

Convergence of the Magnus series

Per Christian Moan* Jitse Niesen†

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Abstract

The Magnus series is an infinite series which arises in the study of linear ordinary differential equations. If the series converges, then the matrix exponential of the sum equals the fundamental solution of the differential equation. The question considered in this paper is: When does the series converge? The main result establishes a necessary condition for convergence, which improves on several earlier results.

1 Introduction

The Magnus series is an infinite series which arises in the study of linear ordinary differential equations of the form $y' = A(t)y$, $y(0) = y_0$, where $y(t)$ is a vector and $A(t)$ is a matrix. The fundamental solution is defined by

$$Y' = A(t)Y, \quad Y(0) = I. \quad (1)$$

If $A(t)$ is constant, then the solution of (1) is given by the matrix exponential $Y(t) = e^{At}$. This suggests the ansatz $Y(t) = e^{\Omega(t)}$, where Ω is a matrix function to be determined, for the nonautonomous equation (1). It turns out that $\Omega(t)$ satisfies the differential equation

$$\Omega' = \operatorname{dexp}_{\Omega}^{-1}(A(t)) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \operatorname{ad}_{\Omega}^k(A(t)), \quad (2)$$

where B_k denote the Bernoulli numbers ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, etc.) and $\operatorname{ad}_{\Omega}$ is the adjoint operator, defined by

$$\operatorname{ad}_{\Omega}(A) = [\Omega, A] = \Omega A - A \Omega.$$

*Centre of Mathematics for Applications, University of Oslo, PO Box 1053 Blindern, NO-0316 Oslo, Norway.

†Department of Mathematics, La Trobe University, Melbourne, Victoria 3086, Australia.
email: j.niesen@latrobe.edu.au

Magnus [12] applied Picard iteration on (2) to find an infinite series for Ω :

$$\begin{aligned} \Omega(t) = & \int_0^t A(\tau) d\tau - \frac{1}{2} \int_0^t \int_0^{\tau_1} [A(\tau_2), A(\tau_1)] d\tau_2 d\tau_1 \\ & + \frac{1}{4} \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} [[A(\tau_3), A(\tau_2)], A(\tau_1)] d\tau_3 d\tau_2 d\tau_1 \quad (3) \\ & + \frac{1}{12} \int_0^t \int_0^{\tau_1} \int_0^{\tau_1} [A(\tau_3), [A(\tau_2), A(\tau_1)]] d\tau_3 d\tau_2 d\tau_1 + \dots \end{aligned}$$

This series has since come to be called the *Magnus series*. We can write it as

$$\Omega(t) = \Omega_1(t) + \Omega_2(t) + \Omega_3(t) + \Omega_4(t) + \dots,$$

where the term $\Omega_n(t)$ is a sum of n -fold integrals of $n - 1$ nested commutators. Explicit expressions for the $\Omega_n(t)$ are given by Białynicki-Birula, Mielnik and Plabański [2], Chacon and Fomenko [4], and Iserles and Nørsett [9].

The Magnus series can be used to derive the Baker–Campbell–Hausdorff (BCH) formula for the product of two matrix exponentials. This formula states that $e^{A_1} e^{A_2} = e^B$ with

$$B = A_1 + A_2 + \frac{1}{2}[A_1, A_2] + \frac{1}{12}[A_1, [A_1, A_2]] + \frac{1}{12}[A_2, [A_2, A_1]] + \dots \quad (4)$$

Indeed, if we define the function $A(\cdot)$ by $A(t) = A_1$ for $t \in (0, 1)$ and $A(t) = A_2$ for $t \in (1, 2)$, then B is the solution of (1) at time $t = 2$.

Magnus derived the expansion (3) in the context of quantum mechanics, where $A(t)$ is a skew-Hermitian matrix. Because the Magnus series is constructed from commutators, $\Omega(t)$ is also skew-Hermitian and $e^{\Omega(t)}$ is unitary, just like the fundamental solution of the original differential equation (1). More generally, if $A(t)$ is in some Lie algebra, then $\Omega(t)$ will be in the same Lie algebra and $e^{\Omega(t)}$ will be in the corresponding Lie group.

In the 1990s, Arieh Iserles and Syvert Nørsett were among the group of mathematicians that established the discipline of *Geometric Integration*. This field concerns itself with numerical integrators that respect the geometric structure of differential equations (see, e.g., Hairer, Lubich and Wanner [7] for an introduction). Iserles and Nørsett were looking for a way to integrate (1) such that the numerical solution evolves on the Lie group if the matrix $A(t)$ is in the Lie algebra. Unaware of Magnus' work, they rederived the Magnus series. They realized that after truncating the infinite series and approximating the integrals by quadrature, a method arises that respects the Lie-algebraic structure of the equation (see Iserles and Nørsett [9] and Iserles [8] for details).

Since the Magnus expansion is an infinite series, it is natural to ask whether it converges. Indeed, Magnus himself gave an example in which the series does not converge [12]. The question of convergence is the subject of this paper. The main result (Theorem 3) gives a sufficient condition for convergence.

2 Previous results on the convergence

Due to the complexity of the expansion (3), several proof strategies have been used to derive bounds on the terms Ω_n , which in turn have led to many different convergence estimates. Magnus [12] gave no convergence estimate but stated that for sufficiently small t the series converges. By the known bound

$$\|Y(t) - \mathbf{I}\|_2 \leq \exp\left(\int_0^t \|A(\tau)\|_2 d\tau\right) - 1$$

one easily arrives at the conclusion that the Magnus series converges whenever $\int_0^t \|A(\tau)\|_2 d\tau < \log 2$.¹

However there are several improvements on this bound. In the following, r denotes a number for which the following statement holds:

$$\text{If } \int_0^t \|A(\tau)\|_2 d\tau < r \text{ then the Magnus expansion converges.}$$

In the field of quantum physics there has been some interest in the convergence issue. In 1966, Pechukas and Light [17] consider particular quantum systems and find convergence conditions, although these are not of the general form we are considering here (see also Fernández [5], Klarsfeld and Oteo [11], and Salzman [18]).

In 1976, Karasëv and Mosolova [10] cite a bound $r = \frac{1}{2} \log 2$.² Agračhev and Gamkrelidze, working in the field of control theory, mention in their 1981 paper [1] a result by Vakhrameev stating that $r = 1.08688$.³ In 1998, this bound was rediscovered independently by Blanes, Casas, Oteo and Ros [3] and by Moan [13], using different methods. Vela [22] states in 2003 that this is a sharp result.

In 1991, Chacon and Fomenko [4] found an alternative expression for Ω_n . They used this expression to prove that $r = 0.57745$.

A few years earlier, in 1987, Strichartz [19] had rediscovered the explicit expression for Ω_n found by Bialynicki-Birula et al. [2], stated in terms of Lie brackets. He used this to prove $r = 1$. The same result was found independently by Vinokurov [23] in 1997. Finally, Moan and Oteo [15] derived the bound $r = 2$ (the best result at the moment) by similar techniques, except that they avoided the use of commutators as they seemed to introduce unnecessary complications in the convergence bound.

Moan [14] found a condition for existence of a real logarithm for real A with $r = \pi$. It was however unclear if this condition is sufficient for convergence of the series expansion. Theorem 3 answers this question affirmatively.

¹V.S. Varadarajan [21, p. 119] gives implicitly the same result for the BCH formula.

²In 1977, Suzuki [20] derives this bound for BCH.

³Newman et al. [16] establish the same bound for the BCH formula in 1988.

3 The existence of a real logarithm

The fundamental solution $Y(t)$ of the differential equation (1) is an invertible matrix. A theorem by Gantmacher [6] states that every invertible matrix has a logarithm, and hence there exists an $\Omega(t)$ such that $e^{\Omega(t)} = Y(t)$. However, the logarithm may fail to be real, even if $Y(t)$ is real. For example, the matrix

$$\begin{bmatrix} -1 & 0 \\ b & -1 \end{bmatrix}, \quad b \in \mathbb{R} \setminus \{0\}, \quad (5)$$

does not have a real logarithm. Since all the terms in the Magnus expansion (3) are real if the original differential equation (1) is real, we conclude that the infinite series cannot converge to a logarithm of $Y(t)$ if $Y(t)$ has no real logarithm. The question we are studying in this section is therefore: Does $Y(t)$ have a real logarithm?

The following lemma is easily proved, for instance by factoring $\Phi = VJV^{-1}$ where J is in Jordan form.

Lemma 1. *Suppose that the invertible matrix Φ has no negative eigenvalues, that is, suppose that the eigenvalues of Φ are contained in $\mathbb{C} \setminus (\infty, 0]$. Then*

$$\log \Phi = (\Phi - I) \int_0^\infty \frac{1}{1 + \mu} (\mu I + \Phi)^{-1} d\mu. \quad (6)$$

It follows that if $Y(t)$ is real and has no negative eigenvalues, then the logarithm of $Y(t)$ is real as well. The next result (which can be found in [14]) gives an easy condition under which $Y(t)$ has no negative eigenvalues and hence a real logarithm.

Theorem 2. *Let $A(t)$ be a real integrable matrix, and let $Y(t)$ denote the solution of $Y' = A(t)Y$, $Y(0) = I$. If $\int_0^t \|A(\tau)\|_2 d\tau < \pi$, then $Y(t)$ has a real logarithm.*

Proof. Choose an arbitrary vector y_0 , and consider the vector $y(t) = Y(t)y_0$ satisfying $y' = A(t)y$ and $y(0) = y_0$. Let $\hat{y}(t) = y(t)/\|y(t)\|_2$ denote the unit vector in the direction $y(t)$. Then

$$y'(t) = \left(\frac{d}{dt}\|y(t)\|_2\right)\hat{y}(t) + \|y(t)\|_2 \hat{y}'(t).$$

Since $\hat{y}(t)$ and $\hat{y}'(t)$ are orthogonal, it follows that

$$\|y'(t)\|_2^2 = \left(\frac{d}{dt}\|y(t)\|_2\right)^2 + \|\hat{y}'(t)\|_2^2 \|y(t)\|_2^2.$$

Therefore, we have

$$\|\hat{y}'(t)\|_2 \|y(t)\|_2 \leq \|y'(t)\|_2 \leq \|A(t)\|_2 \|y(t)\|_2,$$

and thus $\|\hat{y}'(t)\|_2 \leq \|A(t)\|_2$. Integrating this inequality, we find the bound

$$\int_0^t \|\hat{y}'(\tau)\|_2 d\tau \leq \int_0^t \|A(\tau)\|_2 d\tau.$$

The left-hand side is the length of the curve swept out by the unit vector $\hat{y}'(\tau)$ when $\tau \in [0, t]$. Therefore, the angle between $y(t)$ and $y(0)$ is smaller than π if $\int_0^t \|A(\tau)\|_2 d\tau < \pi$. Since $y(t) = Y(t)y(0)$, this implies that $Y(t)$ has no negative eigenvalues, and hence it has a real logarithm. \square

Example 2 in the forthcoming Section 5 shows that the constant π in Theorem 2 is sharp.

4 Proof of convergence

Theorem 2 gives a condition for $Y(t)$ to have a real logarithm. The following theorem states that under the same condition, the Magnus series converges to this logarithm.

Theorem 3. *If $\int_0^t \|A(\tau)\| d\tau < \pi$, then the Magnus series (3) converges and its sum $\Omega(t)$ satisfies $e^{\Omega(t)} = Y(t)$.*

Proof. We write the Magnus series as $\Omega(t) = \sum_{n=1}^{\infty} \Omega_n(t)$, where the term $\Omega_n(t)$ is a sum of n -fold integrals of $n - 1$ nested commutators. If we now introduce a new parameter κ and we replace $A(t)$ by $\kappa A(t)$, then the Magnus series becomes $\Omega(t; \kappa) = \sum_{n=1}^{\infty} \kappa^n \Omega_n(t)$. The idea is to fix t and consider the function f defined by $f(\kappa) = \log Y(t; \kappa)$ where $Y(t; \kappa)$ denotes the solution of $Y' = \kappa A(t)Y$, $Y(0) = I$ with $\kappa \in \mathbb{C}$. We will show that f is analytic and that the Magnus series is the Taylor series of this function around $\kappa = 0$. We can then use the standard result from the theory of complex functions which states that the Taylor series of a function converges in a disc if the function is analytic in that disc.

Set $\gamma = \int_0^t \|A(\tau)\| d\tau$. As stated in Section 2, it is easy to show that if $|\kappa| < \frac{1}{\gamma} \log 2$, the Magnus series converges and its sum $\Omega(t; \kappa)$ satisfies $e^{\Omega(t; \kappa)} = Y(t; \kappa)$. Hence, the power series $\Omega(t; \kappa)$ coincides with $f(\kappa)$ for $|\kappa| < \frac{1}{\gamma} \log 2$, and the Magnus series is the Taylor series expansion of f around $\kappa = 0$.

We say that a matrix-valued function (like f) is analytic if all the matrix entries are analytic functions. We now want to prove that f is analytic in the disc with radius $\frac{\pi}{\gamma}$. Firstly, the fundamental matrix $Y(t; \kappa)$ is analytic as a function of κ . The proof of Theorem 2 shows that $Y(t; \kappa)$ has no eigenvalues in $(-\infty, 0]$ if $|\kappa| < \frac{\pi}{\gamma}$, so the logarithm is given by (6). Hence, the derivative of f is given by

$$\begin{aligned} f'(\kappa) &= \frac{\partial}{\partial \kappa} Y(t; \kappa) \int_0^{\infty} \frac{1}{\mu + 1} (\mu I + Y(t; \kappa))^{-1} d\mu \\ &\quad - Y(t; \kappa) \int_0^{\infty} \frac{1}{\mu + 1} (\mu I + Y(t; \kappa))^{-1} \frac{\partial}{\partial \kappa} Y(t; \kappa) (\mu I + Y(t; \kappa))^{-1} d\mu. \end{aligned}$$

The right-hand side is well-defined, thus proving that the function f is analytic in the disc with radius $\frac{\pi}{\gamma}$. Hence, we can expand the entries of the matrix $f(\kappa)$ in a power series, and this series will converge provided that $|\kappa| < \frac{\pi}{\gamma}$. But this power series is precisely the Magnus series. \square

5 Examples

In this section, we study some examples to investigate the connections between the condition $\int_0^t \|A(\tau)\|_2 d\tau < \pi$, the eigenvalues of the fundamental solution, the existence of a real logarithm, and the convergence of the Magnus series.

Example 1. The following simple example suffices to show that the condition $\int_0^t \|A(\tau)\|_2 d\tau < \pi$ is not a necessary condition. Consider the equation $Y' = AY$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a constant matrix. The fundamental matrix is

$$Y(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

When $t = \pi$, this matrix has a double eigenvalue at -1 . Furthermore, we have $\int_0^\pi \|A(t)\| dt = \pi$. Nevertheless, $Y(t)$ has a real logarithm for all t , including $t = \pi$, because $Y(t) = e^{tA}$. The Magnus series converges for all t : the terms in the series are $\Omega_1 = A$ and $\Omega_k = 0$ for $k > 1$. The critical point here seems to be that the double eigenvalue at -1 is not defective (meaning that its algebraic multiplicity equals its geometric multiplicity).

Example 2. This example, taken from Moan [14], shows that the condition $\int_0^t \|A(\tau)\|_2 d\tau < \pi$ is sharp, in the sense that the constant π on the right-hand side cannot be replaced by a bigger constant. Consider the equation $Y' = A(t)Y$ with

$$A(t) = \frac{1}{2} \begin{bmatrix} \sin 2t & -1 - \cos 2t \\ 1 - \cos 2t & -\sin 2t \end{bmatrix}.$$

A simple computation shows that $\int_0^\pi \|A(\tau)\|_2 d\tau = \pi$, and that the solution of the differential equation is given by

$$Y(t) = \begin{bmatrix} t \sin t + \cos t & -\sin t \\ \sin t - t \cos t & \cos t \end{bmatrix}.$$

Hence $Y(\pi) = \begin{bmatrix} -1 & 0 \\ \pi & -1 \end{bmatrix}$, which is of the form (5), and therefore $Y(\pi)$ does not have a real logarithm. This implies that the Magnus series diverges for $t = \pi$.

Example 3. This example shows that the Magnus series may diverge even though the fundamental matrix $Y(t)$ has a real logarithm. Take

$$A(t) = \begin{bmatrix} 2 & t \\ 0 & -1 \end{bmatrix}.$$

The Magnus series is

$$\Omega(t) = \begin{bmatrix} 2t & \frac{1}{2}t^2 \\ 0 & t \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{4}t^3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{80}t^5 \\ 0 & 0 \end{bmatrix} + \dots$$

On the other hand, the solution of $Y' = A(t)Y$ is

$$Y(t) = \begin{bmatrix} e^{2t} & \frac{1}{9}e^{2t} - \left(\frac{1}{9} + \frac{1}{3}t\right)e^{-t} \\ 0 & e^{-t} \end{bmatrix}.$$

The logarithm of this solution is

$$\log Y(t) = \begin{bmatrix} 2t & f(t) \\ 0 & -t \end{bmatrix} \quad \text{where} \quad f(t) = \frac{te^{2t} - (t + 3t^2)e^{-t}}{3(e^{-2t} - e^t)}.$$

So, the Magnus series is the Taylor series expansion of $\log Y(t)$ around $t = 0$ (this is not true in general). The function f has a pole at $t = \frac{2}{3}\pi i$, thus the Magnus series converges up to $t = \frac{2}{3}\pi$.

However, $Y(\frac{2}{3}\pi)$ has eigenvalues at $e^{2\pi/3}$ and $e^{4\pi/3}$, so $Y(\frac{2}{3}\pi)$ has a real logarithm. Nevertheless, the Magnus series diverges. The reason is probably as follows. As in the proof of Theorem 3, let $Y(t; \kappa)$ denote the fundamental solution of $Y' = \kappa A(t)Y$. The eigenvalues of $Y(t; \kappa)$ are $e^{\kappa t}$ and $e^{-2\kappa t}$. When $\kappa = i$, these eigenvalues move in a circle around the origin and collide when $t = \frac{2}{3}\pi$. This collision causes the Magnus series to diverge at $t = \frac{2}{3}\pi$.

Example 4. The preceding examples all have some structure which allowed us to determine where the Magnus series starts to diverge. We close this section with a more-or-less randomly chosen example.

Consider the equation $Y' = A(t)Y$, where the matrix A is defined by

$$A(t) = \begin{bmatrix} -t & 3t & 0 & 3t^2 + t + 2 \\ t^2 - t & -3 & t^2 + 2t + 3 & 0 \\ 3 & 0 & t^2 - 2t & -t^2 - 3 \\ t^2 - t + 3 & 2t^2 - 3t & -3t - 2 & -t + 2 \end{bmatrix}. \quad (7)$$

