Lecture 4

Continuum Limits

Having discussed in Lecture 3 the integrability aspects of quadrilateral PΔEs it is now in order to see what these equations are, and whether they can be identified as discrete analogues or discretisations of PDEs, i.e. equations that live on the space-time continuum. In order to address that issue we will pay some attention in this Lecture to the continuum limits of the lattice equations, i.e. the differential equations that we retrieve from the discrete equations by shrinking the lattice grid to a continuous set of values corresponding to spatial and temporal coordinates.

A few remarks are in order here. First, the conventional picture is the one where discrete equations, namely finite-difference schemes, are used as numerical approximations to differential equations. The choice of difference schemes is sought on the basis of a variety of criteria such as numerical stability and speed of convergence. The continuum limit of the difference scheme is in those situations often a priori clear, as the system that in dominant order of the lattice parameters reproduces the original differential equation. In most cases the continuum limit itself is a reductive procedure: we lose the lattice grid parameters by performing the limit. One may question in this picture what is what: is the differential equation an approximation of a discrete equation, or reversely is it the relevant finite-difference equation which is the approximation?

Since, in these Lectures, we consider the discrete systems as the principal objects of study, it is our perspective that continuum limits of the discrete equations are the degenerations of the latter. One would expect, therefore, that the continuous equations are less rich in parameters than the former. In studying continuum limits of integrable discrete systems, however, it turns out that the latter is only partially true. We will observe that in some respects we do not really lose the “richness” of the relevant equations, as long as we keep intact the structure of parameter families of compatible equations. In its most explicit form the discrete equations can be shown to generate entire infinite families of continuous differential equations which constitute the hierarchies associated with the famous soliton systems.
4.1 Plane-wave factors and linearisation

As an example for our ideas we will study the continuum limits of (1.1), i.e. the \( P\Delta E \) associated with the Bäcklund transformations of the KdV equation, which we constructed in Lecture 2. In Lecture 3 we pointed out that the Bäcklund parameters \( \lambda \) and \( \mu \), which can be associated with the directions of the lattice, could be interpreted as parameters measuring somehow the grid size in each direction. It is, thus, these parameters that can be used as “tuning devices” by which the lattice can be shrunk or expanded in a certain direction, eventually allowing us to shrink the lattice points together to create a continuum of points.

However, there may be many different ways in which this can happen and, thus, in principle there may be various limits that we could perform on a given lattice equation. In a continuum limit involving a lattice parameter (or step-size parameter), say \( h \), the operation of a difference such as \((0.0.4)\) will tend to a derivative, namely by \((0.0.3)\), as was explained in Lecture 1. Thus, it is by doing Taylor expansions on the shift operators like \((0.0.5)\), namely by inserting in the equations expansions of the form:

\[
y(x + h) = y(x) + h \frac{dy}{dx} + \frac{h^2}{2!} \frac{d^2y}{dx^2} + \cdots \tag{4.1.1}
\]

into the difference equations, and then by expanding power-by-power in the lattice parameter \( h \) that we obtain the transition from difference to differential equation. Typically the equation that emerges as “the continuum limit” of the discrete equation is the coefficient of the dominant term in this expansion as \( h \to 0 \).

Performing this sequence of steps on a given lattice equation there are two questions to answer, namely

- among the various parameters present in the lattice equation, how do we identify the parameter (or combination of parameters) to take as the one tending to zero in order to shrink the lattice;
- how do we determine the behaviour of the independent and dependent variables under the limit on this chosen parameter?

In general a brute force or naive continuum limit may easily lead to a total collapse of the equation, where the Taylor expansions applied to the equation lead to mismatch of orders, and hence to a situation where the limit results in no equation at all (due to conflicting constraints emerging from the expansion) or to a trivial equation in leading order in the lattice parameter. To avoid this problem and to answer the questions above, it turns out it is useful to first study the continuum limits of the linearised equation before we attack the full nonlinear equation. We will, thus, first derive the linearised form of the lattice equations under consideration and derive a special class of solutions of these linear equations to use these as a guidance on how to take nontrivial and consistent limits.

By a linearisation of a nonlinear lattice equation such as (1.1) we mean the linear equation obtained by expanding the dependent variable around a specific known solution of the nonlinear equation and taking the dominant term. The simplest linearisations are obtained by taking a trivial solution, such as the zero solution (if it exists). In the case of the example (1.1), taking \( w \equiv 0 \) is not allowed since it does not lead to a solution of that equation, but
we can modify the equation slightly by chaning it into

\[ (p + q + u_{n,m+1} - u_{n+1,m})(p + q + u_{n,m} - u_{n+1,m+1}) = p^2 - q^2, \]

(4.1.2)

which is (3.2.4) in explicit form, by setting as before \( 4\lambda = p^2 \), \( 4\mu = q^2 \), and \( w_{n,m} = u_{n,m} - np - mq \). It is easy to see that (4.1.2) admits the solution \( u_{n,m} \equiv 0 \), i.e. \( u \) vanishes for all \( n, m \). By setting next

\[ u_{n,m} = \epsilon \rho_{n,m}, \]

and expanding up to linear terms in the small parameter \( \epsilon \), we obtain the following linear equation for \( \rho \):

\[ (p + q)(\rho_{n,m+1} - \rho_{n+1,m}) = (p - q)(\rho_{n+1,m+1} - \rho_{n,m}). \]

(4.1.3)

It is easily verified that the linear lattice equation (4.1.3) obeys the consistency-around-the-cube property of section 3.2 in the same way as the full nonlinear equation, and hence we can consistently embed this equation in a higher dimensional lattice by writing the compatible system:

\[ (p + q)(\tilde{\rho} - \bar{\rho}) = (p - q)(\tilde{\rho} - \bar{\rho}), \]

(4.1.4a)

\[ (p + k)(\bar{\rho} - \tilde{\rho}) = (p - k)(\bar{\rho} - \tilde{\rho}), \]

(4.1.4b)

\[ (q + k)(\rho - \tilde{\rho}) = (q - k)(\rho - \tilde{\rho}), \]

(4.1.4c)

where \( \bar{\cdot} \) denotes the shift in the third direction associated with lattice parameter \( k \). The linear system (4.1.4) has many solutions, but we will fix a specific class of solution by demanding that the specific variable \( k \) is associated with a shift \( \rho \mapsto \bar{\rho} \) such that \( \bar{\rho} = 0 \). (One way of thinking of what this means is that the solution \( \rho \) with this constraint is obtained from an (inverse) Bäcklund transformation \( \bar{\rho} \mapsto \rho \) with seed solution \( \bar{\rho} = 0 \).)

Solving the two relations (4.1.4b) and (4.1.4c) with \( \bar{\rho} = 0 \):

\[ \tilde{\rho} = \frac{p - k}{p + k} \rho, \quad \bar{\rho} = \frac{q - k}{q + k} \rho, \]

we obtain the solution:

\[ \rho_{n,m} = \left( \frac{p - k}{p + k} \right)^n \left( \frac{q - k}{q + k} \right)^m \rho_{0,0}, \]

(4.1.5)

with \( \rho_{0,0} \) some arbitrary initial value. It is straightforward to show by direct computation that (4.1.5) is a solution of (4.1.3).

The solutions (4.1.5) we will refer to as lattice plane-wave factors and we will see in Lecture 5 that they play not only a role as approximated solutions (namely as solutions of the linearised version of the nonlinear lattice equation (4.1.2)) but they will also play an important role in the exact solution of the full nonlinear equation. It is these solutions that we will exploit in the next section to formulate the precise limits on parameters and discrete variables to get nontrivial limiting equations.
4.2. THE SEMI-CONTINUOUS LIMITS

4.2 The semi-continuous limits

We will now study various limits we can perform on the plane wave factors $\rho$ of the specific form found in (4.1.5). The guiding principle would be to seek ways in which this solution will approach exponential factors with continuous variables in the exponents. The “trick” to be used is the following limit which is well-known from basic analysis

$$\lim_{n \to \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha.$$  

4.2.1 Straight continuum limit

Focusing on one of the factors in (4.1.5) we can easily see that in the limit

$$m \to \infty, \quad q \to \infty \quad \text{such that} \quad \frac{m}{q} \text{ finite} \quad (4.2.1)$$

we obtain:

$$\lim_{m \to \infty} \left(\frac{q - k}{q + k}\right)^m = \lim_{m \to \infty} \left(1 + \frac{-2k}{q + k}\right)^m = \lim_{m \to \infty} \left(1 + \frac{\xi}{m} \frac{(-2k)}{1 - k\xi/m}\right)^m = e^{-2k\xi},$$  

using the fact the extra term in the denominator within the brackets becomes negligible as $m \to \infty$.

Let us now investigate the effect of this limit on the equations, e.g. the linear equation (4.1.4a). The idea is to re-interpret the dependent variable as

$$\rho = \rho_{n,m} =: \rho_n(\xi), \quad \xi = \xi_0 + \frac{m}{q}, \quad (4.2.3)$$

where $\xi_0$ is some initial value. This means that any shift in the discrete variable $m$ increments in the argument of the function $\rho$ by $1/q$. The next thing is for $1/q$ small, to consider Taylor series expansions of the form:

$$\rho_{n,m+1} = \rho_n(\xi + \frac{1}{q}) = \rho_n(\xi) + \frac{1}{q} \partial_\xi \rho_n(\xi) + \frac{1}{2} \frac{1}{q^2} \partial^2_\xi \rho_n(\xi) + \cdots,$$

and inserting this into the eq. (4.1.4a) we get:

$$\left(1 + \frac{p}{q}\right) \left[\rho_n - \rho_{n+1} + \frac{1}{q} \partial_\xi \rho_n + \frac{1}{2} \frac{1}{q^2} \partial^2_\xi \rho_n + \cdots\right]$$

$$= \left(-1 + \frac{p}{q}\right) \left[\rho_{n+1} - \rho_n + \frac{1}{q} \partial_\xi \rho_{n+1} + \frac{1}{2} \frac{1}{q^2} \partial^2_\xi \rho_{n+1} + \cdots\right],$$

leading in dominant order (in terms of $\frac{1}{q}$) to the differential-difference equation (DDE):

$$\partial_\xi (\rho_{n+1} + \rho_n) = 2p(\rho_{n+1} - \rho_n).$$  

(4.2.4)
This linear equation is a mixed form of differential equation (w.r.t variable $\xi$) and discrete (w.r.t. variable $n$, and is hence a semi-discrete equation. By construction, it can be directly verified that the form:

$$\rho_n(\xi) = \left(\frac{p-k}{p+k}\right)^n e^{-2k\xi} \rho_0(0),$$

is a solution of this equation.

Inspired by this result, let us now turn to the nonlinear equation for $u$, (4.1.2), and perform exactly the same limit there. Thus, introducing in a similar was as for the linear equation the reinterpretation of the discrete variables

$$u_{n,m} =: u_n(\xi), \quad \xi = \xi_0 + \frac{m}{q}, \quad (4.2.5)$$

and we derive by similar Taylor expansions as before:

$$p^2 - q^2 = \left[ p - q + \left( u_n + \frac{1}{q} \partial_\xi u_n + \frac{1}{2} \frac{1}{q^2} \partial^2 u_n + \cdots \right) - u_{n+1} \right] \times$$

$$\times \left[ p + q + u_n - \left( u_{n+1} + \frac{1}{q} \partial_\xi u_{n+1} + \frac{1}{2} \frac{1}{q^2} \partial^2 u_{n+1} + \cdots \right) \right],$$

Expanding in powers of $1/q$, noting that the dominant terms of order $O(q^2)$ and order $O(q)$ cancel identically, we obtain as coefficient of the leading term of order $O(1)$ the following equation

$$\partial_\xi (u_n + u_{n+1}) = 2p(u_{n+1} - u_n) - (u_{n+1} - u_n)^2,$$  \hspace{1cm} (4.2.6)

which is consequently the continuum limit of the lattice equation (4.1.2) under this limit as $q \to \infty$. Eq. (4.2.6) is a nonlinear D\DE, which is of first order in the derivative w.r.t. the continuous variable $\xi$ and of first order in the discrete variable $n$ as well.

**Remark:** By comparing eq. (4.2.6) with eq. (2.2.47a) we recognise that by the change of variables $x = 2\xi$, $p^2 = 4\lambda^2$, $\hat{w} - w = u_{n+1} - u_n - p$, $\partial_\xi u_n = 2w_x$, we recover the spatial part of the BT of the KdV equation.

**Other lattice equations:** We can perform similar limits on the other members of the KdV family of lattice equations, namely on (3.2.5) and (3.2.7), leading to

$$\partial_\xi (v_{n+1} v_n) = p(v_{n+1}^2 - v_n^2)$$  \hspace{1cm} (4.2.7)

and

$$(\partial_\xi z_n)(\partial_\xi z_{n+1}) = p^2(z_n - z_{n+1})^2,$$  \hspace{1cm} (4.2.8)

respectively. We shall show in section 4.3 that all these D\DEs are integrable by virtue of the existence of semi-discrete analogues of the Lax pairs in all cases.
4.2. THE SEMI-CONTINUOUS LIMITS

4.2.2 Skew continuum limit

The limit described above is not the only continuum limit that we can perform on the lattice equation. Instead of taking a limit on one of the variables \( n \) and \( m \) separately, one can also first mix them up, by means of a change of independent variables on the lattice, before taking a limit. As we shall see this will lead to quite different semi-continuous equation as a result of the limit.

To describe this, let us first consider the linearised equation (4.1.3) and the following change of variables:

\[
\rho_{n,m} = R_{n+m,m} \Rightarrow \rho_{n+1,m} = R_{N+1,m+1}, \quad \rho_{n,m+1} = R_{N+2,m},
\]

where \( N = n + m \) which can be visualised in the diagram:

This change of independent variables \( (n, m) \mapsto (N = n + m, m) \) bring the linear equation (4.1.4a) in the form

\[
(p + q)(R_{N+1,m+1} - R_{N+2,m}) = (p - q)(R_{N+2,m+1} - R_{N,m}).
\]

(4.2.10)

Let us investigate what happens on the level of the plane-wave factors \( \rho \) given by (4.1.5) with this change of variables \( (n, m) \mapsto (N = n + m, m) \). Rearranging factors we get:

\[
\rho_{n,m} = \left( \frac{p - k}{p + k} \right)^{n+m} \left( \frac{q - k}{q + k} \right)^m \rho_{0,0} = \left( \frac{p - k}{p + k} \right)^N \left( 1 + \frac{2(q - p)k}{(q + k)(p - k)} \right)^m \rho_{0,0} := R_{N,m}.
\]

(4.2.9)

Keeping \( N \) fixed and setting \( \delta = q - p \), we can now perform the limit

\[
n \to -\infty, \quad m \to \infty, \quad \delta \to 0 \quad \text{such that} \quad N \text { fixed}, \quad \delta m \text{ finite}
\]

and focusing on what happens with the second factor in this limit we observe:

\[
\lim_{m \to \infty} \left( 1 + \frac{2\delta k}{(p + \delta + k)(p - k)} \right)^m = \lim_{m \to \infty} \left( 1 + \frac{2\delta k}{(p^2 - k^2) + (p - k)\tau} \right)^m
\]

\[
= \lim_{m \to \infty} \left( 1 + \frac{\tau}{m} \frac{2k}{(p^2 - k^2) + (p - k)\tau/m} \right)^m = \exp \left( \frac{2k\tau}{p^2 - k^2} \right),
\]

(4.2.11)
using the fact that the extra term in the denominator within the brackets becomes negligible as \( m \to \infty \). Thus, the limit (4.2.11) makes good sense on the level of the plane-wave factors.

To investigate what happens with the linear lattice equation (4.2.10) under the limit (4.2.11), we set

\[
R = R_{N,m} =: R_N(\tau) , \quad \tau = \tau_0 + m\delta ,
\]

(allowing for some constant background value \( \tau_0 \) of the continuous variable), and applying the Taylor expansion:

\[
R_{N,m+1} = R_N(\tau + \delta) = R_N(\tau) + \delta \partial_\tau R_N(\tau) + \frac{1}{2} \delta^2 \partial_\tau^2 R_N(\tau) + \cdots .
\]

Inserting this into the eq. (4.2.10) we get:

\[
(2p + \delta) \left[ (R_{N+1} + \delta \dot{R}_{N+1} + \frac{1}{2} \delta^2 \ddot{R}_{N+1} + \cdots ) - R_{N+1} \right] = -\delta \left[ (R_{N+2} + \delta \dot{R}_{N+2} + \frac{1}{2} \delta^2 \ddot{R}_{N+2} + \cdots ) - R_N \right] ,
\]

which in leading order \( \mathcal{O}(\delta) \) yields as dominant term in the expansion the equation:

\[
2p\dot{R}_N = R_{N-1} - R_{N+1} .
\]

(4.2.13)

It is straightforward to check that

\[
R_N(\tau) = \left( \frac{p - k}{p + k} \right)^N \exp \left( \frac{2k\tau}{p^2 - k^2} \right) R_0(0) ,
\]

(4.2.14)

provides a solution of eq. (4.2.13).

Let us now move to the nonlinear equation and perform a similar limit there. In fact, applying the change of variables \((n, m) \mapsto (N = n + m, m)\) in (4.1.2) with the changes:

\[
u_{n,m} = U_{n+m,m} \Rightarrow u_{n+1,m} = U_{N+1,m} , \quad u_{n,m+1} = U_{N+1,m+1} , \quad u_{n+1,m+1} = U_{N+2,m} , \quad u_{n+1,m+1} = U_{N+2,m} ,
\]

(4.2.15)

we obtain

\[
(p - q + U_{N+1,m+1} - U_{N+1,m})(p + q + U_{N,m} - U_{N+2,m}) = p^2 - q^2 ,
\]

(4.2.16)

and then reinterpreting the variable \( U \) as

\[
U = U_{N,m} =: U_N(\tau) , \quad \tau = \tau_0 + m\delta ,
\]

(4.2.17)

we can perform a similar Taylor expansion on \( U \) as we did for \( R \). Inserting this into (4.2.16) we get

\[
\left[ -\delta + \left( U_{N+1} + \delta \dot{U}_{N+1} + \frac{1}{2} \delta^2 \ddot{U}_{N+1} + \cdots \right) - U_{N+1} \right] \times \left[ 2p + \delta + U_N - \left( U_{N+2} + \delta \dot{U}_{N+2} + \frac{1}{2} \delta^2 \ddot{U}_{N+2} + \cdots \right) \right] = -(2p + \delta)\delta ,
\]
which in leading order yields

\[(\dot{U}_{N+1} - 1)(2p + U_N - U_{N+2}) = -2p\ ,\]

or equivalently

\[\dot{U}_N = 1 - \frac{2p}{2p + U_{N-1} - U_{N+1}} . \tag{4.2.18}\]

In spite of the fact that eq. (4.2.18) was derived starting from the same lattice equation, it is quite a different nonlinear DΔE from (4.2.6), as is evident by inspecting the orders: this equation is first order in the derivative w.r.t. the continuous variable \(\tau\), and second order w.r.t. the discrete variable \(N\), whereas (4.2.6) is first order in both the discrete and continuous variables but the derivative w.r.t. \(\xi\) acts on both \(u_n\) and \(u_{n+1}\). We will assess in section 4.3 that both equations (4.2.6) and (4.2.18) are integrable in the sense that there exists an associated Lax pair in both cases.

**Other examples:** The process described above to obtain a mixed continuum limit we refer to as the skew continuum limit of the lattice equation. It can be applied also to the other members of the lattice KdV family, namely to eqs. (3.2.5) and (3.2.7). In those cases we obtain

\[p \partial_\tau \log V_N = \frac{V_{N-1} - V_{N+1}}{V_{N-1} + V_{N+1}} , \tag{4.2.19}\]

and

\[\dot{Z}_N = \frac{2}{p} \frac{(Z_{N-1} - Z_N)(Z_N - Z_{N+1})}{Z_{N-1} - Z_{N+1}} \tag{4.2.20}\]

respectively.

### 4.3 * Semi-discrete Lax pairs*

In this section we shall show that the integrability characteristics of the equations are conserved when performing the continuum limits which we exhibited in section 4.2. This we will do by applying both the straight as well as the skew continuum limit to the relevant Lax pair on the lattice. We will exemplify this mechanism once again for the case of the lattice potential KdV equation (4.1.2), for which we have the following matricial Lax pair:

\[ (p - k)\phi_{n+1,m} = L_{n,m} \phi_{n,m} \ , \quad (q - k)\phi_{n,m+1} = M_{n,m} \phi_{n,m} \ , \tag{4.3.1a}\]

in which the matrices \(L\) and \(M\) are given by

\[ L_{n,m} = \begin{pmatrix} p - u_{n+1,m} & 1 \\ k^2 - p^2 + (p - u_{n+1,m})(p + u_{n,m}) & p + u_{n,m} \end{pmatrix} , \tag{4.3.1b}\]

\[ M_{n,m} = \begin{pmatrix} q - u_{n,m+1} & 1 \\ k^2 - q^2 + (q - u_{n,m+1})(q + u_{n,m}) & q + u_{n,m} \end{pmatrix} . \tag{4.3.1c}\]
The Lax pair (4.3.1b) can be obtained from the Lax pair (3.3.3) for the form of the equation in terms of \( w \), as was derived in Lecture # 3, by a simple gauge transformation. The prefactors \((p - k)\) and \((q - k)\) included in (4.3.1a) are obviously irrelevant for the nonlinear equation that arises from the Lax pair as the consistency condition (since these factors can be easily absorbed into a redefinition of the vector \( \phi \)), but will turn out to be useful in performing the limits.

### 4.3.1 Straight continuum limit

We will now perform the limit (4.2.1) on the Lax pair itself. Obviously this will not affect the first member of the Lax pair, i.e. the first equation in (4.3.1), since it only affects the dependence on the variable \( m \) and the parameter \( q \). Setting once again \( u_{n,m} = u_n(\xi) \), \( \phi_{n,m} = \phi_n(\xi) \), \( \xi = \xi_0 + \frac{m}{q} \), and expanding both \( \phi_{n,m+1} = \phi_n(\xi + \frac{1}{q}) \) and \( u_{n,m+1} = u_n(\xi + \frac{1}{q}) \) in the entries of the matrix \( bM \) by Taylor series in powers of \( 1/q \), we get from the second member of the Lax pair:

\[
(q - k) \left( \phi_n + \frac{1}{q} \partial_\xi \phi_n + \cdots \right) = q \phi_n + \begin{pmatrix}
-\frac{1}{2} \partial_\xi u_n - \cdots & \frac{1}{q} - 1 \\
1 & k^2 + q(\frac{1}{q} \partial_\xi u_n - \cdots) + u_n
\end{pmatrix} \phi_n,
\]

and retaining only terms of order \( O(1) \) we obtain:

\[
\partial_\xi \phi_n = N_n \phi_n = \left( \begin{array}{cc}
k - u_n & 1 \\
k^2 - \frac{1}{q} \partial_\xi u_n - u_n^2 & k + u_n
\end{array} \right) \phi_n .
\]

(4.3.2a)

It is easy to check that the semi-discrete Lax pair consisting of eq. (4.3.2) and the linear equation which we obtain directly from the first member of (4.3.1a), namely:

\[
(p - k) \phi_{n+1} = L_n \phi_n = \left( \begin{array}{cc}
k^2 - p - u_n & 1 \\
kp + pu_n + (p - u_{n+1})(p + u_n)
\end{array} \right) \phi_n .
\]

(4.3.2b)

In fact, the relevant Lax equations in this case are obtained from:

\[
(p - k) \partial_\xi \phi_{n+1} = \partial_\xi (L_n \phi_n) = (\partial_\xi L_n) \phi_n + L_n N_n \phi_n = (p - k) N_{n+1} \phi_{n+1} = N_{n+1} L_n \phi_n,
\]

leading to the semidiscrete Lax equation

\[
\partial_\xi L_n = N_{n+1} L_n - L_n N_n .
\]

(4.3.3)

It is a straightforward exercise to check that the compatibility condition (4.3.3) indeed gives rise to the nonlinear \( D \Delta E \) (4.2.6). In a similar way the semidiscrete Lax pairs for eqs. (4.2.7) and (4.2.8) can be found.
4.3.2 Skew continuum limit

We now turn to the derivation of the Lax pair for the skew continuum limit of the lattice equation (4.1.2) from (4.3.1). In this case the derivation is slightly more complicated, because the two members of the Lax pair get mixed together in the limit due to the fact that we have to perform the change of variables as in (??). Thus, implementing the same change of variables on the vector \( \phi \), i.e.

\[
\phi_{n,m} := \Phi_N(\tau) \quad \text{with} \quad N = n + m , \quad \tau = \tau_0 + m \delta , \quad \delta = q - p ,
\]

we see again that the first part of the Lax pair (4.3.1a) carries over straightforwardly, replacing the variable \( n \) by the new spatial variable \( N \). The second part of the Lax pair now becomes upon the change of variables:

\[
(q - k)\phi_{n,m+1} = (q - k)\Phi_{N+1}(\tau + \delta) = (p - k + \delta) [\Phi_{N+1}(\tau) + \delta \partial_\tau \Phi_{N+1}(\tau) + \cdots]
\]

\[
= \left( \begin{array}{cc}
p + \delta - U_{N+1} - \delta \partial_\tau U_{N+1} - \cdots \\
-k^2 + (p + \delta)(U_N - U_{N+1} - \delta \partial_\tau U_{N+1} + \cdots) + \frac{1}{p + \delta + U_N}
\end{array} \right) \Phi_N(\tau) .
\]

Using the spatial part of the Lax representation in the form:

\[
(p - k)\Phi_{N+1} = \mathcal{L}_N \Phi_N = \left( \begin{array}{cc}
k^2 - p^2 + \\
(p - U_{N+1})(p + U_N)
\end{array} \right) \Phi_N , \quad (4.3.4)
\]

we see that the dominant term of the expansion is of order \( O(\delta) \) and the resulting term leads to the equation:

\[
\Phi_{N+1} + (p - k)\partial_\tau \Phi_{N+1} = \left( \begin{array}{cc}
1 - \partial_\tau U_{N+1} \\
U_N - U_{N+1} - \rho \partial_\tau U_{N+1} \\
-U_N \partial_\tau U_{N+1}
\end{array} \right) \Phi_N
\]

and using the spatial part of the Lax pair again we arrive after some algebra at the following temporal part of the Lax pair:

\[
\Phi_N = \frac{1}{k^2 - p^2} \left( \begin{array}{cc}
k - U_{N+1} - (p - U_{N+1})\hat{U}_N \\
k^2 - U_{N+1}^2 + (p - U_{N+1})^2 \hat{U}_N \\
k + U_{N+1} + (p - U_{N+1})\hat{U}_N
\end{array} \right) \Phi_N ,
\]

(4.3.5)

in which \( \hat{U}_N = \partial_\tau U_N \), \( \hat{\Phi}_N = \partial_\tau \Phi_N \). Thus, eqs. (4.3.4) and (4.3.5) together form the Lax pair for the D\(\Delta\)E (4.2.18). In fact, calling the matrix on the right-hand side of (4.3.5) \( \mathcal{N}_N \), by a similar argument as given above in the skew continuum limit part, we have in this case the Lax equation:

\[
(k^2 - p^2)\mathcal{L}_N = \mathcal{N}_{N+1} \mathcal{L}_N - \mathcal{L}_N \mathcal{N}_N , \quad (4.3.6)
\]

and it is a tedious but straightforward calculation to show that (4.3.6) is verified if \( U_N \) obeys the semi-discrete KdV equation (4.2.18).
4.4 Full continuum limit

Finding the full continuum limit of lattice equations such as eq. (4.1.2) is a two-stage process: first we must establish a semi-discrete (semi-continuous) limit, as we have established in the previous section, and second to find the full continuum limit from the latter in the next step.

In this section we will perform the second step of the process starting from the skew limit of the equation, and do a second limit in order to turn the remaining discrete variable into a continuous one. In fact, in order to obtain a nontrivial limit, we shall see that the process is slightly more involved, and that we need to mix once again both spatial and temporal variables in order to obtain a nonlinear PDE, retaining the integrability.

As before we shall use the plane-wave factors to guide us in finding the limit we need to impose in order to get nontrivial equations from the semi-discrete one. Taking the skew limit of the plane-wave factor, i.e. the form of the variable $R_N(\tau)$ as given in (4.2.14), we have

$$\left(\frac{p-k}{p+k}\right)^N \exp\left(\frac{2k\tau}{p^2-k^2}\right) = \exp\left\{\frac{2k\tau}{p^2-k^2} + N \left[\ln\left(\frac{1-k}{p}\right) - \ln\left(1 + \frac{k}{p}\right)\right]\right\}$$

$$= \exp\left\{\frac{2k\tau}{p^2} \left[1 + \frac{k^2}{p^2} + \frac{k^4}{p^4} + \cdots \right] - 2N \left[\frac{k}{p} + \frac{1}{3} \frac{k^3}{p^3} + \frac{1}{5} \frac{k^5}{p^5} + \cdots \right]\right\}$$

$$= \exp\left\{2k\left(\frac{\tau}{p^2} - \frac{N}{p}\right) + 2k^3 \left(\frac{\tau}{p^4} - \frac{1}{3} \frac{N}{p^3}\right) + 2k^5 \left(\frac{\tau}{p^6} - \frac{1}{5} \frac{N}{p^5}\right) + \cdots\right\}$$

$$\to e^{kx+k^3t+k^5t'+\cdots}.$$  

This leads to the identification of the variables of the full continuum limit as the coefficients of the various powers of $k$ in the expansion of the exponents of the plane-wave factors, namely

$$x = 2 \left(\frac{\tau}{p^2} - \xi\right), \quad t = 2 \left(\frac{\tau}{p^4} - \frac{1}{3} \frac{\xi}{p^3}\right), \quad (4.4.1)$$

in which as before $\xi = N/p$. Thus, this calculation suggests, first, that the solution (4.2.14) can be identified as follows:

$$R_N(\tau) := \mathcal{R}(\xi, \tau) \Rightarrow R_{N \pm 1}(\tau) = \mathcal{R}(\xi \pm \frac{1}{p}, \tau) = \mathcal{R} + \frac{1}{p} \mathcal{P}_\xi + \frac{1}{2p^2} \mathcal{P}_\xi \xi + \frac{1}{6p^3} \mathcal{P}_\xi \xi \xi + \cdots,$$

and, second, to perform the change of variables (4.4.1), i.e.

$$\mathcal{R}(\xi, \tau) := R(x, t),$$

which implies by chain rule:

$$\frac{\partial \mathcal{R}}{\partial \xi} = \frac{\partial R}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial R}{\partial t} \frac{\partial t}{\partial \xi} = -2Rx - \frac{2}{3p^2} R_t, \quad (4.4.2a)$$

$$\frac{\partial \mathcal{R}}{\partial \tau} = \frac{\partial R}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial R}{\partial t} \frac{\partial t}{\partial \tau} = \frac{2}{p^2} Rx + \frac{2}{p^4} R_t. \quad (4.4.2b)$$

Since the two steps (i.e. Taylor expansion of the shift in $N$ and change of variables $(\xi, \tau) \mapsto (x, t)$) involve the parameter $p$, the expansion in powers of $1/p$ and the selection of the
4.4. FULL CONTINUUM LIMIT

A dominant term can only be done after having done both of these steps. Thus, starting from the semi-discrete equation (4.2.13) we get

\[ 2p \partial_x R = -\frac{2}{p} \partial_x R - \frac{1}{3p^3} \partial_x^3 R - \ldots \Rightarrow \]

\[ \Rightarrow 2p \left( \frac{2}{p^2} \partial_x + \frac{2}{p^4} \partial_t \right) R = -\frac{2}{p} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right) R - \frac{1}{3p^3} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right)^3 R - \ldots, \]

and we observe that while the leading term of order \( O(1/p) \) cancels identically, the next term of order \( O(1/p^3) \) yields precisely the equation:

\[ R_t = R_{xxx}, \]

as expected.

Turning now to the nonlinear equation (4.2.18) and performing the same continuum limit there, setting first

\[ U_N(\tau) := \mathcal{U}(\xi, \tau) \Rightarrow U_{N \pm 1}(\tau) = \mathcal{U}(\xi \pm \frac{1}{p}, \tau) = \mathcal{U} \pm \frac{1}{p} \mathcal{U}_\xi + \frac{1}{2p^2} \mathcal{U}_{\xi\xi} \pm \frac{1}{6p^4} \mathcal{U}_{\xi\xi\xi} + \ldots, \]

which by insertion in (4.2.18) yields:

\[ \partial_\tau U = 1 - \left[ 1 - \frac{1}{2p} \left( \frac{2}{p} U_\xi + \frac{1}{3p^2} U_{\xi\xi} + \ldots \right) \right]^{-1}, \]

and, second, to use similar expressions as (4.4.2) to implement the change of variables \((\xi, \tau) \mapsto (x, t)\) to obtain:

\[ \left( \frac{2}{p^2} \partial_x + \frac{2}{p^4} \partial_t \right) U = 1 - \left[ 1 - \frac{1}{p^2} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right) U - \frac{1}{6p^4} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right)^3 U + \ldots \right]^{-1} \]

\[ = - \left[ \frac{1}{p^2} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right) U + \frac{1}{6p^4} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right)^3 U + \ldots \right] - \left[ \frac{1}{p^2} \left( -2\partial_x - \frac{2}{3p^2} \partial_t \right) U + \ldots \right]^2 - \ldots \]

expanding the denominator on the right-hand side in a power series. Again we observe that the terms of order \( O(1/p) \) cancel identically, and that the next dominant order \( O(1/p^3) \) yields the equation:

\[ U_t = U_{xxx} - 3U_x^2, \]  \quad (4.4.3)

which is the potential KdV equation (coinciding with (2.2.46) in terms of the variable \( w \) up to a change of sign in the independent variables, i.e. after \( x \mapsto -x, t \mapsto -t \) we recover the equation for \( w \)).

We see that we have come now full circle: we started out with the continuous KdV equation, derived its Bäcklund transformations, which using the permutability theorem led to the construction of a lattice of solutions, the relations between these solutions being reinterpreted as a partial difference equation on the two-dimensional lattice. As a dynamical
equation the latter was seen as a discretisation of some continuum equations, both semi-discrete as well as fully discrete, and the full continuum limit now turns out to be the equation we started out from, namely the KdV itself. Perhaps this is not so surprising, but what is remarkable is that the equations at all levels are compatible with each other: the continuous and discrete equations can be imposed simultaneously on one and the same variable $u$, and, in fact, the variable $U$ of (2.2.46) is nothing else than the variable $w$ of Lecture # 2.

**Remark:** Performing similar continuum limits on the semi-discrete equations (4.2.19) and (4.2.20) we recover the following fully continuous equations, namely

$$V_t = V_{xxx} - 3 \frac{V_x V_{xx}}{V}, \quad (4.4.4)$$

which is the potential MKdV equation, and

$$Z_t = Z_{xxx} - 3 \frac{Z_{xx}^2}{2 Z_x}, \quad (4.4.5)$$

which is the Schwarzian KdV equation.

**Literature**


