Lecture 2

From Continuous to Discrete Equations via Transformations

In this chapter we will discuss how difference equations arise from transformations applied to differential equations. Thus we find that some functions can be defined both by a differential equation and by a difference equation. Of course the independent variables of these two types of equations are not the same, but rather we find interesting duality between parameters and independent variables.

Often functions defined by a differential equation possess special transformations. One particularly important class of transformations are those that act on the differential equation by changing the values of its parameters. When such transformations are iterated, we obtain a sequence of differential equations (and often a sequence of solutions) that are strung together by the changing sequence of parameter values.

As particularly simple example consider the differential equation

$$x w' = \alpha w. \tag{2.0.1}$$

Here $x$ is the independent (continuous) variable, the prime stands for $x$-derivative ($w' = \frac{dw}{dx}$) and $\alpha$ is a parameter of the equation. It is easy to find the solution of this equation, it is

$$w_\alpha(x) = k x^\alpha, \tag{2.0.2}$$

where $k$ is required integration constant. From the form of the solution (2.0.2) we can see, that the simple transformation of multiplying the function $w_\alpha$ by $x$ changes $\alpha$, that is

$$w_{\alpha+1} = x w_\alpha. \tag{2.0.3}$$

Equation (2.0.3) can now be interpreted as a discrete or difference equation, where $\alpha$ is the (discrete) independent variable and $x$ is a parameter. Thus we have found, that the equations (2.0.1) and (2.0.3) both describe the same function $w_\alpha(x)$ of (2.0.2) but from two different points of view.

We will first discuss the case of such procedures for ordinary differential equations (ODEs), using the examples of equations for certain classical special functions, such as
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the Bessel, Legendre, and hypergeometric functions. These can be shown not only linear
ODEs, but also some ordinary difference equations (OΔEs) in terms of certain parameters.
There are also nonlinear special functions, such as the elliptic functions and the Painlevé
transcendents, that satisfy nonlinear differential equations, for which similar ideas apply,
but we will not treat them in this Lecture. Next, we will move from ODEs to partial differ-
etial equations (PDEs), and treat briefly some basic theory of soliton equations, and show
how similar ideas there lead to the construction of integrable partial difference equations
(PΔEs).

2.1 Special Functions and linear equations

Special functions [Whittaker-Watson, 1927] such as the Bessel, Parabolic Cylinder, Hyper-
geometric functions satisfy ODEs with independent variable $x$ and at least one parameter
$\alpha$. They also satisfy recurrence relations in which $\alpha$ is shifted. We will now study some of these
differential equations and derive the corresponding transformations and related difference
equations. (Further properties of these special functions can be found in Appendix.)

2.1.1 Weber Functions and Hermite Polynomials

Consider the differential equation

$$w'' + \left(\alpha + \frac{1}{2} - \frac{1}{4}x^2\right)w = 0,$$  \hspace{1cm} (2.1.4)

where the primes denote differentiation in $x$. This equation arises in many physical applica-
tions, for example in the study of quantum mechanics in harmonic potential. A particular
special solution of this equation is the parabolic cylinder or Weber function $w =: D_{\alpha}(x)$,
uniquely specified by the asymptotic behaviour

$$D_{\alpha}(x) = x^\alpha e^{-x^2/4} \left(1 - \frac{\alpha(\alpha - 1)}{2x^2} + O\left(\frac{1}{x^4}\right)\right), \quad x \to +\infty.$$  \hspace{1cm} (2.1.5)

The simplest special case arises when $\alpha$ is an integer $n$, this yields the Hermite polynomials
$H_n(x)$, defined by

$$D_{n}(x) = 2^{-n/2} \exp(-x^2/4) H_n(z), \quad z = x/\sqrt{2}.$$  \hspace{1cm} (2.1.6)

Instead of thinking of Equation (2.1.4) as one equation specified by one fixed value of
$\alpha$, it is more productive to think of it as an infinite sequence of equations, each of which is
specified by successive values of $\alpha = n + \alpha_0$. This alternative perspective has a wonderful
consequence: we can generate new solutions of each successive equation by knowing solutions
of an earlier equation in the sequence. To see how to do this, note that the differential
operator in Equation (2.1.4) factorizes:

$$\left(\partial_x - x/2\right)\left(\partial_x + x/2\right)w = -\alpha w.$$  \hspace{1cm} (2.1.7)

Assume that $w = w_{\alpha}(x)$ is a general solution corresponding to a given value of $\alpha$. Let us define

$$\bar{w} := (\partial_x + x/2)w.$$  \hspace{1cm} (2.1.8)
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Then we have from (2.1.7) the system of two equations

\[
\begin{cases}
(\partial_x + x/2)w = \tilde{w}, \\
(\partial_x - x/2)\tilde{w} = -\alpha w.
\end{cases}
\] (2.1.9)

If we use the first to eliminate \(\tilde{w}\) we immediately get (2.1.7), while eliminating \(w\) yields

\[
(\partial_x + x/2)(\partial_x - x/2)\tilde{w} = -\alpha \tilde{w},
\] (2.1.10)

or after expanding

\[
\tilde{w}'' + ((\alpha - 1) + \frac{1}{2} - \frac{1}{4}x^2) \tilde{w} = 0.
\] (2.1.11)

which is the same as (2.1.4) except for a new parameter value \(\tilde{\alpha} = \alpha - 1\), and thus \(\tilde{w}_{\tilde{\alpha}(x)} \propto w_{\alpha-1}(x)\).