Since A is polynomial in t , all the commutators in the Magnus series are also polynomials and hence the integrals can be computed exactly. The first two terms of the series are given by

$$\Omega_1 = \begin{bmatrix} -\frac{1}{2}t^2 & \frac{3}{2}t^2 & 0 & -t^3 + \frac{1}{2}t^2 + 2t \\ \frac{1}{3}t^3 - \frac{1}{2}t^2 & -3t & \frac{1}{3}t^3 + t^2 + 3t & 0 \\ 3t & 0 & \frac{1}{3}t^3 - t^2 & -\frac{1}{3}t^3 - 3t \\ \frac{1}{3}t^3 - \frac{1}{2}t^2 + 3t & \frac{2}{3}t^3 - \frac{3}{2}t^2 & -\frac{3}{2}t^2 - 2t & -\frac{1}{2}t^2 + 2t \end{bmatrix}$$

$$\Omega_2 = \begin{bmatrix} -\frac{1}{60}t^5 - \frac{11}{12}t^4 + \frac{5}{12}t^3 & \frac{7}{60}t^5 - \frac{1}{3}t^4 - \frac{1}{4}t^3 & \frac{1}{10}t^5 + \frac{1}{2}t^4 + \frac{13}{12}t^3 & -\frac{1}{2}t^4 + \frac{1}{6}t^3 \\ -\frac{1}{60}t^5 + \frac{1}{2}t^4 + \frac{1}{4}t^3 & \frac{1}{20}t^5 & -\frac{1}{15}t^5 + t^3 & \frac{1}{6}t^4 - \frac{2}{3}t^3 \\ \frac{1}{60}t^5 + \frac{1}{4}t^4 - \frac{1}{2}t^3 & \frac{1}{20}t^5 + \frac{1}{2}t^4 - \frac{3}{2}t^3 & \frac{1}{20}t^5 + \frac{1}{6}t^4 - \frac{3}{4}t^3 & -\frac{1}{60}t^5 + \frac{1}{3}t^4 \\ \frac{1}{60}t^5 - \frac{1}{6}t^4 - \frac{7}{12}t^3 & \frac{1}{12}t^5 - \frac{5}{6}t^4 + \frac{1}{2}t^3 & \frac{1}{6}t^5 + \frac{2}{3}t^4 - \frac{5}{12}t^3 & -\frac{1}{12}t^5 + \frac{3}{4}t^4 + \frac{1}{3}t^3 \end{bmatrix}$$

We computed the first thirty terms of the Magnus series with the help of a computer algebra system. The recursive formulas given by Blanes, Casas, Oteo and Ros [3] proved to be useful for this purpose; other formulations require a very long time to evaluate. In Figure 1, we plot the sums of the first fifteen, twenty, twenty-five and thirty terms of the Magnus series (3). It seems that the series starts to diverge between $t = 0.7$ and $t = 0.8$, though it is of course not possible to pinpoint the location precisely.

Figure 2 shows how the eigenvalues of the fundamental solution $Y(t)$ move around as t increases from 0 to 1. The eigenvalues do not collide in the left

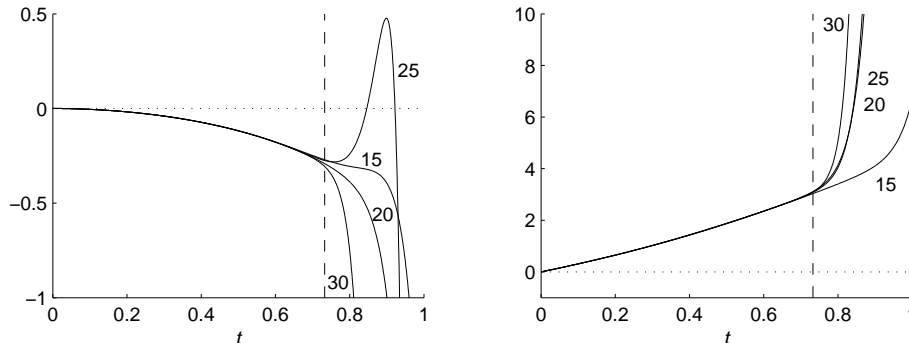


Figure 1: The left plot shows the partial sums of the first 15, 20, 25 and 30 terms of the (1,1) element in the Magnus expansion (3), when A is as given in (7). The dash line indicates $t = 0.733$. The right panel shows the same for the (2,3) element.

half-plane, so $Y(t)$ has a real logarithm for $t \in [0, 1]$. However, as the right panel in the figure shows, the eigenvalues of $Y(t; e^{i\alpha_*})$ with $\alpha_* = 1.805\dots$ do collide. The collision takes place at $t_* = 0.733\dots$ and $\lambda_* \approx -0.485 + 0.0249i$. The value of t_* is shown by the dash line in Figure 1. It seems plausible that the Magnus series starts to diverge around this point.

Incidentally, the condition $\int_0^t \|A(\tau)\|_2 d\tau < \pi$ is satisfied for $t < 0.56\dots$. However, the Magnus series continues to converge for slightly larger values of t , showing again that this condition is not necessary for convergence. At $t = t_*$, the integral is approximately 4.36.

6 Conclusion

The main result of the paper is Theorem 3, which states that the Magnus series converges if $\int_0^t \|A(\tau)\|_2 d\tau < \pi$. This condition is in the same form as various earlier results mentioned in Section 2. Example 2 shows that our result is sharp in the sense that the constant π is the largest number for which the result holds. However, the other examples show that the condition is not necessary for convergence.

A necessary and sufficient condition for convergence is given in the following conjecture, which is motivated by both the proof and the examples in this paper.

Conjecture 4. *Let $Y(t; \kappa)$ denote the solution of $Y' = \kappa A(t)Y$, $Y(0) = I$. Denote the eigenvalues of $Y(t; \kappa)$ by $\lambda_n(t; \kappa)$, where the eigenvalues are to be ordered so that λ_n is a continuous function of t . Let t_* be the smallest $t > 0$ for which there exists a $\kappa \in \mathbb{C}$ with $|\kappa| = 1$ such that there is a multiple eigenvalue, say $\lambda_i(t, \kappa) = \lambda_j(t, \kappa)$ with $i \neq j$, for which the geometric multiplicity is smaller*

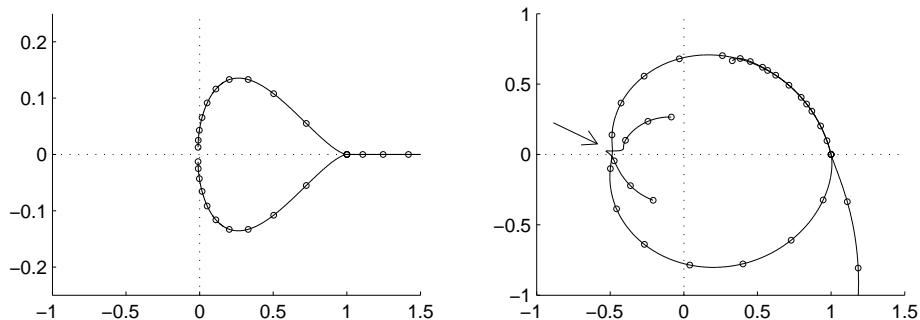


Figure 2: The left plot shows the location in the complex plane of the eigenvalues of the fundamental solution of $Y' = A(t)Y$ with A as in (7), for $t \in [0, 1]$. All four eigenvalues are at 1 when $t = 0$. As t increases, two eigenvalues move to the right, while the other two form a complex pair. The small circles show the locations at $t = 0, 0.1, 0.2, \dots, 1$. It is clear that no eigenvalues collide on the negative real axis for $t \in [0, 1]$.

The right plot shows the eigenvalues for $Y' = e^{i\alpha_*} A(t)Y$ with $\alpha_* \approx 1.805$. Two eigenvalues encircle the origin and collide near the arrow.

than the algebraic multiplicity, and the loop

$$\{\lambda_i(\tau, \kappa) \mid \tau \in [0, t]\} \cup \{\lambda_j(\tau, \kappa) \mid \tau \in [0, t]\}$$

encircles the origin. Then the Magnus series converges if and only if $t < t_*$.

This conjecture suggests two tasks. The first is obviously whether we can find a proof for this conjecture. However, even if the conjecture is true, it is difficult to apply in practice because the condition is not easy to check. Therefore, the second task is to find a more practical condition for convergence.

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