MATH3491/5491:
DISCRETE SYSTEMS AND INTEGRABILITY

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Acknowledgement:
Parts of these lecture notes are adaptations of a text which is intended as a monograph by J. Hietarinta, N. Joshi and F.W. Nijhoff on the same subject.
These Lectures deal with discrete systems, i.e. models of (physical, biological, economic, etc.) phenomena that are mathematically modelled through equations that involve finite (as opposed to infinitesimal) operations.

When modelling continuous phenomena in nature we are often using differential equations as the mathematical tools to encode the essence of what is going on in the system we are studying. The differential equations are believed to encapsulate the main mechanisms behind the phenomena, based on either microscopic theory or general principles relevant to the subject. Whatever the source of the model, solving the basic differential equations is then believed to lead to predictions on the behaviour of the system.

In many circumstances, the phenomena of nature are most adequately described not in terms of continuous but in terms of discontinuous steps (for instance many models in biology, economy, neural processes, decision processes, etc.). In that case we are dealing with discrete-time systems, which can mathematically be described through:

- recurrence relations;
- difference equations;
- dynamical mappings.

In addition to these types of systems we will also consider:

- functional relations

which are of a slightly different character, but will also arise in connection to discrete systems. All of the above will form the main mathematical tools in these Lectures.

To make a comparison with the continuous theory, let us recall the types of differential equations we know. These fall into two classes:

1. Ordinary differential equations (ODEs), which are of the form:

\[ \mathcal{F}(y(x), y'(x), y''(x), \ldots; x) = 0, \]

in which \( \mathcal{F} \) is some expression (a function of its arguments) and where \( x \) is called the independent variable, whilst \( y(x) \) is the dependent variable since it depends on \( x \). The primes denote differentiation w.r.t. \( x \):

\[ y'(x) = \frac{dy}{dx}, \quad y''(x) = \frac{d^2y}{dx^2}, \quad \ldots \]
and the aim is to try and solve $y$ as a function of $x$, wherever possible, from (0.0.1). The solution, if we can find it at all, is not unique, and needs further data in order to render it unique. For instance, this can be done by imposing, in addition to the ODE itself, a number of initial data, e.g. by fixing values of $y$ and some of its derivatives at a given value of $x$, say at $x = 0$, namely:

$$ y(0) , \quad y'(0) \quad \ldots $$

2. Partial differential equations (PDEs), in which case we have more than one independent variable. For instance, if we have two independent variables, say $x$ and $t$, a PDE takes on the general form:

$$ \mathcal{F}(y, y_x, y_t, y_{xx}, y_{xt}, \ldots; x, t) = 0 , \quad (0.0.2) $$

where $\mathcal{F}$ denotes again some expression, and we have in its argument the function $y(x,t)$, the dependent variable depending on both independent variables, and its partial derivatives

$$ y_x = \frac{\partial y}{\partial x} , \quad y_t = \frac{\partial y}{\partial t} , \quad y_{xx} = \frac{\partial^2 y}{\partial x^2} , \quad y_{xt} = \frac{\partial^2 y}{\partial x \partial t} , \quad y_{tt} = \frac{\partial^2 y}{\partial t^2} , \quad \ldots $$

The theory of PDEs is quite different in character from the theory of ODEs. In fact, PDEs may possess many solutions and the determination of auxiliary data is much more complicated. These data may comprise initial values as well as boundary values, and a finite number of auxiliary data (to fix the solution) is no longer sufficient.

Let us turn now to the discrete situation. Let us recall that the derivatives, on which differential equations are built, come from the limit:

$$ \frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} , \quad (0.0.3) $$

and that without taking the limit we actually first encounter the difference operation:

$$ \Delta_h y(x) = \frac{y(x+h) - y(x)}{h} . \quad (0.0.4) $$

The derivative (0.0.3) and the differential (0.0.4) are, in fact, quite distinct operations: the first is a local operation, involving the function at one point $x$, whilst the difference operation (0.0.4) is inherently nonlocal and involves two points $x$ and $x+h$ at a distance $h$. By taking the limit $h \to 0$, we actually throw away information and thus the difference operator $\Delta_h$ is in a sense a more general construct. There is good reason to study continuous systems in which the derivative plays the main role: the operations are local, instantaneous, involving smooth functions, thus apparently most suitable to describe the apparent smoothness and continuity that we observe in Nature.

The study of difference equations seem often to come in hindsight: in those cases where one cannot calculate solutions of differential equations by analytic means, finite-difference schemes are used to obtain approximate solutions by numerical methods. Here the discrete equations seem to come as an artefact and a tool rather than being fundamental to the
phenomena under study. However, in view of eq. (0.0.3) this may be considered as a logical reversal: since we get the derivative (0.0.3) by the reductive procedure of a limit from the finite operation (0.0.4) of a difference, shouldn’t we consider the latter as being the more fundamental?

In modern physics, this question gains even more weight. The underlying structure of subatomic physics, based on quantum mechanics, leaves no doubt that there is a fundamental width (given in terms of Planck’s constant $\hbar$) beyond which a “continuum” is no longer perceptible, and even may make no longer sense. Thus, maybe the continuum aspect of Nature is an illusion, only of validity on the scale of the macroscopic world. Rather than considering the finite-difference to be the approximation, one actually should consider the derivative to be the approximation (or simplification) of a more fundamental set of tools. A famous citation by A. Einstein\textsuperscript{1} declares:

“To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraic description of nature, that is to the elimination of continuum functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At the present time, however, such a program looks like an attempt to breathe in empty space.”

So what is the problem? Well, in contrast to the theory of (ordinary or partial) differential equations, the theory of difference equations is even today, in the beginning of the 21st century, still in its infancy. Whereas the theory of differential equations has steadily moved forward, and even though there are big areas that need to be developed, a fundamental theory does exist, the state-of-the-art for difference equations is a far from reaching a similar point of development. The fact of the matter is simply that, in spite of the simple appearance of the basic operation (0.0.4), to develop a comprehensive theory has proved to be far more difficult. Part of the blame lies with the history of the subject; up to the beginning of the 20th century, mathematicians considered the theory of differential equations and the theory of difference equations very much as two aspects of one and the same endeavour. It is mostly after the two world wars that a generation of mathematicians working systematically on difference equations (the famous school of G.D. Birkhoff) had disappeared and scientists had moved on. But apart from this historical mishap, it is admittedly much more difficult to classify difference equations than is the case for differential equations: there are just way too many possibilities of writing down difference equations, whereas for differential equations one can easily distinguish them by their order, their degree, their well-posedness, their amenability to boundary and initial value problems, etc.

What has fundamentally changed in the recent two decades is the surge in interest in another issue, namely that of integrable systems. Without preempting what an integrable systems is, it should just be mentioned that the theory of integrable difference equations

has undergone a revolution since the early 1980’s, and that it is in this context that the above-mentioned aspects of a theory has achieved a major development. In fact, the notion of integrability (of which more in the following Lectures) has proven to be a very strong guiding principle which has driven the subject forward. The subjects which it has strongly affected can be listed as follows

- theory of ordinary difference equations (OΔEs);
- theory of partial difference equations (PΔEs);
- theory of integrable discrete-time systems (mappings);
- discrete and “difference” geometry;
- theory of linear and nonlinear special functions;
- representation theory of (infinite-dimensional and quantum) algebras & groups;
- combinatorics and graph theory.

Some of these developments go back to results from the 19th century and even earlier, but it is fair to say that in combining classical results (from Leibnitz, Bernouilli, Gauss, Lagrange, etc.), via the turn of the 19/20th century (Poincaré, Birkhoff, Nörlund, etc.) with the results of the modern age, the study of integrability in discrete systems forms at the present time the most promising route towards a general theory of difference equations and discrete systems.

Types of Difference Equations (ΔEs)

In these Lectures we are going to consider various types of discrete systems. From a general perspective let us briefly mention the various types of difference equations that we might encounter.

First, as with differential equations, they divide up between ordinary difference equations (OΔEs) and partial difference equations (PΔEs), depending on whether there is one or more than one independent variable. The independent variable is discrete, but now this can have different interpretations:

a) the independent variable can take on only integer values, so we denote it by \( n \in \mathbb{Z} \), and thus the dependent variable \( y \) can be denoted by \( y_n \);

b) the independent variable \( x \) can take on all values (real or perhaps even complex), but in the equation the dependent variable is evaluated at integer shifts of the independent variable. Thus, the difference equation will involve evaluations of the dependent variable \( y(x) \) at shifts of \( x \) by multiples of a fixed quantity \( h \):

\[
y(x), \quad y(x + h), \quad y(x + 2h), \ldots \quad x, h \in \mathbb{R}.
\]
Remark: These two cases are quite different in nature and the associated problems are quite distinct. However, case a) can be viewed as specialisation of case b), by setting

\[ y_n \equiv y(x_0 + nh) \]

and by fixing \( x_0 \) as a starting value, we can consider the values of the independent variable \( y_n \) as a *stroboscope* of the second case.