Since the equations are linear the proportionality constant is in principle free, and is determined by some other considerations, like normalization. In this particular case the solutions of (2.1.4) (the parabolic cylinder functions \(D_{\alpha}(x)\)) are conventionally normalized to have the asymptotic behaviour as given in (2.1.5). Using (2.1.8) on (2.1.5) we get

\[
(\partial_x + x/2)D_{\alpha}(x) = \alpha x^{\alpha-1}e^{-x^2/4} (1 + O(1/x^2)) = \alpha D_{\alpha-1},
\]

and therefore we should use the normalization

\[ w_{\alpha} = D_{\alpha}, \quad \tilde{w}_{\tilde{\alpha}(x)} = \alpha D_{\alpha-1}(x). \]

Equation (2.1.9) can be written in two additional ways, using this normalization. First by solving for the term that appears alone:

\[
\begin{cases}
D_{\alpha-1}(x) = \frac{1}{2}[D'_{\alpha}(x) + \frac{1}{2}xD_{\alpha}(x)], \\
D_{\alpha}(x) = \frac{1}{2}[D'_{\alpha-1}(x) - \frac{1}{2}xD_{\alpha-1}(x)].
\end{cases}
\] (2.1.12)

and secondly by solving for the derivative terms:

\[
\begin{cases}
D'_{\alpha}(x) = -\frac{1}{2}xD_{\alpha}(x) + \alpha D_{\alpha-1}(x), \\
D'_{\alpha-1}(x) = \frac{1}{2}xD_{\alpha-1}(x) - D_{\alpha}(x).
\end{cases}
\] (2.1.13)

The first pair allows us to travel up and down in the chain of \(D_{\alpha}(x)\)'s for different \(\alpha\)'s separated by integers. The second form (2.1.13) allows us to derive a fully discrete equation: substituting \(\alpha \rightarrow \alpha + 1\) into the second equation of (2.1.13) and then subtracting the two equations we get

\[
D_{\alpha+1}(x) - xD_{\alpha}(x) + \alpha D_{\alpha-1}(x) = 0.
\] (2.1.14)

This is a difference equation, where \(\alpha\) is the independent variable and \(x\) the parameter.

Let us now reflect on what we have obtained. The conventional theory of ordinary differential equations regards the parameters \(\alpha\) as given and fixed, while \(x\) varies in some domain. By using the Darboux transformation (2.1.8) we can change the parameter \(\alpha\). In the fully dual point of view, we would keep \(x\) is fixed while \(\alpha\) changes. At an intermediate step also have recurrence relations that relate \(w(x; \alpha), w(x; \tilde{\alpha})\) and their \(x\)-derivatives. In the theory of integrable equations, such recurrence relations are called (auto-) Bäcklund transformations.
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Exercise 2.1.1. Note that we could have factorized the operator in Weber’s equation in a
different order to get $(\partial_x + x/2)(\partial_x - x/2)w = -(\alpha + 1)w$. Find the equation satisfied by
$\tilde{w} = (\partial_x - x/2)w$ and deduce another Darboux transformation. Also show that $\tilde{w} \propto w_{\alpha+1}$.

Exercise 2.1.2. Use the relationship between Weber function and Hermite polynomials
(2.1.6) to derive the difference equation for Hermite polynomials $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, starting from (2.1.14).

Quantum mechanical interpretation: Equation (2.1.4) looks like a Schrödinger equation for a particle moving in a harmonic potential, so the discussion above has a quantum mechanical interpretation: $\partial_x + x/2$ is a lowering operator. It kills the lowest energy state: $(\partial_x + x/2)\psi = 0$, from which we find $\psi_0 \propto e^{-x^2/4}$. $\partial_x - x/2$ is the raising operator, which generates excited states starting from the ground state $\psi_0$.

2.1.2 Darboux and Bäcklund transformation in general

Jean Gaston Darboux (1842 - 1917) was one of the pioneers of classical differential geometry. Furthermore, he made important contributions to many other fields of mathematics including differential equations. In his book [Darboux, 1914] (p.210, Section 408) a theorem, which has been revived in the modern theory of integrable systems, leading to what in the modern literature, cf. e.g. [Matveev & Sall, 1991; Rogers & Schief, 2002], is called a Darboux transformation. Related to it is the notion of a so-called Bäcklund transformation, which we describe briefly below and in the next section in more detail (for the case of PDEs). Whereas Darboux’ theorem applies primarily to linear equations, the Bäcklund transformation, as we shall see, is most relevant to nonlinear equations.

Darboux’ theorem can be described as follows:

**Theorem 2.1.1. (Darboux transformation)** Let us consider the differential equation

$$y'' = [\phi(x) + h]y,$$  \hspace{1cm} (2.1.15)

with parameter $h$. Suppose $f(x)$ is a particular solution of this equation for some specific value $h_1$ of $h$. Let us define a new function $\tilde{y}$ by

$$\tilde{y} := [\partial_x - (\log f)']y.$$  \hspace{1cm} (2.1.16)

Then $\tilde{y}$ solves the equation

$$\tilde{y}'' = [\tilde{\phi}(x) + h]\tilde{y},$$  \hspace{1cm} (2.1.17)

where $\tilde{\phi} := \phi - 2(\log f)'$.

We say that $\tilde{\phi}$ is the Darboux transform of the potential and $\tilde{y}$ of the wave-function.

What is behind this theorem is a factorisation property. In fact, given that the “potential” function $\phi(x)$ does not depend on the (spectral) parameter $h$, the solution $y(x, h_1) = f(x)$ of (2.1.15) allows on to express this potential as

$$\phi(x) = -h_1 + \partial_x^2 \log f + (\partial_x \log f)^2,$$
and hence eq. (2.1.15) for general $h$ can be rewritten as:

$$y'' = (h - h_1 + v' + v^2)y, \quad \text{where} \quad v = \partial_x \log f.$$

This can be factorised as follows:

$$(\partial_x + v)(\partial_x - v)y = (h - h_1)y,$$

and setting $\tilde{y} = (\partial_x - v)y$, we obtain by interchanging the factors:

$$(\partial_x - v)(\partial_x + v)\tilde{y} = (h - h_1)\tilde{y},$$

the equation (??) with a new potential $\tilde{\phi}(x) = \phi(x) - 2v'$ given by (2.1.16).

We note that the result on the Weber function of the previous section can be viewed as an application of Darboux' theorem: In comparison with (2.1.4) we see that we should choose $\phi = x^2/4$, $f = e^{x^2/4}$, $h_1 = 1/2$. Then we find that $\tilde{\phi} = \phi - 1$ in agreement with (2.1.11).

The pair of equations (2.1.12) are a special case of a construction named after Albert Victor Bäcklund (1845 - 1922) who worked on transformations of surfaces in differential geometry. In the modern era the connection between the latter subject and the theory of differential equations has become more prominent. In fact, there is close relation between transformations between special surfaces in terms of coordinates on these surfaces and transformations between solutions of (linear and nonlinear) differential equations. As we shall see in the next section these transforms will form the basis of the construction of exact discretizations of those very same differential equations. We defer the precise definition of a Bäcklund transformation to later when we deal with PDEs, but only present here a loose definition of this concept as follows:

**Definition 2.1.1** (Bäcklund transformation (loose definition)). Suppose we have a pair of equations depending on two dependent variables $u$ and $v$ and possibly on their partial derivatives:

$$\begin{cases} F(u, u_x, \ldots, v, v_x, \ldots) = 0, \\
G(u, u_x, \ldots, v, v_x, \ldots) = 0. \end{cases}$$

(2.1.18)

If upon eliminating $v$ we obtain the equation $R(u, u_x, u_{xx} \ldots) = 0$ and upon eliminating $u$ we obtain $S(v, v_x, v_{xx} \ldots) = 0$ then (2.1.18) is called a **Bäcklund transformation (BT)** between the equations $R = 0$ and $S = 0$. If $R$ and $S$ differ only through some parameters the transformation is called an **auto-Bäcklund transformation (aBT)**.

**2.1.3 Bessel Functions**

As an application of the general DT method let us consider the Bessel functions, which also occur in many applications, e.g., is the study of waves in circular domains.
Bessel functions are defined as solutions of
\[ x^2 w'' + x w' + (x^2 - \nu^2) w = 0, \] (2.1.19)
where \( \nu \) is a parameter, with specific asymptotic behaviours: The standard Bessel function of the first kind is defined by
\[ J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k! \Gamma(\nu + k + 1)}. \] (2.1.20)

We will again consider Bessel’s equation as an infinite sequence of equations in the space of the parameter value \( \nu \). To allow us to iterate in this parameter space, we need recurrence relations for Bessel functions. To find Darboux transformations from which recurrence relations follow, it is easier to start by transforming equation (2.1.19) into the Liouville-Green-Schrödinger form without the first derivative term.

**Exercise 2.1.3.** Transform variables by taking \( w(x;\nu) = p(x)y(x;\nu) \). Substitute into Equation (2.1.19) and show that the coefficient of the first derivative term vanishes iff
\[ 2xp'(x) + p(x) = 0, \] which implies \( p(x) = 1/\sqrt{x} \).

Transforming variables by taking \( w(x;\nu) = y(x;\nu)/\sqrt{x} \), we get
\[ y'' + (1 - (\nu^2 - 1/4)x^{-2})y = 0 \] (2.1.21)

Following Darboux’ theorem 2.1.1 we should first construct a suitable function \( f \). In the above equation \( h = -1 \), and clearly \( f := x^\sigma \) works, if \( \sigma = \pm(\nu - 1/2) \), \( h_1 = 0 \). Taking the lower sign yields the transformation
\[ \tilde{y} := (\partial_x + (\nu - 1/2)/x) y, \] (2.1.22)
\[ \tilde{\phi} := -(1 - (\nu^2 - 1/4)x^{-2}) - 2(\log f)' \]
\[ = -(1 - [(\nu - 1)^2 - 1/4|x^{-2}}) . \] (2.1.23)

Thus again DT implies an integer step in some parameter, in this case \( \nu \), and therefore we can set
\[ \tilde{y} = y_{\nu-1}(x), \] (2.1.24)
with proportionality constant 1 as determined from (2.1.20).

Using (2.1.22) we find easily the factorization
\[ (\partial_x - (\nu - 1/2)^2/x)(\partial_x + (\nu - 1/2)^2/x) = (\partial_x^2 - (\nu^2 - 1/4)x^{-2}) \] (2.1.25)
and the Bäcklund pair
\[ \begin{cases} [\partial_x + (\nu - \frac{1}{2})^2/x] y = \tilde{y}, \\ [\partial_x - (\nu - \frac{1}{2})^2/x] \tilde{y} = -y, \end{cases} \] (2.1.26)
where eliminating \( \tilde{y} \) yields (2.1.21) for \( y \), while eliminating \( y \) yields (2.1.21) with \( \nu \to \nu - 1 \) for \( \tilde{y} \).

\(^1\)Recall the definition of the \( \Gamma \)-function, which is used in this formula, as the function obeying the difference equation \( \Gamma(z + 1) = z\Gamma(z) \).
Returning to the original variables with \( y = \sqrt{x} w \) the above equations become

\[
\begin{align*}
\left[ \partial_x + \left( \frac{1}{2} \right) \nu + \frac{1}{4} \right] w &= \tilde{w}, \\
\left[ \partial_x - \left( \nu - \frac{1}{2} \right) \right] \tilde{w} &= -w.
\end{align*}
\tag{2.1.27}
\]

Now solving for the derivative terms with (2.1.24) and shifting \( \nu \rightarrow \nu + 1 \) in the second equation we obtain

\[
\begin{align*}
x w'_\nu &= -\nu w_\nu + x w_{\nu-1}, \\
x w''_\nu &= \nu w_\nu - x w_{\nu+1},
\end{align*}
\tag{2.1.28}
\]

and after subtracting we get the difference equation in \( \nu \) with \( x \) a parameter:

\[
x w_{\nu+1} - 2\nu w_\nu + x w_{\nu-1} = 0.
\tag{2.1.29}
\]

The equations (2.1.28) and (2.1.29) hold for the Bessel functions of the first kind \( J_\nu(x) \).

### 2.1.4 * Hypergeometric functions

As a final example of Darboux and Bäcklund transformation for linear equations we consider the hypergeometric functions. They form a very rich class of functions that have intricate connections to geometry and complex analysis. The Gauss’ hypergeometric function \( {}_2F_1(a, b; c; x) \) is defined by the series

\[
{}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n
\tag{2.1.30}
\]

for \( |x| < 1 \) and by analytic continuation elsewhere. The Pochhammer symbol \( (a)_n \) was defined in section 1.3.2, \( (a)_n = a (a + 1) (a + 2) \ldots (a + n - 1) \). If we let the coefficients in the series (2.1.30) be denoted by \( \alpha_n \), we get

\[
\frac{\alpha_{n+1}}{\alpha_n} = \frac{(n + a) (n + b)}{(n + c)}
\]

The subscripts 2 and 1 in the notation \( {}_2F_1(a, b; c; x) \) come from the respective degrees of the numerator and denominator in this ratio.

We will derive recurrence relations for \( {}_2F_1(a, b; c; x) \) by the method of Darboux-Bäcklund transformations. Many of these relations are called *contiguity relations* because they relate two hypergeometric functions having nearby (=contiguous) parameter values. Note that there are now several parameters and we expect different transformations to change the parameters differently.

As a function of \( x \), \( {}_2F_1(a, b; c; x) \) satisfies the differential equation

\[
x (1 - x) y'' + (c - (a + b + 1) x) y' - a b y = 0.
\tag{2.1.31}
\]

Our previous experience suggests that a Bäcklund transformation of the form

\[
\begin{align*}
\left[ a(x) \partial_x + b(x) \right] w &= \tilde{w}, \\
\left[ c(x) \partial_x + d(x) \right] \tilde{w} &= w.
\end{align*}
\]
should work. However this leads to rather complicated equations for the functions \(a, b, c, d\).
Let us instead consider the ansatz

\[
\begin{align*}
\frac{x^{\varphi_1}(1-x)^{\varphi_2}}{\kappa x^{\varphi_3}(1-x)^{\varphi_4}} \left( \frac{\partial_x - \frac{\mu x + \nu}{2x(x-1)}}{\frac{\rho x + \sigma}{2x(x-1)}} \right) w &= \tilde{w} \\
\kappa x^{\varphi_3}(1-x)^{\varphi_4} \left( \frac{\partial_x + \frac{\rho x + \sigma}{2x(x-1)}}{\frac{\mu x + \nu}{2x(x-1)}} \right) \tilde{w} &= w,
\end{align*}
\]

(2.1.32)

which is sufficiently general to allow us to derive several contiguity relations.

In order to recover (2.1.31) after eliminating \(\tilde{w}\) we must set

\[
\varphi_1 = c + \frac{1}{2}(\sigma - \nu), \quad \varphi_2 = a + b + \frac{1}{2}(\mu - \rho) - \varphi_1 + 1, \quad \varphi_3 = -\varphi_1 + 2 - \omega_1, \quad \varphi_4 = -\varphi_2 + 2 - \omega_2,
\]

where \(\omega_i\) are two integers satisfying \(0 \leq \omega_1 + \omega_2 \leq 2\). With different choices of \(\omega_i\) we can derive different relations. We will furthermore restrict to cases where the exponents \(\varphi_i\) are all integers.

For example, if \(\omega_1 = 0, \omega_2 = 1\) we can obtain, among other, Bäcklund transformation pairs that only change \(a\) (or \(b\)). If we take \(\mu = -2a, \nu = 2a, \rho = 2b, \sigma = 2(a - c + 1), \kappa = -1/(a(a - c + 1))\) we get the pair

\[
\begin{align*}
\frac{1}{a-c+1} [x(x-1)\partial_x + a - c + 1 + bx] y(a+1,b,c;x) &= ay(a+1,b,c;x) \\
[1-(x-1)\partial_x - c] y(a+1,b,c+1;x) &= \frac{a(b-c)}{c} y(a+1,b,c+1;x).
\end{align*}
\]

(2.1.33)

(The normalization \(\tilde{w} = ay(a+1,b,c;x)\) can be derived from the first terms of the power series expansion (2.1.30).) After shifting \(a \mapsto a - 1\) in the second equations and then eliminating \(y'(a,b,c;x)\) we get one of the Gauss’ relations for contiguous functions (AS 15.2.10)

\[
a(x-1)_2 F_1(a+1,b,c;x) + ((b-a)x+2a-c)_2 F_1(a,b,c;x) \\
+ (c-a)_2 F_1(a-1,b,c;x) = 0.
\]

(2.1.34)

Another transformation is obtained with \(\omega_1 = 1, \omega_2 = 0, \mu = -2a, \nu = 0, \rho = 2b, \sigma = -2c, \kappa = 1/(a(b - c)):\)

\[
\begin{align*}
\frac{1}{a-c+1} [x(x-1)\partial_x - a] y(a,b,c;x) &= \frac{a(b-c)}{c} y(a+1,b,c+1;x), \\
\frac{1}{a-c+1} [x(x-1)\partial_x + (c-bx)] y(a+1,b,c+1;x) &= y(a,b,c;x).
\end{align*}
\]

(2.1.35)

After eliminating the derivative terms, we get the difference equation

\[
a(b-c)_2 F_1(a+1,b,c+1;x) + c(x-a-b+c-1)_2 F_1(a,b,c;x) \\
+ c(1-c)_2 F_1(a-1,b,c-1;x) = 0.
\]

(2.1.36)

The difference equations (2.1.34,2.1.36) involve three terms on a line in the 3D-space of the parameters \(a, b, c\). One can also have corner relations. For example by taking the first equations of the pairs (2.1.33,2.1.35) and eliminating the derivative between them one obtains (AS 15.2.20 with \(a \mapsto a + 1\))

\[
(b-c)_2 F_1(a+1,b,c+1;x) + c(x-1)_2 F_1(a+1,b,c;x) \\
+ c_2 F_1(a,b,c;x) = 0.
\]

(2.1.37)

Numerous other contiguity relations can be derived this way.
2.2 Bäcklund transformations for non-linear PDEs

Many non-linear integrable PDEs possess Bäcklund transformations and their consistency condition leads to integrable nonlinear partial difference equations. In this section we will mainly discuss the Korteweg–de Vries (KdV) equation, which is given by the PDE:

\[ u_t = u_{xxx} + 6uu_x . \]  

(2.2.38)

This so-called nonlinear evolution equation was derived in 1895 by the two people whose names it bears, in a study on shallow water waves and where an exact “solitary wave” solution was presented. The equation was a key milestone in a big controversy on the nature of waves, not least following the famous “real life” observation of a solitary wave by John Scott Russell in 1834. It is not the place here to recite the whole history of the soliton, which can be found in several of the existing monographs on solitons and integrable systems, see e.g. [Ablowitz & Segur, 1982; Calogero & Degasperis, 1982; Newell, 1985; Drazin & Johnson, 1989; Ablowitz & Clarkson, 1991]. We just mention that the KdV equation was revived 70 years after Korteweg and de Vries’ paper, when in a celebrated study by C. Gardner, J. Greene, M. Kruskal and R. Miura, it was shown that nonlinear PDE (2.2.38) can be exactly solved by an ingenious method, which is nowadays referred to as the inverse scattering transform method. Although this method is only applicable to very special equations, which we refer to as soliton equations or exactly integrable equations, we now know entire infinite families of such equations to which the method can be applied to find exact solutions of the nonlinear equations (this being in stark contrast with the generic situation that nonlinear PDEs in the general case cannot be exactly solved and that typically we have to resort to either qualitative studies or perturbative and numerical methods to study their solutions).

2.2.1 Lax pair for KdV

A very important property of this equation is there is an underlying overdetermined system of linear equations:

\[ \psi_{xx} + u\psi = \lambda \psi , \]  

(2.2.39a)

\[ \psi_t = 4\psi_{xxx} + 6u\psi_x + 3u_x \psi , \]  

(2.2.39b)

whose consistency condition leads to (2.2.38). The first equation (2.2.39a) has the form of a linear spectral problem for the differential operator

\[ \mathcal{L} = \partial_x^2 + u , \]

where the coefficient \( u = u(x,t) \) plays the role of a potential. The parameter \( \lambda \) is an eigenvalue of the operator \( \mathcal{L} \) and can, in principle, depend on \( t \) if \( u \) depends on it, however, by definition, it is independent of \( x \).
The second equation (2.2.39b) describes the (linear) time-evolution of the function $\psi(x,t)$, where again the same $u$ enters in the coefficients. The system (2.2.39) is overdetermined: the two linear equations can only be compatible with each other if additional conditions hold for the coefficients, which are all expressed in terms of $u$.

**Theorem 2.2.1.** Under the assumption of isospectrality, i.e., $\lambda_t = 0$, the linear system (2.2.39) is self-consistent, i.e., $(\psi_{xx})_t = (\psi_t)_{xx}$, iff either $\psi \equiv 0$ or the potential $u = u(x,t)$ obeys the Korteweg-de Vries (KdV) equation (2.2.38).

**Proof.** The proof is by direct computation, namely computing $(\psi_t)_{xx} - (\psi_{xx})_t$ while back-substituting $\psi_{xx} = (\lambda - u)\psi$, from (2.2.39a). Using also $\lambda_t = 0$, one finds that the only remaining term is proportional to $(u_t - u_{xxx} - 6wu_x)\psi$.

**Exercise 2.2.1.** Perform the calculation in the above proof, which can be slightly simplified by rewriting (2.2.39b) as:

$$\psi_t = (4\lambda + 2u)\psi_x - u_x\psi.$$ 

The linear system of equations associated with a non-linear PDE is symptomatic of its integrability through the inverse scattering method. Such a linear system, consisting of a spectral problem and an equation for the time evolution, is called a **Lax pair**, after P.D. Lax who gave a systematic framework for describing such linear problems in his celebrated paper [Lax,1968]. Although for virtually all soliton equations Lax pairs have been found, there is no fully algorithmic method known to produce a Lax pair for a given equation.

### 2.2.2 Miura transformation

The KdV equation possesses a remarkable transformation, called the Miura transformation after its inventor, which gives rise to many insights about its solutions, and in particular can be used to derive a Bäcklund transformation for KdV. To find it, consider how we can eliminate the function $u$ from the system (2.2.39) and obtain a PDE in terms of the “eigenfunction” $\psi$ itself.

From (2.2.39a) we find

$$u = \lambda - \psi_{xx}/\psi,$$

and inserting this into (2.2.39b) we obtain the following equation for $\psi$:

$$\psi_t = \psi_{xxx} - 3\frac{\psi_x\psi_{xx}}{\psi} + 6\lambda\psi_x.$$  \hfill (2.2.40)

Introducing the variable:

$$v := \partial_x \log \psi,$$  \hfill (2.2.41)

we easily obtain from (2.2.39a) the **Miura transformation**

$$u = \lambda - v_x - v^2.$$  \hfill (2.2.42)

Furthermore, after taking derivatives w.r.t. $x$ on both sides of (2.2.40) and expressing all terms using $v$ we get a PDE governing $v$:

$$v_t = v_{xxx} - 6v^2v_x + 6\lambda v_x.$$  \hfill (2.2.43)
The latter equation (for $\lambda = 0$) is known as the modified KdV equation (MKdV), and it differs from the KdV equation (2.2.38) notably in the nonlinear term. The differential substitution (2.2.42) allows one to find a solution of the KdV equation given a solution of the MKdV equation: if $v$ solves the MKdV (2.2.43) and $u$ is defined by (2.2.42), then $u$ solves KdV (2.2.38).

### 2.2.3 The Bäclund transformation

Let us now turn to the derivation of Bäclund transformations using the above. We start with the Miura transformation (2.2.42), and combine it with the simple observation that the equation (2.2.43) is invariant under the replacement $v \mapsto -v$. The idea is to use one sign in transforming from $u$ to $v$ and another in transforming from $v$ to $\tilde{u}$, that is, we will have the Miura transformations

\begin{align*}
  \tilde{u} &= \lambda + v_x - v^2, \\
u &= \lambda - v_x - v^2.
\end{align*}

(2.2.44a)

(2.2.44b)

It is surprising that the trivial transformation $v \mapsto -v$ implies a highly nontrivial transformation $u \mapsto \tilde{u}$ on the solutions of the KdV equation.

Adding and subtracting the two relations above we obtain

\begin{align*}
  \tilde{u} + u &= 2(\lambda - v^2) \\
  \tilde{u} - u &= 2v_x.
\end{align*}

(2.2.45a)

(2.2.45b)

The latter can be integrated if we introduce the variable $w$ by taking $u = w_x$. For the KdV equation this change of variables leads to

\begin{equation}
  w_t = w_{xxx} + 3w_x^2,
\end{equation}

(2.2.46) after one integration in $x$. (Note that we have omitted an irrelevant integration constant.) This equation for $w$ is called the potential KdV equation (PKdV).

In terms of this new dependent variable the equation (2.2.45b) can be integrated to $\tilde{w} - w = 2v$, and inserting it into the first relation (2.2.45a) we obtain

\begin{equation}
  (\tilde{w} + w)_x = 2\lambda - \frac{1}{2} (\tilde{w} - w)^2.
\end{equation}

(2.2.47a)

written entirely in terms of $w$.

Equation (2.2.47a) provides us with the $x$-dependent part of the Bäclund transformation. To fully characterize the solution $\tilde{w}$ we need also a $t$-dependent equation, which can be readily found by using the PKdV equation itself. Adding (2.2.46) for $w$ and $\tilde{w}$ and using (2.2.47a) to reduce $w_{xxx} + \tilde{w}_{xxx}$ we obtain the relation

\begin{equation}
  (\tilde{w} + w)_t = (\tilde{w} - w)(w_{xx} - \tilde{w}_{xx}) + 2(w_x^2 + w_x \tilde{w}_x + \tilde{w}_x^2),
\end{equation}

(2.2.47b)

and the relations (2.2.47a), (2.2.47b) together constitute the Bäclund transformation for the KdV equation. (Note that in practice, one could use the PKdV equation itself rather than (2.2.47b) to implement the BT.)

**Exercise 2.2.2.** Differentiating (2.2.47a) by $t$ and (2.2.47b) by $x$ and subtracting the results, show that the two parts of the BT guarantee that if $w$ is a solution of (2.2.46), it follows that $\tilde{w}$ is again a solution of (2.2.46).
2.2.4 Using BTs to generate multisoliton solutions

Note that the Bäcklund pair (2.2.47) is different from the one we had before in that it contains a parameter that does not appear at all in its base equation (2.2.46). This parameter can be used to generate more complicated solutions from simpler ones.

Suppose we know a given “seed solution” solution \( w \) of the PKdV, then inserting this into (2.2.47a) we obtain a first order nonlinear ODE for \( w \). This ODE will always be of the form:

\[
\tilde{w}_x = -\frac{1}{2} \tilde{w}^2 + a(x)\tilde{w} + b(x),
\]

where the right-hand side is a quadratic in \( \tilde{w} \) (with \( x \)-dependent coefficients). This is a well-known type of differential equation called a Riccati equation. These equations are generally solvable through a linearisation procedure. After solving this equation we have some integration constants that may depend on \( t \), they can be determined from (2.2.47b).

Example: As a specific example, consider the simplest case where the seed solution \( w \) of the PKdV equation (2.2.46) is the trivial solution \( w \equiv 0 \). Setting \( w \equiv 0 \) in (2.2.47a) yields

\[
\tilde{w}_x = 2\lambda - \frac{1}{2} \tilde{w}^2,
\]

which can be integrated by separation of variables and yields

\[
\tilde{w}(x, t) = 2k \tanh (kx + c(t)) , \quad \lambda = k^2.
\]

Substituting this expression into the (2.2.47b) (with \( w = 0 \)) reveals that we must take \( c_t = 4k^3t + c_0 \), were \( c_0 \) is a constant. Thus we have obtained the solution:

\[
\tilde{w}(x, t) = 2k \tanh (kx + 4k^3t + c_0).
\]

2.2.5 Permutability property of BTs

The solution we obtained above can now be regarded as the starting point for applying the BT once again to obtain yet another solution of the PKdV equation. Carrying this out further we can iteratively obtain an infinite sequence of increasingly complicated solutions of the same nonlinear PDE. The procedure of solving a Riccati equation at each stage obviously becomes increasingly more cumbersome as we go along. However, there is a powerful new ingredient that can be used to simplify the iteration, namely the permutability property of the BTs.

Suppose we want to compose two different BTs, one with a parameter \( \lambda \), as in (2.2.47), and one with another parameter, say \( \mu \), given by

\[
\begin{align*}
BT_\lambda : w &\rightarrow \tilde{w} \\
(\tilde{w} + w)_x &= 2\lambda - \frac{1}{2} (\tilde{w} - w)^2,
\end{align*}
\]

\[
\begin{align*}
BT_\mu : w &\rightarrow \hat{w} \\
(\hat{w} + w)_x &= 2\mu - \frac{1}{2} (\hat{w} - w)^2,
\end{align*}
\]

where we have used the notation \( \tilde{w}, \hat{w} \) to denote the solution obtained by applying the BT with parameter \( \lambda, \mu \), respectively.
There are now two ways to compose these BTs: either start with $BT_\lambda$ and and subsequently apply $BT_\mu$, or the other way around. In this way we get iterated solutions which we can denote by $\tilde{\tilde{w}}$ and $\tilde{\tilde{w}}$ respectively,

$$\tilde{\tilde{w}} = BT_\mu \circ BT_\lambda w, \quad \tilde{\tilde{w}} = BT_\lambda \circ BT_\mu w.$$ 

The highly nontrivial result is that both ways of composing BTs lead to the same result: $\tilde{\tilde{w}} = \tilde{\tilde{w}}$, and hence the two BTs commute. This is the famous permutable property of the BTs.

**Theorem 2.2.2.** The BTs given by (2.2.47) for different parameters $\lambda$ and $\mu$ commute, and hence we have the following commutation diagram of BTs:

\[ 
\begin{array}{ccc}
BT_\lambda & \stackrel{w}{\longrightarrow} & \tilde{w} \\
\downarrow & & \downarrow \\
BT_\mu & \stackrel{\tilde{w}}{\longrightarrow} & \tilde{\tilde{w}}
\end{array}
\]

$$BT_\mu : \tilde{w} \mapsto \tilde{\tilde{w}} \quad \left( \tilde{\tilde{w}} + \tilde{w} \right)_x = 2\mu - \frac{1}{2} \left( \tilde{\tilde{w}} - \tilde{w} \right)^2, \\
BT_\lambda : \tilde{w} \mapsto \tilde{\tilde{w}} \quad \left( \tilde{\tilde{w}} + \tilde{w} \right)_x = 2\lambda - \frac{1}{2} \left( \tilde{\tilde{w}} - \tilde{w} \right)^2. 
$$

(2.2.49c) (2.2.49d)

Now setting $\tilde{\tilde{w}} = \tilde{w}$, and eliminating all the derivatives from the four eqs. (2.2.49) we obtain a purely algebraic equation of the form:

$$\left( \tilde{\tilde{w}} - w \right) \left( \tilde{\tilde{w}} - \tilde{w} \right) = 4(\mu - \lambda).$$

(2.2.50)

This allows us to obtain directly the iterated BT transformed variable $\tilde{w}$ without having to derive the solution through the Riccati equations of the BT.

**Exercise 2.2.3.** Construct a 2-soliton solution by starting from the seed solution $w \equiv 0$, and two 1-soliton solutions of the type (2.2.48) (with different parameters $k$ and $l$, where $\lambda = k^2$, $\mu = l^2$, respectively) and solving for $\tilde{\tilde{w}}$ from (2.2.50). Note that the phases $c_0$ may be taken to be different for $\tilde{w}$ and $\tilde{\tilde{w}}$. Verify explicitly that the $\tilde{w}$ so constructed actually solves (2.2.46).
2.2.6 Bäcklund transformation for the sine-Gordon equation

BTs exist for other integrable evolution equations as well. In fact the first BT, the one proposed by Bäcklund himself, is associated with the sine-Gordon equation,

$$\theta_{xt} = \sin \theta,$$

(2.2.51)

For this equation the Bäcklund transformation \( \theta \xrightarrow{\lambda} \tilde{\theta} \) is given by the following relations:

$$\left( \tilde{\theta} - \theta \right)_x = 2\lambda \sin \left( \frac{\tilde{\theta} + \theta}{2} \right),$$

(2.2.52a)

$$\left( \tilde{\theta} + \theta \right)_t = \frac{2}{\lambda} \sin \left( \frac{\tilde{\theta} - \theta}{2} \right),$$

(2.2.52b)

connecting a variable \( \theta(x,t) \) to a new variable \( \tilde{\theta}(x,t) \). By calculating the \( t \)-derivative (2.2.52a) and the \( x \)-derivative (2.2.52b) and then taking a sum or difference, one can easily derive (2.2.51) for \( \tilde{\theta} \) or \( \theta \), respectively. Thus (2.2.52) is a one-parameter auto-Bäcklund transformation for the sine-Gordon equation.

As before, we can now introduce a second BT \( \theta \xrightarrow{\mu} \hat{\theta} \) of the form (2.2.52) with parameter \( \mu \), namely

$$\left( \hat{\theta} - \theta \right)_x = 2\mu \sin \left( \frac{\hat{\theta} + \theta}{2} \right),$$

(2.2.53a)

$$\left( \hat{\theta} + \theta \right)_t = \frac{2}{\mu} \sin \left( \frac{\hat{\theta} - \theta}{2} \right).$$

(2.2.53b)

We can also apply \( BT_{\mu} \) on \( \tilde{\theta} \), and \( BT_{\lambda} \) on \( \hat{\theta} \), and if among these 8 equations we eliminate all derivatives (under the assumption of the permutability of the BTs, i.e., \( \tilde{\theta} = \hat{\theta} \)), we obtain the following permutability property:

$$\sin \left( \frac{\tilde{\theta} + \tilde{\theta} - \hat{\theta} - \hat{\theta}}{4} \right) = \frac{\lambda}{\mu} \sin \left( \frac{\tilde{\theta} + \hat{\theta} - \tilde{\theta} - \hat{\theta}}{4} \right).$$

(2.2.54)

If, for simplification, we denote \( e^{\frac{1}{2}\theta} = w \), we can write (2.2.54) as

$$\lambda(\tilde{w}\tilde{w} - \hat{w}\hat{w}) = \mu(\tilde{w}\hat{w} - \tilde{w}\hat{w}).$$

(2.2.55)

**Exercise 2.2.4.** Starting from the trivial solution of the sine-Gordon equation \( \theta \equiv 0 \), use the BT (2.2.52) to obtain a solution \( \bar{\theta}(x,t) \) of the same equation containing the parameter \( \lambda \), namely:

$$\bar{\theta}(x,t) = 2\tan^{-1} \left\{ \exp \left( \lambda x + \frac{t}{\lambda} + \varphi_0 \right) \right\},$$

(2.2.56)

where \( \varphi_0 \) is a (constant) phase. (Hint: Use the integral

$$\int \frac{d\varphi}{\sin \varphi} = \ln \left( \tan \frac{\varphi}{2} \right) + c.$$
2.2.7 Transition to lattice equations

It is obvious from the above, that by iterating the BTs with two different parameters we obtain from one seed solution $w$ an entire lattice of solutions, see Figure 2.4. Note that building this lattice of solutions crucially depends on the validity of the permutability property!

We have derived the permutability equations (2.2.50) and (2.2.55) from the properties of the PKdV and SG, respectively, thus these equations are descriptive, they describe yet another property of the sequence of functions derived using BTs. We can introduce an enumeration of the solutions as follows:

$$w_{n,m} = BT^n \circ BT^m w,$$

(2.2.57)

after which we can write (2.2.50) and (2.2.55) as difference equations of the form:

$$(w_{n+1,m+1} - w_{n,m})(w_{n,m+1} - w_{n+1,m}) = 4(\mu - \lambda),$$

(2.2.58)

and

$$\lambda(w_{n+1,m+1}w_{n,m+1} - w_{n+1,m}w_{n,m}) = \mu(w_{n+1,m+1}w_{n+1,m} - w_{n,m+1}w_{n,m}),$$

(2.2.59)

respectively, the shifts along the lattice $w_{n,m} \mapsto w_{n+1,m}$ and $w_{n,m} \mapsto w_{n,m+1}$ corresponding to the application of the Bäcklund transformations $BT_\lambda$ and $BT_\mu$.

We will now change our point of view: at each elementary plaquette of the above lattice of solutions we have a relation of the form (2.2.58) or (2.2.59) (or something else for other equations), and we will now elevate these equations as being the main equations of interest. In a sense having reached this point we can “forget” about the original PDE’s from whence the construction originated, and place the permutability equations at the centre of our focus.

Later we will indeed consider lattice equations, i.e., PDE’s like (2.2.58) or (2.2.59) on their own merit and study their remarkable properties. In particular we will show that these equations are integrable in some precise sense.
2.2. BÄCKLUND TRANSFORMATIONS FOR NON-LINEAR PDES

Literature:


