate interest rates and discount factors. They should be able to apply these concepts to the pricing of simple, fixed-income financial instruments and the assessment of investment projects.

Syllabus

- Interest rates. Simple interest rates. Present value of a single future payment. Discount factors.
- Effective and nominal interest rates. Real and money interest rates. Compound interest rates. Relation between the time periods for compound interest rates and the discount factor.
- Compound interest functions. Annuities and perpetuities.
- Loans.
- Introduction to fixed-income instruments. Generalized cashflow model.
- Net present value of a sequence of cashflows. Equation of value. Internal rate of return. Investment project appraisal.
- Examples of cashflow patterns and their present values.
- Elementary compound interest problems.

Reading list

These lecture notes are based on the following books:

1. Samuel A. Broverman, Mathematics of Investment and Credit, 4th ed., ACTEX Publications, 2008. ISBN 978-1-56698-657-1.

2. The Faculty of Actuaries and Institute of Actuaries, Subject CT1: Financial Mathematics, Core Technical. Core reading for the 2009 examinations.

3. Stephen G. Kellison, The Theory of Interest, 3rd ed., McGraw-Hill, 2009. ISBN 978-007-127627-6.

4. John McCutcheon and William F. Scott, An Introduction to the Mathematics of Finance, Elsevier Butterworth-Heinemann, 1986. ISBN 0-7506-0092-6.

5. Petr Zima and Robert L. Brown, Mathematics of Finance, 2nd ed., Schaum's Outline Series, McGraw-Hill, 1996. ISBN 0-07-008203.

The syllabus for the MATH1510 module is based on Units 1–9 and Unit 11 of book 2. The remainder forms the basis of MATH2510 (Financial Mathematics II). The book 2 describes the first exam that you need to pass to become an accredited actuary in the UK. It is written in a concise and perhaps dry style.

These lecture notes are largely based on Book 4. Book 5 contains many exercises, but does not go quite as deep. Book 3 is written from a U.S. perspective, so the terminology is slightly different, but it has some good explanations. Book 1 is written by a professor from a U.S./Canadian background and is particularly good in making connections to applications.

All these books are useful for consolidating the course material. They allow you to gain background knowledge and to try your hand at further exercises. However, the lecture notes cover the entire syllabus of the module.

Organization for 2011/12

Lecturer	Jitse Niesen
E-mail	jitse@maths.leeds.ac.uk
Office	Mathematics 8.22f
Telephone	35870 (from outside: 0113 3435870)
Lectures	Tuesdays 10:00 – 11:00 in Roger Stevens LT 20 Wednesdays 12:00 – 13:00 in Roger Stevens LT 25 Fridays 14:00 – 15:00 in Roger Stevens LT 17
Example classes	Mondays in weeks 3, 5, 7, 9 and 11, see your personal timetable for time and room.
Tutors	Niloufar Abourashchi, Zhidi Du, James Fung, and Tongya Wang.
Office hours	Tuesdays .... (to be determined) or whenever you find the lecturer and he has time.
Course work	There will be five sets of course work. Put your work in your tutor's pigeon hole on Level 8 of School of Mathematics. Due dates are Wednesday 1 February, 15 February, 29 February, 14 March and 25 April.
Late work	One mark (out of ten) will be deducted for every day.
Copying	Collaboration is allowed (even encouraged), copying not. See the student handbook for details.
Exam	The exam will take place in the period 14 May – 30 May; exact date and location to be announced.
Assessment	The course work counts for 15%, the exam for 85%.
Lecture notes	These notes and supporting materials are available in the Blackboard VLE.

Chapter 1

The time value of money

Interest is the compensation one gets for lending a certain asset. For instance, suppose that you put some money on a bank account for a year. Then, the bank can do whatever it wants with that money for a year. To reward you for that, it pays you some interest.

The asset being lent out is called the *capital*. Usually, both the capital and the interest is expressed in money. However, that is not necessary. For instance, a farmer may lend his tractor to a neighbour, and get 10% of the grain harvested in return. In this course, the capital is always expressed in money, and in that case it is also called the *principal*.

1.1 Simple interest

Interest is the reward for lending the capital to somebody for a period of time. There are various methods for computing the interest. As the name implies, *simple interest* is easy to understand, and that is the main reason why we talk about it here. The idea behind simple interest is that the amount of interest is the product of three quantities: the rate of interest, the principal, and the period of time. However, as we will see at the end of this section, simple interest suffers from a major problem. For this reason, its use in practice is limited.

Definition 1.1.1 (Simple interest)**.** The interest earned on a capital C lent over a period n at a rate i is niC.

Example 1.1.2. How much interest do you get if you put 1000 pounds for two years in a savings acount that pays simple interest at a rate of 9% per annum? And if you leave it in the account for only half ar year?

Answer. If you leave it for two years, you get

$$2 \cdot 0.09 \cdot 1000 = 180$$

pounds in interest. If you leave it for only half a year, then you get $\frac{1}{2} \cdot 0.09 \cdot 1000 =$ 45 pounds.

As this example shows, the rate of interest is usually quoted as a percentage; 9% corresponds to a factor of 0.09. Furthermore, you have to be careful that the rate of interest is quoted using the same time unit as the period. In this example, the period is measured in years, and the interest rate is quoted per annum ("per annum" is Latin for "per year"). These are the units that are used most often. In Section 1.5 we will consider other possibilities.

Example 1.1.3. Suppose you put £1000 in a savings account paying simple interest at 9% per annum for one year. Then, you withdraw the money with interest and put it for one year in another account paying simple interest at 9%. How much do you have in the end?

Answer. In the first year, you would earn $1 \cdot 0.09 \cdot 1000 = 90$ pounds in interest, so you have £1090 after one year. In the second year, you earn $1 \cdot 0.09 \cdot 1090 = 98.1$ pounds in interest, so you have £1188.10 ($= 1090 + 98.1$) at the end of the two years.

Now compare Examples 1.1.2 and 1.1.3. The first example shows that if you invest £1000 for two years, the capital grows to £1180. But the second example shows that you can get £1188.10 by switching accounts after a year. Even better is to open a new account every month.

This inconsistency means that simple interest is not that often used in practice. Instead, savings accounts in banks pay compound interest, which will be introduced in the next section. Nevertheless, simple interest is sometimes used, especially in short-term investments.

Exercises

1. (*From the 2010 exam*) How many days does it take for £1450 to accumulate to £1500 under 4% p.a. simple interest?
2. (*From the sample exam*) A bank charges simple interest at a rate of 7% p.a. on a 90-day loan of £1500. Compute the interest.

1.2 Compound interest

Most bank accounts use *compound interest*. The idea behind compound interest is that in the second year, you should get interest on the interest you earned in the first year. In other words, the interest you earn in the first year is combined with the principal, and in the second year you earn interest on the combined sum.

What happens with the example from the previous section, where the investor put £1000 for two years in an account paying 9%, if we consider compound interest? In the first year, the investor would receive £90 interest (9% of £1000). This would be credited to his account, so he now has £1090. In the second year, he would get £98.10 interest (9% of £1090) so that he ends up with £1188.10; this is the same number as we found before. The capital is multiplied by 1.09 every year: $1.09 \cdot 1000 = 1090$ and $1.09 \cdot 1090 = 1188.1$.

More generally, the interest over one year is iC, where i denotes the interest rate and C the capital at the beginning of the year. Thus, at the end of the year, the capital has grown to $C + iC = (1 + i)C$. In the second year, the principal is $(1 + i)C$ and the interest is computed over this amount, so the interest is $i(1 + i)C$ and the capital has grown to $(1 + i)C + i(1 + i)C = (1 + i)^2C$. In the third year, the interest is $i(1+i)^2C$ and the capital has grown to $(1+i)^3C$.

This reasoning, which can be made more formal by using complete induction, leads to the following definition.

Definition 1.2.1 (Compound interest)**.** A capital C lent over a period n at a rate i grows to $(1+i)^n C$.

Example 1.2.2. How much do you have after you put 1000 pounds for two years in a savings acount that pays compound interest at a rate of 9% per annum? And if you leave it in the account for only half ar year?

Answer. If you leave it in the account for two years, then at the end you have

$$(1+0.09)^2 \cdot 1000 = 1188.10,$$

as we computed above. If you leave it in the account for only half a year, then at the end you have

$$(1+0.09)^{1/2} \cdot 1000 = \sqrt{1.09} \cdot 1000 = 1044.03$$

pounds (rounded to the nearest penny). This is 97p less than the 45 pounds interest you get if the account would pay simple interest at the same rate (see Example 1.1.2).

Example 1.2.3. Suppose that a capital of 500 dollars earns 150 dollars of interest in 6 years. What was the interest rate if compound interest is used? What if simple interest is used?

Answer. The capital accumulated to \$650, so in the case of compound interest we have to solve the rate i from the equation

$$\begin{aligned}(1+i)^6 \cdot 500 = 650 &\iff (1+i)^6 = 1.3 \\ &\iff 1+i = 1.3^{1/6} = 1.044698\ldots \\ &\iff i = 0.044698\ldots\end{aligned}$$

Thus, the interest rate is 4.47%, rounded to the nearest basis point (a basis point is 0.01%). Note that the computation is the same, regardless of the currency used.

In the case of simple interest, the equation to solve $6 \cdot i \cdot 500 = 150$, so $i = \frac{150}{6 \cdot 500} = 0.05$, so the rate is 5%.

Example 1.2.4. How long does it take to double your capital if you put it in an account paying compound interest at a rate of $7\frac{1}{2}\%$? What if the account pays simple interest?

Answer. The question is for what value of n does a capital C accumulate to $2C$ if $i = 0.075$. So we have to solve the equation $1.075^n C = 2C$. The first step is to divide by C to get $1.075^n = 2$. Then take logarithms:

$$\log(1.075^n) = \log(2) \iff n\log(1.075) = \log(2) \iff n = \frac{\log(2)}{\log(1.075)} = 9.58\ldots$$

So, it takes 9.58 years to double your capital. Note that it does not matter how much you have at the start: it takes as long for one pound to grow to two pounds as for a million pounds to grow to two million.

The computation is simpler for simple interest. We have to solve the equation $n \cdot 0.075 \cdot C = C$, so $n = \frac{1}{0.075} = 13\frac{1}{3}$, so with simple interest it takes $13\frac{1}{3}$ years to double your capital.

More generally, if the interest rate is i, then the time required to double your capital is

$$n = \frac{\log(2)}{\log(1+i)}.$$

We can approximate the denominator by $\log(1+i) \approx i$ for small i; this is the first term of the Taylor series of $\log(1+i)$ around $i = 0$ (note that, as is common in mathematics, "log" denotes the *natural* logarithm). Thus, we get $n \approx \frac{\log(2)}{i}$. If instead of the interest rate i we use the percentage $p = 100i$, and we approximate $\log(2) = 0.693\ldots$ by 0.72, we get

$$n \approx \frac{72}{p}.$$

This is known as the *rule of 72*: To calculate how many years it takes you to double your money, you divide 72 by the interest rate expressed as a percentage. Let us return to the above example with a rate of $7\frac{1}{2}\%$. We have $p = 7\frac{1}{2}$ so we compute $72/7\frac{1}{2} = 9.6$, which is very close to the actual value of $n = 9.58$ we computed before.

The rule of 72 can already be found in a Italian book from 1494: *Summa de Arithmetica* by Luca Pacioli. The use of the number 72 instead of 69.3 has two advantages: many numbers divide 72, and it gives a better approximation for rates above 4% (remember that the Taylor approximation is centered around $i = 0$; it turns out that it is slightly too small for rates of 5–10% and using 72 instead of 69.3 compensates for this).

Remember that with simple interest, you could increase the interest you earn by withdrawing your money from the account halfway. Compound interest has the desirable property that this does not make a difference. Suppose that you put your money m years in one account and then n years in another account, and that both account pay compount interest at a rate i. Then, after the first m years, your capital has grown to $(1+i)^m C$. You withdraw that and put it in another account for n years, after which your capital has grown to $(1+i)^n(1+i)^m C$. This is the same as what you would get if you had kept the capital in the same account for $m+n$ years, because

$$(1+i)^n(1+i)^m C = (1+i)^{m+n} C.$$

This is the reason why compound interest is used so much in practice. Unless noted otherwise, interest will always refer to compound interest.

Exercises

1. The rate of interest on a certain bank deposit account is $4\frac{1}{2}\%$ per annum effective. Find the accumulation of £5000 after seven years in this account.
2. (*From the sample exam*) How long does it take for £900 to accumulate to £1000 under an interest rate of 4% p.a.?

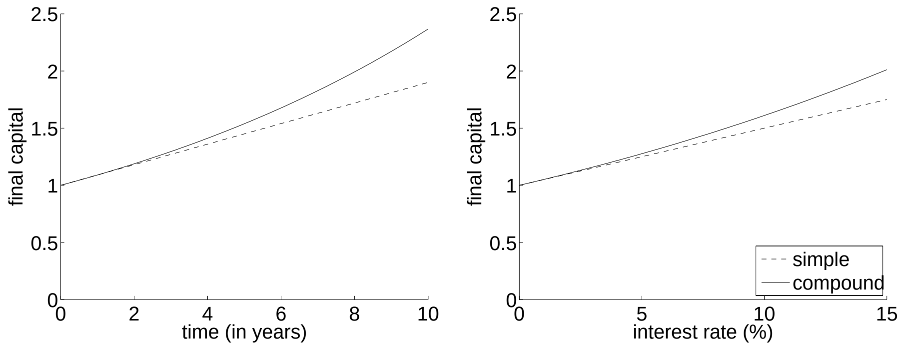

Figure 1.1: Comparison of simple interest and compound interest. The left figure plots the growth of capital in time at a rate of 9%. The right figure plots the amount of capital after 5 years for various interest rates.

1.3 Comparing simple and compound interest

Simple interest is defined by the formula "interest = inC." Thus, in n years the capital grows from C to $C + niC = (1 + ni)C$. Simple interest and compound interest compare as follows:

$$\begin{aligned} \text{simple interest:} \quad & \text{capital after } n \text{ years} = (1 + ni)C \\ \text{compound interest:} \quad & \text{capital after } n \text{ years} = (1 + i)^n C \end{aligned}$$

These formulas are compared in Figure 1.1. The left plot shows how a principal of 1 pound grows under interest at 9%. The dashed line is for simple interest and the solid curve for compound interest. We see that compound interest pays out more in the long term. A careful comparison shows that for periods less than a year simple interest pays out more, while compound interest pays out more if the period is longer than a year. This agrees with what we found before. A capital of £1000, invested for half a year at 9%, grows to £1045 under simple interest and to £1044.03 under compound interest, while the same capital invested for two years grows to £1180 under simple interest and £1188.10 under compound interest. The difference between compound and simple interest get bigger as the period gets longer.

This follows from the following algebraic inequalities: if i is positive, then

$$\begin{aligned} (1 + i)^n &< 1 + ni \quad \text{if} \quad n < 1, \\ (1 + i)^n &> 1 + ni \quad \text{if} \quad n > 1. \end{aligned}$$

These will not be proven here. However, it is easy to see that the formulas for simple and compound interest give the same results if $n = 0$ and $n = 1$. Now consider the case $n = 2$. A capital C grows to $(1 + 2i)C$ under simple interest and to $(1 + i)^2C = (1 + 2i + i^2)C$ under compound interest. We have $(1 + 2i + i^2)C > (1 + 2i)C$ (because C is positive), so compound interest pays out more than simple interest.

The right plot in Figure 1.1 shows the final capital after putting a principal of 1 pound away for five years at varying interest rates. Again, the dashed line corresponds to simple interest and the solid curve corresponds to compound

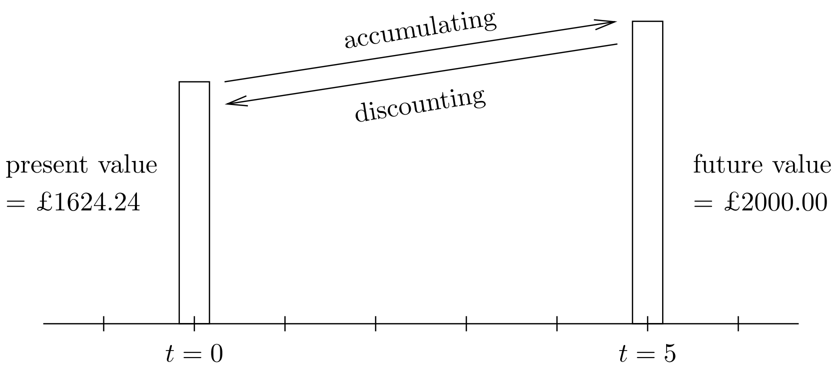

Figure 1.2: The time value of money: £1624.24 now is equivalent to £2000 in five years at a rate of $4\frac{1}{4}\%$.

interest. We see again that compound interest pays out more, as $n = 5$ is greater than 1. However, the plot also shows that the difference is smaller if the interest rate is small.

This can be explained with the theory of Taylor series. A capital C will grow in n years to $(1+i)^n C$. The Taylor series of $f(i) = (1+i)^n C$ around $i = 0$ is

$$f(0) + f'(0)i + \tfrac{1}{2}f''(0)i^2 + \cdots = C + niC + \tfrac{1}{2}n(n-1)i^2C + \cdots .$$

The first two terms are $C + niC = (1 + ni)C$, which is precisely the formula for simple interest. Thus, you can use the formula for simple interest as an approximation for compound interest; this approximation is especially good if the rate of interest is small. Especially in the past, people often used simple interest instead of compound interest, notwithstanding the inconsistency of simple interest, to simplify the computations.

1.4 Discounting

The formula for compound interest relates four quantities: the capital C at the start, the interest rate i, the period n, and the capital at the end. We have seen how to calculate the interest rate (Example 1.2.3), the period (Example 1.2.4), and the capital at the end (Example 1.2.2). The one remaining possibility is covered in the next example.

Example 1.4.1. How much do you need to invest now to get £2000 after five years if the rate of interest is $4\frac{1}{4}\%$?

Answer. One pound will accumulate to $(1 + 0.0425)^5 = 1.2313466$ in five years, so you need to invest $2000/1.2313466 = 1624.24$ pounds.

We say that £1624.24 now is equivalent to £2000 in five years at a rate of $4\frac{1}{4}\%$. We call £1624.24 the *present value* and £2000 the *future value*. When you move a payment forward in time, it *accumulates*; when you move it backward, it is *discounted* (see Figure 1.2).

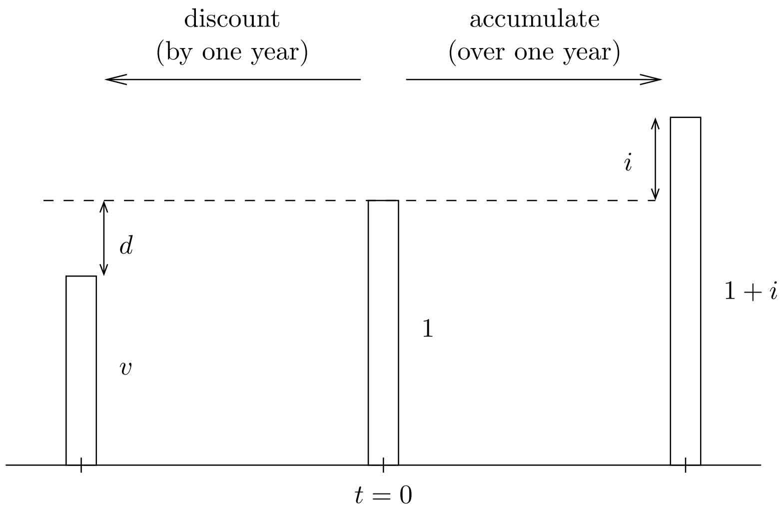

Figure 1.3: The relation between the interest rate i, the rate of discount d and the discount factor v.

This shows that money has a *time value*: the value of money depends on the time. £2000 now is worth more than £2000 in five years' time. In financial mathematics, all payments must have a date attached to them.

More generally, suppose the interest rate is i. How much do you need to invest to get a capital C after one time unit? The answer is $\frac{1}{1+i}C$. The factor

$$v = \frac{1}{1+i}. \tag{1.1}$$

is known as the discount factor. It is the factor with which you have to multiply a payment to shift it backward by one year (see Figure 1.3). If the interest rate is $4\frac{1}{4}\%$, then the discount factor is $\frac{1}{1.0425} = 0.95923$.

Provided the interest rate is not too big, the discount factor is close to one. Therefore people often use the rate of discount $d = 1 - v$, usually expressed as a percentage (compare how the interest rate i is used instead of the "accumulation factor" $1 + i$). In our example, the rate of discount is 0.04077 or 4.077%.

Example 1.4.2. Suppose that the interest rate is 7%. What is the present value of a payment of €70 in a year's time?

Answer. The discount factor is $v = 1/1.07 = 0.934579$, so the present value is $0.934579 \cdot 70 = 65.42$ euro (to the nearest cent).

Usually, interest is paid *in arrears*. If you borrow money for a year, then at the end of the year you have to pay the money back plus interest. However, there are also some situations in which the interest is paid in advance. The rate of discount is useful in these situations, as the following example shows.

Example 1.4.3. Suppose that the interest rate is 7%. If you borrow €1000 for a year and you have to pay interest at the start of the year, how much do you have to pay?

Answer. If interest were to be paid in arrears, then you would have to pay $0.07 \cdot 1000 = 70$ euros at the end of the year. However, you have to pay the interest one year earlier. As we saw in Example 1.4.2, the equivalent amount is $v \cdot 70 = 65.42$ euros.

There is another way to arrive at the answer. At the start of the year, you get €1000 from the lender but you have to pay interest immediately, so in effect you get less from the lender. At the end of the year, you pay €1000 back. The amount you should get at the start of the year should be equivalent to the €1000 you pay at the end of the year. The discount factor is $v = 1/1.07 = 0.934579$, so the present value of the €1000 at the end of the year is €934.58. Thus, the interest you have to pay is €1000 − €934.58 = €65.42.

In terms of the interest rate $i = 0.07$ and the capital $C = 1000$, the first method calculates ivC and the second method calculates $C - vC = (1 - v)C = dC$. Both methods yield the same answer, so we arrive at the important relation

$$d = iv. \tag{1.2}$$

We can check this relation algebraically. We found before, in equation (1.1), that the discount factor is

$$v = \frac{1}{1+i}.$$

The rate of discount is

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i}. \tag{1.3}$$

Comparing these two formulas, we find that indeed $d = iv$.

We summarize this discussion with a formal definition of the three quantities d, i and v.

Definition 1.4.4. The *rate of interest* i is the interest paid at the end of a time unit divided by the capital at the beginning of the time unit. The *rate of discount* d is the interest paid at the beginning of a time unit divided by the capital at the end of the time unit. The *discount factor* v is the amount of money one needs to invest to get one unit of capital after one time unit.

This definition concerns periods of one year (assuming that time is measured in years). In Example 1.4.1, we found that the present value of a payment of £2000 due in five years is £1624.24, if compound interest is used at a rate of $4\frac{1}{4}\%$. This was computed as $2000/(1 + 0.0425)^5$. The same method can be used to find the present value of a payment of C due in n years if compound interest is used at a rate i. The question is: which amount x accumulates to C in n years? The formula for compound interest yields that $(1 + i)^n x = C$, so the present value x is

$$\frac{C}{(1+i)^n} = v^n C = (1 - d)^n C. \tag{1.4}$$

This is called *compound discounting*, analogous with compound interest.

There is another method, called *simple discounting* (analogous to simple interest) or *commercial discounting*. This is defined as follows. The present value of a payment of C due in n years, at a rate of simple discount of d, is $(1 - nd)C$.

Simple discounting is not the same as simple interest. The present value of a payment of C due in n years, at a rate of simple interest of i, is the amount x that accumulates to C over n years. Simple interest is defined by $C = (1+ni)x$, so the present value is $x = (1 + ni)^{-1}C$.

Example 1.4.5. What is the present value of £6000 due in a month assuming 8% p.a. simple discount? What is the corresponding rate of (compound) discount? And the rate of (compound) interest? And the rate of simple interest?

Answer. One month is $\frac{1}{12}$ year, so the present value of is $(1 - \frac{1}{12} \cdot 0.08) \cdot 6000 =$ 5960 pounds. We can compute the rate of (compound) discount d from the formula "present value $= (1 - d)^n C$":

$$\begin{aligned} 5960 = (1-d)^{1/12} \cdot 6000 &\implies (1-d)^{1/12} = \tfrac{5960}{6000} = 0.993333 \\ &\implies 1 - d = 0.993333^{12} = 0.922869 \\ &\implies d = 0.077131. \end{aligned}$$

Thus, the rate of discount is 7.71%. The rate of (compound) interest i follows from

$$\frac{1}{1+i} = 1 - d = 0.922869 \implies 1 + i = 1.083577$$

so the rate of (compound) interest is 8.36%. Finally, to find the rate of simple interest, solve $5960 = (1 + \frac{1}{12}i)^{-1}6000$ to get $i = 0.080537$, so the rate of simple interest is 8.05%.

One important application for simple discount is U.S. Treasury Bills. However, it is used even less in practice than simple interest.

Exercises

1. In return for a loan of £100 a borrower agrees to repay £110 after seven months.

    (a) Find the rate of interest per annum.

    (b) Find the rate of discount per annum.

    (c) Shortly after receiving the loan the borrower requests that he be allowed to repay the loan by a payment of £50 on the original settlement date and a second payment six months after this date. Assuming that the lender agrees to the request and that the calculation is made on the original interest basis, find the amount of the second payment under the revised transaction.

2. The commercial rate of discount per annum is 18% (this means that simple discount is applied with a rate of 18%).

    (a) We borrow a certain amount. The loan is settled by a payment of £1000 after three months. Compute the amount borrowed and the effective annual rate of discount.

    (b) Now the loan is settled by a payment of £1000 after *nine* months. Answer the same question.

1.5 Interest payable monthly, quarterly, etc.

Up to now, we assumed that interest is paid once a year. In practice interest is often paid more frequently, for instance quarterly (four times a year). This is straightforward if the interest rate is also quoted per quarter, as the following example shows.

Example 1.5.1. Suppose that you save £1000 in an account that pays 2% interest every quarter. How much do you have in one year, if the interest is paid in the same account?

Answer. We can use the formula for compound interest in Definition 1.2.1, which says that a capital C accumulates to $(1+i)^n C$ over a period n, if the rate is i. The rate $i = 0.02$ is measured in quarters, so we also have to measure the period n in quarters. One year is four quarters, so the capital accumulates to $1.02^4 \cdot 1000 = 1082.43$ pounds.

However, interest rates are usually not quoted per quarter even if interest is paid quarterly. The rate is usually quoted per annum (p.a.). In the above example, with 2% per quarter, the interest rate would be quoted as 8% p.a. payable quarterly. This rate is called the *nominal interest rate* payable quarterly. You may also see the words "convertible" or "compounded" instead of "payable".

It may seem more logical to quote the rate as 8.243%. After all, we computed that £1000 accumulates to £1082.43 in a year. The rate of 8.243% is called the *effective interest rate*. It often appears in advertisements in the U.K. as the Annual Equivalent Rate (AER). The effective interest rate corresponds to the interest rate i as defined in Definition 1.4.4: the interest paid at the end of a time unit divided by the capital at the beginning of the time unit.

Definition 1.5.2. The *interest conversion period* is the period between two successive interest payments. Denote the quotient of the time unit and the interest conversion period by p. Let $i_{[p]}$ denote the interest rate per conversion period. The *nominal interest rate*, denoted $i^{(p)}$, is then p times $i_{[p]}$.

Common values for p include $p = 365$ (interest payable daily) and $p = 12$ (interest payable monthly). The term "interest payable $p$thly" is used if we do not want to specify the conversion period. In the example, the interest conversion period is a quarter and the time unit is a year, so $p = 4$. The interest rate per quarter is 2%, meaning that $i_{[4]} = 0.02$, so the nominal interest rate is $i^{(4)} = 4 \cdot 0.02 = 0.08$ or 8%, and the effective interest rate is $i = 0.08243$.

To compute the effective interest rate from the nominal interest rate $i^{(p)}$, remember that the interest rate per conversion period is $i_{[p]} = i^{(p)}/p$. There are p conversion periods in a time unit. Thus, by the formula for compound interest, a capital C accumulates to $(1 + i_{[p]})^p C = (1 + i^{(p)}/p)^p C$ in a time unit. However, if the effective interest rate is i, then a capital C accumulates to $(1+i)C$ in a time unit. Thus, a nominal interest rate $i^{(p)}$ payable $p$thly is equivalent to an effective interest rate i if

$$1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p. \tag{1.5}$$

Example 1.5.3. Suppose that an account offers a nominal interest rate of 8% p.a. payable quarterly. What is the AER? What if the nominal rate is the same, but interest is payable monthly? Weekly? Daily?

Answer. For interest payable quarterly, we put $p = 4$ and $i^{(4)} = 0.08$ in (1.5) to find

$$1 + i = \left(1 + \frac{0.08}{4}\right)^4 = 1.08243,$$

so the AER is 8.243%. This is the example we considered above. In the other cases, we find:

$$\begin{aligned}
&\text{monthly } (p = 12): \quad 1 + i = \left(1 + \frac{0.08}{12}\right)^{12} = 1.08300 \\
&\text{weekly } (p = 52): \quad 1 + i = \left(1 + \frac{0.08}{52}\right)^{52} = 1.08322 \\
&\text{daily } (p = 365): \quad 1 + i = \left(1 + \frac{0.08}{365}\right)^{365} = 1.08328
\end{aligned}$$

So, the AER is 8.300% for interest payable monthly, 8.322% for interest payable weekly, and 8.328% for interest payable daily.

It looks like the numbers converge to some limit as the conversion period gets shorter. This idea will be taken up at the end of the module.

There is an alternative but equivalent definition of the symbol $i^{(p)}$, which leads naturally to the valuation of annuities described in the next chapter. In Example 1.5.1, we assumed that the interest is paid in the account so that it generates more interest. If this is not the case, but you use the interest for other purposes, then the amount in the account will remain constant at £1000. You will get £20 interest after each quarter. This is equivalent to receiving £82.43 at the end of the year, given an (effective) interest rate of 8.243% p.a., as the following computation shows:

- £20 at the end of the first quarter is equivalent to $1.08243^{3/4} \cdot 20 = 21.22$ pounds at the end of the year.
- £20 at the end of the second quarter is equivalent to $1.08243^{1/2} \cdot 20 = 20.81$ pounds at the end of the year.
- £20 at the end of the third quarter is equivalent to $1.08243^{1/4} \cdot 20 = 20.40$ pounds at the end of the year.

Thus, £20 at the end of each quarter is equivalent to $21.22 + 20.81 + 20.40 + 20.00 = 82.43$ pounds at the end of the year.

More generally, a capital of 1 generates $i^{(p)}/p$ interest per conversion period. We can either leave the interest in the account, in which case the capital accumulates to $1 + i = (1 + i^{(p)}/p)^p$ at the end of the year, as we computed above, so we get a payment of i at the end of the year. Or we can take the interest as soon as it is paid, so we get p payments of $i^{(p)}/p$ each at times $\frac{1}{p}, \frac{2}{p}, \ldots, 1$. The payment of $i^{(p)}/p$ at time $\frac{k}{p}$ is equivalent to

$$(1 + i)^{(p-k)/p} \frac{i^{(p)}}{p}$$

at the end of the year, because it needs to be shifted $p - k$ periods forward. Thus, the series of p payments is equivalent to

$$\sum_{k=1}^{p} (1+i)^{(p-k)/p} \frac{i^{(p)}}{p}$$

at the end of the year. If we make the substitution $n = p - k$, we get

$$\sum_{k=1}^{p} (1+i)^{(p-k)/p} \frac{i^{(p)}}{p}. = \sum_{n=0}^{p-1} (1+i)^{n/p} \frac{i^{(p)}}{p}.$$

This sum can be evaluated with the following formula for a geometric sum:

$$1 + r + r^2 + \cdots + r^n = \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}. \tag{1.6}$$

Thus, we find that the series of p payments is equivalent to

$$\begin{aligned} \sum_{n=0}^{p-1} (1+i)^{n/p} \frac{i^{(p)}}{p} &= \frac{\big((1+i)^{1/p}\big)^p - 1}{(1+i)^{1/p} - 1} \frac{i^{(p)}}{p} \\ &= \frac{i}{\Big(1 + \frac{i^{(p)}}{p}\Big) - 1} \frac{i^{(p)}}{p} = i \end{aligned}$$

at the end of the year, where in the last line we used that $1+i = (1+i^{(p)}/p)^p$, as stated in (1.5). Thus, a series of p payments of $i^{(p)}/p$ each at times $\frac{1}{p}, \frac{2}{p}, \ldots, 1$ is equivalent to a payment of i at time 1.

This is illustrated in Figure 1.4, which shows four equivalent ways to pay interest on a principal of 1. The top two rows show that a payment of d now is equivalent to a payment of i in a year's time. Indeed, the present value of the latter payment is iv, and in Section 1.4 we found that $iv = d$. The discussion in the preceding paragraph shows that a total payment of $i^{(p)}$ in p equal installments, one at the end of every period of $1/p$ year.

A similar discussion can be had for discounting instead of accumulating interest. A rate of discount of 2% compounded quarterly gives rise to a *nominal rate of discount* of 8% per annum. However, the present value of a payment of C due in one year is $(1 - 0.02)^4 C = 0.9224C$, see (1.4). Thus, the effective rate of discount is $d = 0.0776$ or 7.76%.

Definition 1.5.4. The *nominal rate of discount* compounded $p$thly, denoted $d^{(p)}$, is p times the rate of discount per conversion period.

A similar computation as the one leading to (1.5) yields that

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p. \tag{1.7}$$

In Section 1.4, we concluded that the rate of discount arises in two situations: when computing the present value of a payment and when interest is paid in advance. Indeed, if the principal at the end of a time unit is 1 and interest is paid in advance, then the interest is d by Definition 1.4.4. Analogously to the

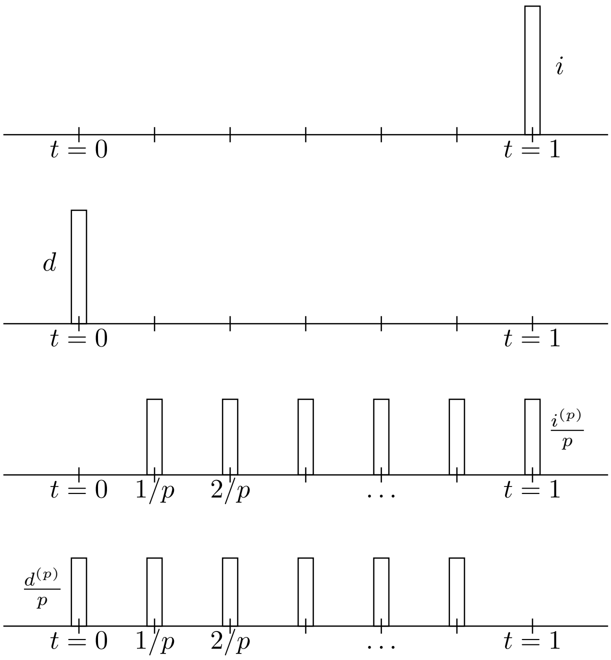

Figure 1.4: The following four situations are equivalent: A payment of i at the end of the year, a payment of d at the beginning of the year, a series of p payments of $i^{(p)}/p$ each at the end of every $1/p$ of a year, and a series of p payments of $d^{(p)}/p$ each at the beginning of every $1/p$ of a year.

discussion under (1.5), it can be shown that if interest is paid $p$thly in advance, then the total interest is $d^{(p)}$. In other words, p payments of $d^{(p)}/p$ each at the beginning of every period of $1/p$ time unit is equivalent to one payment of d at the beginning of the time unit. This follows from the computation

$$\sum_{k=0}^{p-1}(1-d)^{k/p}\frac{d^{(p)}}{p} = \frac{\big((1-d)^{1/p}\big)^p - 1}{(1-d)^{1/p}-1}\frac{d^{(p)}}{p} = \frac{d}{\Big(1-\frac{d^{(p)}}{p}\Big)-1}\frac{d^{(p)}}{p} = d.$$

This is illustrated in the fourth row of Figure 4.1.

Example 1.5.5 (Kellison, p. 22)**.** Compare the following three loans: a loan charging an annual effective rate of 9%, a loan charging $8\frac{3}{4}\%$ compounded quarterly, and a loan charging $8\frac{1}{2}\%$ payable in advance and convertible monthly.

Answer. We will convert all rates to annual effective rates. For the second loan, we use (1.5) with $p = 4$ and $i^{(4)} = 0.0875$ to get $1+i = (1+i^{(p)}/p)^p = 1.0904$, so the annual effective rate is 9.04%. For the third loan, we use (1.7) with $p = 12$ and $d^{(12)} = 0.085$ to get $1 - d = (1 - d^{(p)}/p)^p = 0.91823$. Then, we use (1.1) and (1.3) to deduce that $1 + i = \frac{1}{v} = \frac{1}{1-d} = 1.0890$, so the annual effective rate is 8.90%. Thus, the third loan has the most favourable interest rate.

Consider again the equivalent payments in Figure 1.4. A payment of i at the end of the year is equivalent to a payment of d at the start of the year. However, a payment made later is worth less than a payment made earlier. It follows that i has to be bigger than d. Similarly, the p payments of $i^{(p)}/p$ each in the third row are done before the end of the year, with the exception of the last payment. Thus $i^{(p)}$ has to be smaller than i. Continuing this reasoning, we find that the discount and interest rates are ordered as followed.

$$d < d^{(2)} < d^{(3)} < d^{(4)} < \cdots < i^{(4)} < i^{(3)} < i^{(2)} < i.$$

Exercises

1. Express $i^{(m)}$ in terms of $d^{(\ell)}$, ℓ and m. Hence find $i^{(12)}$ when $d^{(4)} = 0.057847$.

2. (*From the 2010 exam*) How many days does it take for £1450 to accumulate to £1500 under an interest rate of 4% p.a. convertible monthly?

3. (*From the sample exam*) Compute the nominal interest rate per annum payable monthly that is equivalent to the simple interest rate of 7% p.a. over a period of three months.