Timescales and lengthscales from noise

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Noise and nonlinearity can combine to produce surprising and counterintuitive effects. One such is noise-controlled dynamics, where very small amplitude additive noise dramatically changes the dynamics. The phenomenon has been reported in a one-dimensional map [16], and in ordinary differential equations describing the resonant interaction of wave modes [4, 14], the intermittent destabilisation of convection by shear [3, 7], pulsating laser oscillations [8] and plane Poiseuille flow [5]. Noise-controlled dynamics has also been found in a set of stochastic *partial* differential equations describing the shear instability of thermohaline convection [18, 14]. Experimental results exhibit the phenomenon in a YAG:ND^{3+} laser subject to a periodically modulated pump: oscillations of the laser output intensity are sensitive to noise during long intervals separating intensity pulses [2]. We shall describe the latter example in a little more detail below, but first we shall begin with a fairly simple and general set of equations.

In many systems exhibiting invariance under translation and reflection, it is useful to understand the parameter space with reference to the Hopf-Hopf codimension-2 normal form. After rescaling, the differential equations for the coupled evolution of two (positive) mode amplitudes are

$$\begin{aligned} \dot{x} &= \mu x(1-\delta x^2) - xr^2 \\ \dot{r} &= r(x^2 - 1 + \gamma r^2). \end{aligned} \tag{1}$$

Let us take $0 < \mu \ll 1$ fixed and describe the dynamics as a function of δ and γ. There are fixed points of (1) at the origin, at $(x,r) = (0, \gamma^{-\frac{1}{2}})$ and at $(\delta^{-\frac{1}{2}}, 0)$. We restrict ourselves to $\delta < 1$ and $\gamma < \frac{1}{\mu}$, when there is also a fixed point at $\mathcal{P} = (x_f, r_f)$ where

$$x_f^2 = \frac{1-\gamma\mu}{1-\delta\gamma\mu} \qquad r_f^2 = \mu(1-\delta x^2). \tag{2}$$

The fixed point $\mathcal{P}$ is stable for $1 < \delta < \delta_{\rm H}$ where

$$\delta_{\rm H} = \frac{\gamma}{1+\gamma-\gamma\mu}. \tag{3}$$

At $\delta = \delta_{\rm H}$, it loses stability in a Hopf bifurcation. (In the original complex equations, the Hopf bifurcation at $\mathcal{P}$ would be described as a *secondary* Hopf bifurcation, giving birth to torus.) At at the same parameter values, there is a heteroclinic bifurcation, in which the stable manifold, $\mathcal{M}$, of the fixed point at $(0, \gamma^{-\frac{1}{2}})$ connects with the fixed point at $(\delta^{-\frac{1}{2}}, 0)$. See Figure 1. Consequently, there are no periodic orbits for $\delta < \delta_{\rm H}$; all trajectories that start with x, $r > 0$ escape to infinity. When very-small-amplitude noise is added to the differential equations (1), however, noise-controlled dynamics are found in a range $\delta_n < \delta < \delta_{\rm H}$, where δ_n decreases as $\mu|\ln\epsilon|$ is decreased. See Figure 1. Noise-controlled dynamics consists of orbits whose size and period vary from cycle to cycle, with a well defined mean and distribution about the mean [14]. Each orbit consists of a long "slow phase" where the trajectory is close to an invariant manifold, followed by a short fast phase.

Given the existence of a small(ish) parameter μ and the occurrence of a degenerate bifurcation in the normal forms, it is perhaps not surprising that noise can have a large effect. What is counter-intuitive is that the noise acts to *simplify* the dynamics, selecting a particular (noisily) periodic solution. These orbits persist when the parameters differ by $\mathcal{O}(1)$ from their values at the degenerate bifurcation, and are produced by a noise intensity ϵ that is much smaller than any other small parameters in the equations. The sensitivity to noise is due not to the proximity of a *local* bifurcation or to any special properties of the noise, but to the existence of a slow invariant manifold.

The dynamics of a slow phase is analogous to that of a slow passage through a pitchfork bifurcation in the presence of additive noise with amplitude ϵ. There, time-dependence of the

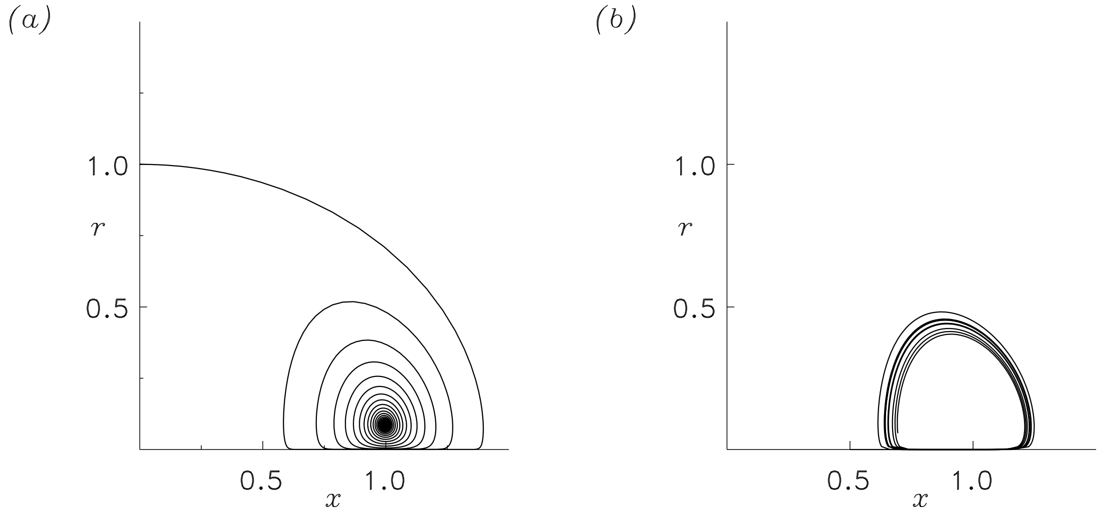

Figure 1: (a) The unstable manifold of the fixed point $\mathcal{P}$, *i.e.* the locus of all points (x, r) such that a trajectory that passes through (x, r) tends to $(0, \gamma^{-\frac{1}{2}})$ as $t \to \infty$. When noise is sufficiently small, trajectories escape to infinity. (b) A trajectory with noise amplitude $\epsilon = 10^{-10}$. Each loop consists of a slow phase close to $r = 0$ with x increasing, followed by a rapid decrease of x. ($\gamma = 1.0$, $\delta = 0.25$ and $\mu = 0.01$.) Trajectories perform noisily-periodic oscillations.

critical parameter produces a delay that is sensitive to noise. The characteristic delay in such a dynamic bifurcation is $\sqrt{2\mu|\ln\epsilon|}$, where μ is the rate of change of the parameter [14, 11, 15, 20, 19]. Analogously, the bifurcation diagram of an ODE systems exhibiting noise-controlled dynamics is conveniently drawn in terms of the parameter $\mu|\ln\epsilon|$. In a spatially-extended system of size L, described by coupled stochastic partial differential equations, the appropriate parameter is [14]

$$\alpha = \frac{L^2}{4\pi^2}\mu|\ln\epsilon|.$$

We now return to the experimental results, obtained using a single mode YAG:Nd^{3+} laser subject to a periodically modulated pump. The response of the laser is well described by rate equations for the electrical field E and the inversion of population F and are given by

$$E' = (F-1)E, \tag{4}$$

$$F' = \gamma[P(t) - F - F|E|^2], \tag{5}$$

where $\gamma \sim 2 \times 10^{-6}$ and $P(t) = A + \Delta A\cos(\omega t)$. Analysis of these equations (and experiments on many practical lasers) reveals a bistable response curve for the laser output intensity as a function of the modulation frequency. By increasing and decreasing the forcing frequency, sudden jump transitions between low and large intensity branches occur at precise frequencies that may be determined either experimentally or numerically from the model equations. The area of the hysteresis loop at first increases with the amplitude of the modulation but then exhibits a plateau whose height is a function of the square root of the logarithm of the noise amplitude in the electric field. Thus a change of $\mathcal{O}(1)$ in the area of the hysteresis loop is produced by tiny amounts of noise.

Finally, let us consider the analogue of a dynamic bifurcation in a spatially-dependent system, or a dynamic transition. To be specific, let us consider an SPDE in the following dimensionless form:

$$\mathrm{d}\mathbf{Y}_t(x) = (g(t)\mathbf{Y}_t(x) - \mathbf{Y}_t(x)^3 + \frac{\partial^2}{\partial x^2}\mathbf{Y}_t(x))\mathrm{d}t + \epsilon\,\mathrm{d}\mathbf{W}_t(x). \tag{6}$$

Here $\mathbf{Y} : [0, L] \times [-\frac{1}{\mu}, \frac{1}{\mu}] \times \Omega \to \mathcal{R}$, Ω is a probability space and $\mathbf{W}_t(x)$ is the Brownian sheet. The equations are solved as initial value problems, with

$$g(t) = \mu t$$

slowly increased from -1 to 1 and with periodic boundaries in x. The constants μ, ϵ and $\frac{1}{L}$ are all $\ll 1$. The solution of the SPDE (6) is a stochastic process with values in a space of continuous functions [21]. That is, for fixed $\omega \in \Omega$ and $t \in [-\frac{1}{\mu}, \frac{1}{\mu}]$, one obtains a configuration, $\mathbf{Y}_t(x)$, that is a continuous function of x. This can be pictured as the shape of a string at time t that is constantly subject to small random impulses all along its length.

For $g < 0$ fixed, the linearised SPDE is known as the infinite-dimensional Ornstein-Uhlenbeck process [21, 6]. Its spatial correlation function is exponential with characteristic length proportional to $g^{-\frac{1}{2}}$. For $g > 0$ fixed, one sees a pattern of regions in which $\mathbf{Y}_t(x)$ is positive and regions in which $\mathbf{Y}_t(x)$ is negative (domains) separated by narrow transition layers (kinks). Although the equation is fully nonlinear, the correlation function can be calculated exactly using the transfer integral. The method applies to arbitrarily nonlinear SPDEs in one space dimension, provided they have a stationary density [13]. At late times there is a dynamic balance between nucleation and annihilation of kink-antikink pairs [9].

The first observation that facilitates analytical solution of the non-autonomous SPDE (6) is that $\mathbf{Y}_t(x)$ remains everywhere small for $g < g_c$ where

$$g_c = \sqrt{2\mu|\log\epsilon|}; \tag{7}$$

thus the emerging pattern of domains can be studied without the cubic term. From the analytical solution [12, 17] of the linearized SPDE, the correlation function is

$$c(x) = \langle \mathbf{Y}_t(x')\mathbf{Y}_t(x'+x)\rangle = \epsilon^2 \int_{t_0}^{t} \frac{\mathrm{e}^{\mu(t^2-s^2)}\mathrm{e}^{-\frac{x^2}{8(t-s)}}}{(8\pi(t-s))^{\frac{1}{2}}}\,\mathrm{d}s. \tag{8}$$

Once $g > \sqrt{\mu}$, we can approximate (8) using Laplace's formula:

$$c(x) \simeq \frac{\epsilon^2}{\sqrt{\mu}}\frac{\mathrm{e}^{\mu t^2}}{(8t)^{1/2}}\mathrm{e}^{-\frac{x^2}{8t}}. \tag{9}$$

Thus typical values of $\mathbf{Y}_t(x)$ increase exponentially fast and the correlation length at time t is $\sqrt{8t}$. Once $c(0) > \mathcal{O}(\epsilon)$, the noise no longer greatly influences the evolution; its effect can be thought of as wiping out the memory of the initial condition at $g < 0$ and replacing it with an effective random initial condition. There is a coarsening, reflected in a decrease in the mean number of zero-crossings per unit length [1, 10], r/L:

$$r/L = \frac{1}{2\pi}\sqrt{\frac{-c''(0)}{c(0)}}. \tag{10}$$

From (9), we see that, at $g = \sqrt{2\mu|\log\epsilon|}$, $c(0) = \mathcal{O}(1)$ and the cubic nonlinearity can no longer be ignored. The second observation that enables the characteristic domain size to be estimated is that the effect of the cubic nonlinearity, when it finally makes itself felt, is to freeze in the spatial structure. This is indeed the case in numerical simulations: no perceptible changes occur between $g = g_c$ and $g = 1$ [12, 17]. Thus the correlation length at $g = g_c$,

$$\lambda = \sqrt{8g_c/\mu} = 2^{7/4}\left(\frac{|\log\epsilon|}{\mu}\right)^{\frac{1}{4}}, \tag{11}$$

becomes the characteristic length for spatial structure, of domains separated by narrow defects, after $g = g_c$.

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