Remark: Before describing various types of difference equations, let us first make the obvious observation that prior to the difference operator (0.0.4) there is an even more basic operation, namely that of the *lattice* shift. This is simply the operation of evaluating the dependent variable at a shifted value of the independent variable, and it is useful to introduce an operator associated with this operation:

\[ T_h y(x) = y(x + h) \]

(0.0.5)

which we will use from time to time. It is easy to see that the difference operator and its higher orders can be simply expressed in terms of the shifts \( T_h \):

\[
\begin{align*}
\Delta_h y(x) &= \frac{1}{h} (T_h - \text{id}) y(x), \\
\Delta_h^2 y(x) &= \frac{1}{h^2} (T_h^2 - 2T_h + \text{id}) y(x), \\
\vdots \\
\Delta_h^n y(x) &= \frac{(-1)^n}{h^n} \sum_{j=0}^{n} \binom{n}{j} (-T_h)^{n-j} y(x)
\end{align*}
\]

and thus the \( n \)th order difference operator acting on \( y(x) \) can be expressed in terms of the shifted variables \( y(x), y(x + h), \ldots, y(x + nh) \). Thus an equation of the form:

\[ \mathcal{F}(y(x), \Delta_h y(x), \Delta_h^2 y(x), \ldots; x) = 0 \]

can be rewritten in the form:

\[ \tilde{\mathcal{F}}(y(x), y(x + h), y(x + 2h), \ldots; x) = 0, \]

where the expression \( \tilde{\mathcal{F}} \) can be straightforwardly be obtained from \( \mathcal{F} \) by using the above mentioned substitutions.

Taking into account this remark, we can distinguish the following types of discrete systems:
a) Ordinary finite difference equations: by which we mean recurrence relations of the form
\[ \mathcal{F}(y_{n-k}, y_{n-k+1}, \ldots, y_{n+l-1}, y_{n+l}; n) = 0, \quad n \in \mathbb{Z}, \quad (0.0.6) \]
where \( k \) and \( l \) are fixed integers (see Figure). We can think of (0.0.6) as an iterative system where we wish to solve \( y_n \) at discrete points only. Thus, giving initial values at a sufficient number of points, generically \( y_0, y_1, \ldots, y_{k+l} \), we can hope to iterate the equation and calculate step-by-step \( y_{k+l+1} \) and subsequent values. The equation may depend explicitly on \( n \), in which case the equation is nonautonomous, but if \( \mathcal{F} \) depends only on \( n \) through the dependent variables then the equation (0.0.6) is autonomous. The order of the equation is generically given by \( k+l \), i.e. the number of initial data required to achieve a well-defined iteration process. We will sometime refer to an equation of the form (0.0.6) as an \((k+l+1)\)-point map (assuming all “points” at which \( y \) is evaluated appear in the expression \( \mathcal{F} \)).

Finite difference equations of this type can also be viewed as a dynamical mapping as follows. Assuming that we can solve \( y_{n+l} \) uniquely from (0.0.6), leading to an expression of the form
\[ y_{n+l} = F_n(y_{n-k}, \ldots, y_{n+l-1}) , \]
and introducing the \( k+l \)-component vector
\[ y_n = (y_{n-k}, \ldots, y_{n-1})^t \]
we can rewrite (0.0.6) as a system of equations through the dynamical map:
\[ y_n \mapsto \tilde{y}_n = y_{n+1} = F_n(y_n) , \quad (0.0.7) \]
with \( F \) denoting the vector-valued function with components \( F_j \) (viewed as a function of the vector \( y_n \)).

b) “analytic” (ordinary) difference equations: i.e. equations of the type
\[ \mathcal{F}(y(x-kh), y(x-(k-1)h), \ldots, f(x+lh); x) = 0 , \quad (0.0.8) \]
where now, even though the equation only involves integer shifts (by an increment \( h \)) in the argument of the dependent variable \( y(x) \), the independent variable \( x \) is meant to be a continuous variable, and we would like to solve (0.0.10) for functions \( y(x) \) in some appropriate class of functions. Clearly, there is indeterminacy in this problem: the solutions cannot be determined fully without additional information, and the solution is determined only up to a shift over \( h \). This implies that the initial data have to be given over an entire interval on the real line, or alternatively that the role of “integration constants” is played by periodic functions obeying:
\[ \pi(x+h) = \pi(x) . \]

c) Functional equations: This is a different class of equations altogether, taking the form
\[ \mathcal{F}(f(x), g(x), f(x+y), g(x+y), f(x+y+z), \ldots; x, y, z, \ldots) = 0 , \quad (0.0.9) \]
which is supposed to hold for arbitrary values of the arguments of the functions $f$, $g$, etc. Under general assumptions on these functions (such as continuity, differentiability, etc.), the imposition that the eq. (0.0.9) should hold for arbitrary values for $x, y, \ldots$, is often sufficient to almost uniquely fix the functions that solve the functional equation. The notion of initial values is irrelevant in this context.

Exercise # 1: Consider the functional equation:

$$\mathcal{F}(f(x), f(y), f(x + y)) = f(x + y) - f(x)f(y) = 0.$$  

Under the assumption that $f$ is differentiable for all real values of its argument, and setting $f'(0) = \alpha \neq 0$, show that the unique solution is given by $f(x) = e^{\alpha x}$.

Exercise # 2: Consider the functional equation:

$$\mathcal{F}(f(x), f(y), g(x), g(y), f(x + y)) = f(x + y) - f(x)f(y) + g(x)g(y) = 0.$$  

Under the assumption that $f, g$ are both differentiable for all real values of their arguments, and that $f$ is an even function of its argument, determine the general solution of the functional equation.

c) Partial difference equations: in this case we have more than one independent variable $x, t \in \mathbb{R}$ or in the finite difference case $n, m \in \mathbb{Z}$. The equations can take the form:

$$\mathcal{F}(y(x - k_1 \delta, t - k_2 \varepsilon), \ldots, y(x + l_1 \delta, t + l_2 \varepsilon); x, t) = 0 \quad (0.0.10)$$

with $k_1, k_2, l_1, l_2$ fixed integers, in the case of partial analytic difference equations, or in the finite difference case:

$$\mathcal{F}(y_{n - k_1, m - k_2}, \ldots, y_{n + l_1, m + l_2}; n, m) = 0. \quad (0.0.11)$$

The dependent variable $y(x, t)$ and $y_{n, m}$ respectively depend here on two (or more) independent variables with discrete shifts. The second case can be reduced to the first case by setting

$$y_{n, m} = y(x_0 + n\delta, t_0 + m\varepsilon),$$

but obviously the two types of problems are quite different in nature, since in the analytic case we want to solve for $y(x, t)$ as a function of a continuous range of values of its arguments. The theory of the latter type of equations is still very under-developed.
One of the amazing facts is that in the study of integrable difference equations, we will encounter all these type of equations, but moreover these types turn out to be quite interconnected. In fact, we will see that in many examples we can interpret one and the same system either as a dynamical map or finite difference equation, or as an analytic difference equation, or even as functional equation, without damaging the main integrability aspect of the system. In the latter interpretation we are often dealing with solutions in terms of (nonlinear discrete) special functions.

The subject is also related to a new branch of geometry, called difference geometry, which is the discrete analogue of the classical differential geometry of curves and surfaces. It turns out that integrable PΔEs describe discrete quadrilateral surfaces like the one depicted in the figure below:

The development of this new theory has been one of the more intriguing applications of the subject.
Outline of this Module

We will first start with rehearsing in the first lectures some aspects of the general theory of difference equations, such as going over some simple facts from the calculus of finite differences, treat some elements of the theory of linear difference equations and treat some specific examples. Then our focus will turn to the theory of partial difference equations, from which we will develop the notions of integrability. To motivate the partial difference equations under consideration, we will introduce them through the consideration of Bäcklund transformations for some of the well-known integrable PDEs. Once we have established the PΔEs, we will study their discrete integrability aspects from a modern perspective. Next, we will look at a variety of special solutions, which allow us to get a kaleidoscopic view on the various techniques used in integrable systems. The soliton solutions will allow us to introduce a powerful structure in terms of infinite recurrence relations which help us to find the connections between different integrable equations. Furthermore, these structures can be generalised to obtain PΔEs in higher dimensions related to the famous KP (Kadomtsev-Petviashvili) hierarchy. Other solutions are obtained through the consideration of initial-value problems on the lattice, which lead to reductions in terms of integrable dynamical mappings. This brings us to the theory of (finite-dimensional) integrable discrete-time systems. Furthermore, we will make a deviation into elliptic functions which we need, not only to paramatrise solutions of the latter, but also to find yet richer classes of integrable systems. Finally, we will discuss the theory of symmetries on the lattice and the possible special solutions we can obtain through the corresponding reduction techniques. This will lead us in particular to nonlinear special functions and discrete analogues of the famous Painlevé transcendental equations. Another aspect is the connection with the theory of (semi-classical) orthogonal polynomials and Padé approximants.

General Literature:


