Slowly Passing through Resonance Strongly Depends on Noise

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The response of a periodically driven nonlinear oscillator slowly passing through resonance has long been suspected but never been shown to be dramatically affected by noise. The slow passage in the presence of noise is studied analytically and experimentally using a periodically modulated laser. Hysteresis in the magnitude of the oscillations of the laser output intensity is produced by sweeping back and forth the modulation frequency near resonance with the laser relaxation oscillations. The area of the hysteresis loop at first increases with the amplitude of the modulation but then exhibits a plateau determined by the noise level.

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Passage through resonance as the forcing frequency slowly approaches the natural frequency of a nonlinear oscillator is a classic problem with practical impacts in mechanical and electrical systems [1]. A slowly varying parameter passing accidentally through resonance produces undesired large amplitude oscillations, possibly causing serious damage. For mechanical systems, the problem has been investigated since the beginning of the century [2,3]. Mitropolsky [4] called these rapid oscillations that are slowly varying in amplitude and frequency “nonstationary oscillations.” Although a slow passage through resonance is generally an undesired phenomenon for mechanical systems, resonance may be desired in other systems. This is typically the case for lasers or optically bistable devices because the output intensity is maximum at or near resonance. Of particular interest is the effect of small amplitude noise which is always present in experiments. Noise has long been suspected to play a determinant role in slow passage experiments [7–11] but its effects have never been investigated experimentally. Lasers are ideal systems for combined theoretical and experimental studies of slow passage problems and have been used to study simple steady state bifurcation problems [12,13]. In this Letter, we investigate the slow transition through resonance using a periodically modulated yttrium-aluminum-garnet (YAG) laser. The periodically modulated YAG laser, like many other practical lasers (CO2 and semiconductor lasers), exhibits a bistable response curve for the laser output intensity as a function of the modulation frequency. By increasing and decreasing the forcing frequency, sudden jump transitions between low and high intensity branches occur at precise frequencies that we may determine either experimentally or numerically from model equations. By comparing out observations for these points, we discover a significant deviation between experiments and deterministic theory for one of the two transitions. We also observed numerically that noise has drastically different effects on the upper and lower branches of the nonlinear resonance curve. More precisely, we note that sweeping the modulation frequency in the presence of noise leads to an anticipation rather than a delay of the switching transition from the upper to the lower branch. Modeling such experiments usually takes advantage of a simple normal form equation, as in [13], but is then restricted to qualitative comparisons. In the present paper, we consider the laser equations and obtain quantitative agreements between experimental and analytical results.

Specifically, we consider a single mode YAG:Nd3+ laser subject to a periodically modulated pump. The response of the laser is well described by rate equations for the electrical field E and the inversion of population F, given by

\[ E' = (F - 1)E, \]

\[ F' = \gamma [P(t) - F - F|E|^2]. \]

In these equations, time \( t \equiv \kappa t' \) has been scaled by the cavity constant and \( \gamma \equiv \gamma_0/\kappa \) is estimated as \( \gamma \sim 2 \times 10^{-6} \) for all of our experiments. The pump parameter \( P(t) = A + \Delta A \cos(\omega t) \) is periodic with respect to time \( t \) but has a slowly varying frequency of the form \( \omega = \omega_0 + c \nu t \), where \( \nu \ll 1 \) is the sweeping rate and \( c = \pm 1 \) (forward or backward passage). The initial frequency \( \omega_0 \) is close to the laser relaxation oscillation frequency defined by \( \omega_R \equiv \sqrt{2 \gamma (A - 1)} \). It is mathematically convenient to introduce the scaled time \( T \) and the new dependent variables \( x \) and \( y \), defined as

\[ T = \omega_R t, \quad F = 1 + \frac{\omega_R}{2} x, \]

\[ E^2 = (A - 1)(1 + y). \]
Time is now measured in units of the relaxation oscillation period $T_R = 2\pi/\omega_R$, and the new variables $x$ and $y$ are deviations of $F$ and $E^2$ from the nonzero intensity steady state $(F, I) = (1, A - 1)$. Inserting (3) into Eqs. (1) and (2) gives
\begin{equation}
x' = -y + \delta \cos(fT) - \epsilon x[1 + (A - 1)(1 + y)],
\end{equation}
\begin{equation}
y' = x(1 + y).
\end{equation}
Note that the number of parameters has been reduced from 4 to 3. They are defined by
\begin{equation}
\delta = \frac{\Delta A}{A - 1}, \quad \epsilon = \frac{\gamma}{\sqrt{2(a - 1)}}, \quad f = \frac{\omega(T)}{\omega_R}.
\end{equation}
\(\delta \sim 10^{-3} - 10^{-1}\) is the normalized amplitude of the modulation. Its small value limits the bifurcation diagram to the primary resonances. \(\epsilon\) is proportional to \(\sqrt{\gamma} \sim 10^{-3}\) and measures the natural damping of the laser relaxation oscillations. The normalized slowly varying frequency is described by \(f = f_0 + c \eta T\), where \(f_0 \equiv \frac{\omega_0}{\omega_0^2} \approx 1\) and \(\eta \equiv \frac{\nu}{\omega_0^2} \approx 10^{-5} - 10^{-2}\).

In the experiments, the laser is monomode and is pumped by a diode laser emitting up to 500 mW with a wavelength $\lambda = 808$ nm. An intracavity Brewster window selects a linear polarization state for the oscillations. The cavity optical length is 3.8 cm, giving a photon lifetime $\kappa^{-1}$ of 28 ns. Pump modulation is achieved by directly modulating the diode laser near the relaxation frequency $\omega_R$ which is measured as 250 kHz for $A = 2$. The amplitude of the laser output is monitored by an intensity detector followed by envelope detection as the frequency of the modulation is swept across resonance at rates in the $10^4$ to $10^6$ Hz/s range.

Before we investigate the effects of a slowly varying frequency, we determine the size of the hysteresis cycle for fixed values of the frequency ($f = f_0$). In Fig. 1, we show the experimentally obtained amplitude of the periodic solutions as a function of the frequency $f$. We begin with the case where $\delta$, $\epsilon$, and $f$ are small and the oscillations of the laser intensity are then nearly harmonic in time. We determine a periodic solution of Eqs. (4) and (5) using a multiple-scale perturbation method [1,5,6]. We find that a bistable hysteresis curve appears as soon as the modulation surpasses a critical amplitude given by
\begin{equation}
\delta_c = 4.3 \times (\epsilon A)^{3/2}.
\end{equation}
If $\delta > \delta_c$, the hysteresis curve is bounded by two limit points near which jump transitions between branches are observed. The up-switching point $f_{up}$ corresponds to the jump transition between the low and high intensity branches. Our analysis shows that its deviation from the linear resonance frequency $f = 1$ is given by
\begin{equation}
f_{up} - 1 = -0.4 \times \delta^{2/3}.
\end{equation}
Similarly, we find the down-switching point $f_{down}$ corresponding to the transition from the high to the low intensity branches as
\begin{equation}
f_{down} - 1 = -0.04 \times \left(\frac{\delta}{\epsilon A}\right)^2.
\end{equation}

The expression (9) is valid provided that the amplitude of the oscillations at the high intensity branch remains small. We now consider large amplitude oscillations ($\delta$ is no longer assumed to be small). We use a different method appropriate for large amplitude oscillations [14] and obtain the approximation
\begin{equation}
f_{down} - 1 = \frac{A \pi^2}{3} \epsilon \delta^{-1} - 1
\end{equation}
for large $\delta/\epsilon$, which means that $|f_{down} - 1| \rightarrow 1$ as $\delta \rightarrow \infty$. Figure 2 represents $1 - f_{up}$ and $1 - f_{down}$ as a function of $\delta$. The solid lines represent the expressions (8) and (9) while dots correspond to experiments. We observe that the experimental data quantitatively follow the $\delta^{2/3}$ law for $1 - f_{up}$ but deviate from the $\delta^2$ law for $1 - f_{down}$ as soon as $\delta$ increases. This results from the fact that the amplitude of the oscillations remains small for the low intensity branch but increases dramatically for the high intensity branch. Furthermore, we note that the experimental data for the jump-down transition saturate at a plateau. However, this plateau is not located at $1 - f_{down} = 1$, as suggested by (10), but is located at a much lower value. This implies that noise—always present in the experiment—may contribute significantly to an earlier jump transition (anticipation). The effect of noise on the high amplitude intensity branch can be suspected by noting that the laser oscillations consist of short
Two main sources of noise appear in our laser experiments. Spontaneous emission primarily affects the electrical field and forces the laser to jump down to low amplitude oscillations (destabilizing effect). On the other hand, noise in the pump acts parametrically and will shift the onset of the hysteresis domain to large δ (stabilizing effect). We model these two sources of noise by modifying the equation for the field and the expression of the pump

\[ E' = (F - 1)E + \sigma_E \dot{\gamma}, \]

\[ P(t) = A + \Delta A \cos(\omega t) + \sigma_P \dot{\delta}, \]

where \( \dot{\gamma} \) and \( \dot{\delta} \) denote white noise source terms. The effect of noise on the field can be investigated analytically by taking advantage of the fact that the laser oscillations consist of short intensity pulses separated by long periods of almost zero intensity. During these long intervals of period close to \( T_R = 2\pi/\omega_R = O(\gamma^{-1/2}) \), \( E \sim 0 \), and we may solve Eq. (2) for \( F \) and then Eq. (11) for \( E \). From its solution, we determine the point \( f_{\text{down}} \) as

\[ \langle 1 - f_{\text{down}} \rangle \sim \frac{\pi}{2\sqrt{-\ln(\sigma_E)}}. \]

In the \( 1 - f \) vs δ diagram, the expression (13) corresponds to a constant controlled by the level of noise \( \sigma_E \). Figure 3 shows the two limits of the hysteresis cycle obtained numerically by simulating the stochastic differential Eqs. (11) and (12). A logarithmic dependence as illustrated by (13) explains how small amplitude noise may have a large effect.

By simulating separately the effect of noise on the field and noise on the pump, we may differentiate their specific effects. Noise on the field leads to an anticipation of the jump-down transition, sensitively dependent on the noise level [see (13)]. By contrast, noise on the pump shifts the hysteresis cycle to higher δ. The change of δc is expected since δc, given by (7), is a function of the pump parameter \( A \). We may explore this particular effect experimentally by progressively adding noise to the pump (Fig. 4). As the level of noise is increased, the jump-up limit does not change but the jump-down limit is progressively shifted to higher δ. Compared to the effect of spontaneous emission, we note that the system is much less sensitive to the level of noise in the pump parameter: relatively large amounts of noise are needed to produce a significant effect.

Many experimental studies of bifurcation problems are done by slowly changing a key control parameter. The naive expectation that the system will adiabatically follow its stable states is often confounded by the observation of earlier or delayed bifurcation or jump transitions which depend on small amplitude noise. In this Letter, we have investigated the effect of noise on two coexisting branches of time-periodic states in a periodically modulated YAG laser. We showed that the effect of noise is small for the low amplitude branch but is dramatic on the high amplitude branch. This results from the fact that the
FIG. 4. Experimental measurements of the limit points \( \Delta f_{\text{up}} \) (\( \bigcirc \)) and \( \Delta f_{\text{down}} \) (\( \bigtriangleup \)) versus \( \Delta A \) for various amplitudes of the pump noise. Added rms noise modulation in 10 MHz bandwidth \( \delta_{\text{rms}} \) (a) 0, (b) 0.12, (c) 0.17, (d) 0.24.

Large amplitude oscillations that characterize the high amplitude branch of the hysteresis cycle become sensitive to noise during long intervals separating successive intensity pulses. This property will be shared by most lasers used in laboratories (solid state, \( \text{CO}_2 \), and fiber lasers) because they all exhibit a small \( \gamma \). However, the effect of noise can be reduced for systems showing nearly harmonic relaxation oscillations, e.g., close to Hopf bifurcation points.

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