Lecture A.1

Elliptic Functions

A.1 Apology

The excuse for these notes is the need I felt to collect together a concise number of formulae for elliptic functions in one coherent notation and from one constructive point of view. The idea is as much as possible to try to derive all possible identities from one single formula, or –where that is not possible– from a handful of formulae, thereby emphasising the connections between the various identities. In my view the subject of elliptic functions has suffered from the plethora of relations, identities and equations, the internal strcuture of which might be lost to all but the initiated. The idea behind these notes is that a lot of the theory of elliptic functions can actually be developed from only a handful of key identities, and memorising these are in principle sufficient to control the subject. In this respect a student might compare this with the situation of the trigonometric functions, where one single identity, e.g. the addition formula

$$\cos(x+y) = \cos x\, \cos y - \sin x\, \sin y$$

plus some information on the periodicities and analytic behaviour around 0, is sufficient to derive all trigonometric identities: addition formulae, differential relations, including the geometry of the underlying curve (i.e. the circle).

Obviously, the theory behind the elliptic functions is much richer and the number of "interesting" or "useful" relations is substantially larger than in the case of the trigonometric functions. Even more so, for the working mathematician, the need to put some order in the abundance of identities. Historically, the subject of elliptic functions has developed through various paths, stemming from the great fathers of the subject (Abel, Weierstrass, Jacobi, Frobenius, and many others), but unfortunately this has also resulted in a rather unsatisfactory abundance of notations which through history have taken their shape. Strangely, some of the notations in the theory of elliptic functions have not been really standardised: whereas the Jacobi functions sn, cn and dn have become standard, the related functions ds, cs, dc, nc and sc, cf. e.g. [Whittaker-Watson], have greatly dropped out of use. Similarly, the Weierstrassian functions σ, ζ and $\wp$ have become standard, but still at the time of Frobenius, cf. [Frobenius], and well into the beginning of the 20th century various different notations for these functions were in use. The situation with the theta functions is even much less satisfactory, where even today not all agree on the same notation (some people using θ_1, θ_2, θ_3, θ_4, whilst others use θ_{11}, θ_{10}, θ_{00}, θ_{01}, and some definitions include factors π in the argument), cf. e.g. [Mumford]. When we consider the situation in higher genus (hyperelliptic funcions, Abelian functions), the situation is even worse.

It has become a fine tradition in modern texts to build the theory of elliptic functions from the point of view of geometry. This being very satisfactory and beautiful for theorists (e.g. algebraic geometers) but it is less useful for practicioners (applied mathematicians, physicists, engineers), i.e. to those who really need to work with the formulae and identities. It is somewhat a tragedy that the subject of elliptic functions has been dropped from the regular curriculum, and consequently out of the conscience of the working mathematician, due to the in this respect negative influence of the far-reaching level of abstractness with which the subject has become endowed in the course of this century. Elliptic functions being seen nowadays as a subject for specialists only, it is a far cry from the times that every mathematics student was supposed to be able to work with elliptic functions with the same agility as with trigonometric functions. The erroneous impression has been created that elliptic functions have no bearing on "real-life problems" and problems of a practical nature has had the reverse effect that people nowadays only tend to think of those applied problems where they are not needed as realistic. Obviously this amounts to a dramatic impoverishment of the scope of our thnking, which flies in the face of progress: the truly modern problems in the applied sciences will inevitably require increasingly sophisticated tools (in particular increasingly more complicated special functions) not only to solve the problems but even to formulate the problems in the first place.

One of the areas where we have seen some modest revival of elliptic functions is notably in the theory of integrable systems. Actually, in the last few decades one has realised that many of the structures behind integrable systems share deep common features with the beautiful structure of the theory elliptic functions and elliptic curves. In a sense elliptic functions form a microcosm, a paradigm, for the wider theory of integrable systems.

A.2 Some Definitions

We need to give a definition of what an elliptic function is, but the aim of these notes is to avoid in the first place any theoretical groundwork that would trail us in the direction of the general theory of elliptic curves. That theory can be found in many textbooks, and I would consider the monographs by F. Kirwan and by McKean and Moll the best for a first, yet comprehensive, encounter. (The latter covers in a nice and pedestrian way a great variety of ramifications of the theory into other areas of pure mathematics.) The treatment I prefer here is closest to the one in the book by Akhiezer, namely very much a "constructive" approach highlighting very much the inteconnection between the various functions. In the treatment given here it is the *addition formulae* which will play a central role (and which connect it also to the main subject of the course, namely Discrete Integrable Systems).

Analytic and meromorphic functions

We will consider in what follows functions of a single complex variable (in addition to parameters which we consider to be fixed). We recall from complex function theory that a function $f : \mathcal{D} \to \mathbb{C}$, defined on some open domain $\mathcal{D}$ in the complex plane, is called *analytic* or *holomorphic* if the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} =: f'(z_0) \ ,$$

exists for all $z_0 \in \mathcal{D}$, in other words if f is differentiable in the complex sense in the domain $\mathcal{D}$. The complex derivative of a function (when it is applicable) follows very similar rules as the usual derivative of a real-valued function:

$$\frac{df}{dz} = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \ . \tag{A.2.1}$$

In fact, the usual rules such *sum rule, product rule, chain rule* apply very similarly to complex differentiation as well as to the usual differentiation. The big difference is that in the complex case, in taking the limit $h \to 0$ the variable h itself is complex valued, and hence is supposed to approach 0 from any direction in the complex plane requiring the limit $f'(z)$ to be well-defined and unique coming from all directions[1]. As a consequence, the condition of complex differentiability on a function f in the complex plane is much more restrictive than in the real-valued case. In fact, decomposing the function $f(z)$ in its real and imaginary part, and viewed as functions of the two real variables which are the real and imaginary part of z, namely:

$$f(z) = u(x,y) + iv(x,y) \quad , \quad z = x + iy \ ,$$

it can easily be shown that the complex differentiability at a given point $z \in \mathbb{C}$ implies the following conditions on the functions u and v, namely:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ , \tag{A.2.2}$$

which are called the *Cauchy-Riemann equations*. These, in turn, imply that both u and v obey Laplace's equation at the given point (x, y) corresponding to z, namely

$$\Delta u = \Delta v = 0 \quad , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ . \tag{A.2.3}$$

Another consequence of f being analytic in a domain $\mathcal{D} \subset \mathbb{C}$ is that in that domain f is automatically not only once complex differentiable, but actually can be differentiated an

[1] This can be formulated more precisely, by using the strict definition of the complex limit, namely:

$$\lim_{z\to z_0} g(z) = a$$

to mean that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $z \in D(z_0; r) - \{0\}$, ($D(z_0; r)$ denoting the disk of radius r in the complex plane centred at z_0) for some finite nonzero r, $|z - z_0| < \delta$ implies that $|g(z) - a| < \varepsilon$.

arbitrary number of times (i.e., not only $f'(z)$ exists, but als $f''(z)$, $f'''(z)$ and generally $f^{(n)} = \frac{d^n f}{dz^n}$ as well), and admits a *convergent* Taylor series expansion:

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \quad , \quad z_0 \in \mathcal{D} \; . \tag{A.2.4}$$

One of the main consequences of a function being analytic is *Cauchy's theorem*, which is one of the centre points of analytic function theory. This theorem states the following:

Theorem: If the function f is analytic in an open domain $\mathcal{D} \subset \mathbb{C}$, then on any closed curve Γ which is homotopic to a point[2] in $\mathcal{D}$ then the contour integral

$$\int_\Gamma f(z) dz = 0 \; .$$

A corollary of Cauchy's theorem is *Cauchy's integral formula* which states that for f analytic in a domain $\mathcal{D} \subset \mathbb{C}$ and Γ a closed curve homotopic to a point in $\mathcal{D}$, then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, dz \; , \tag{A.2.5}$$

for $z_0 \in \mathcal{D}$ in the interior of the curve Γ, and where the curve is parametrised in such a way that in the contour integral we go around the curve only once in the counter clock-wise direction. This formula tells us that all values of the analytic function f for points inside the curve can be found from knowing solely the values of f *on* the curve! In a similar way we have an integral formula for all the derivatives of the analytic function f as well.

Thus, analyticity (complex differentiability) is a very strong property. However, it can happen that complex functions are not everywhere in $\mathbb{C}$ differentiable. If a function is analytic everywhere in $\mathbb{C}$ then we call the function an *entire function*, examples being functions like:

$$e^z \quad , \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad , \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \; .$$

It can also happen that functions have singular behaviour at isolated points, such as:

$$\frac{1}{z} \quad , \quad \frac{1}{z - z_0} \quad , \quad \frac{1}{(z - z_0)^n} \quad (n \text{ integer}) \; .$$

These isolated points are called *poles* of the function, and usually these kind of singularities are somewhat benign. Worse singular behaviour happens if we consider functions such as $\log z$, $\sqrt{z}$ or z^p with p noninteger, in which case the maximal domains on which these fuctions can be regarded as analytic are the ones where we take out from $\mathbb{C}$ an entire infinite line (called a *branch cut*). Furthermore, in these cases the problem of *multivaluedness* arises, i.e. the problem that there are ambiguities arising in order to define the function as a single-valued object. We refer to the textbooks on analytic function theory for discussions on how to incorporate such functions into the theory.

[2]Homotopic to a point means that the curve can be *continuously* deformed (shrunk) to a point, i.e. with the shrinking curves being all entirely in the domain of interest.

Functions which are analytic apart from the appearance of isolated singularities (poles) are called *meromorphic functions*. If f is meromorphic in a domain $\mathcal{D}$ then in a neighborhood of any regular (nonsingular) point it will admit a Taylor expansion of the form (A.2.4), but more generally in the neighborhood of a pole z_0 it admits the more general *Laurent series expansion* which is of the form:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \ , \tag{A.2.6}$$

which converges uniformly on any *annulus* (annular region) surrounding z_0 where f is analytic. The coefficients in (A.2.6) are given by

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \quad , \quad n = \ldots, -1, 0, 1, 2, \ldots \ .$$

If $a_n = 0$ for $n < -p$, we say that z_0 is a pole *of order* p, and if $a_n = 0$ for $n < 0$ the function is analytic at z_0 (in which case the coefficient a_n for n positive equals $f^{(n)}(z_0)/n!$). The coefficient a_{-1} (if it exists) is called the *residue* of the function f at z_0. These values, denoted by $\operatorname*{Res}_{z=z_0} f(z)$, play a major role in the socalled *residue calculus*, which is a complex analytic technique (based on Cauchy's theorem) to evaluate definite integrals and infinite sums.

Periodic and multiply periodic functions

We will restrict ourselves from now on to meromorphic functions, which as explained above are functions having as only singularities poles (and zeroes, which can be considered as being singularities at $z = \infty$). Often it will be useful to consider function, rather than on $\mathbb{C}$, as functions on the *compactified complex plane* $\mathbb{C} \cup \{\infty\}$, where the point $z = \infty$, which is identified with the set $\{\zeta \in \mathbb{C} \,|\, |\zeta| \geq R \,, \text{ for all } R > 0\}$, is added to the complex plane.

Let f be a complex function, then if for any regular point z we have

$$f(z + \Omega) = f(z) \quad , \quad \Omega \in \mathbb{C} \text{ fixed and nonzero} \ ,$$

then f is called *periodic* with period Ω. Clearly, in that case we have $f(z + m\Omega) = f(z)$ for any integer m. If two functions f, g are periodic with the same period Ω, then pointwise sums and differences $f \pm g$, products fg and quotients f/g, and derivatives f', are periodic as well with the same period. Any integer multiple of a period Ω is also a period. A period Ω is called *primite* if any other period of the function is an integer multiple of the primitive period.

A *multiply periodic function* is a function that has more than one primitive periods, namely $\Omega_1, \Omega_2, \ldots, \Omega_n$, with

$$f(z + m_1\Omega_1 + \cdots + m_n\Omega_n) = f(z) \ ,$$

with the Ω_i all independent (meaning that the only nontrivial integral linear combination of these periods adding up to zero is the one with all coefficients equal to zero). The set of all points of the form $m_1\Omega + \cdots + m_n\Omega_n$, with the m_i integer, is called the *period lattice*.

An interesting question is whether functions exist with more than one nontrivial independent period. For $n = 1$ we know many examples, such as

$$f(z) = e^{2\pi i z/\Omega} ,$$

which is manifestly periodic with primitive period Ω. The answer is given by the following statement:

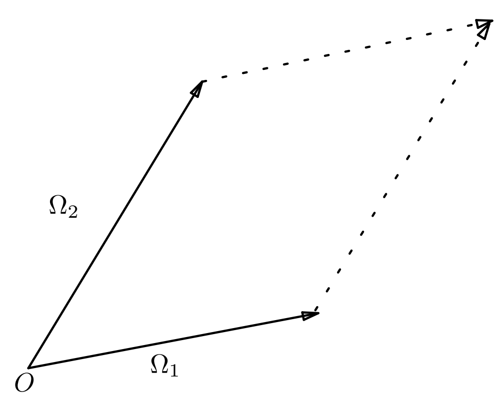
Proposition: *1.* There does *not* exist a nonconstant function with $n \geq 3$ primitive periods; *2.* there exist nonconstant functions with $n = 2$ given primitive periods iff the ratio of these periods is not real.

This statement is rephrasing of a theorem of Jacobi. A proof can be found in [Akhiezer].

Elliptic functions and elliptic curves

Definition: A complex function $f(z)$ is called an *elliptic function* if it is meromorphic and doubly periodic, i.e. it admits two independent primitive periods.

At least one of the two prinitive periods Ω_1, Ω_2 of an elliptic function should be complex since the ratio Ω_2/Ω_1 should be nonreal. Hence the form a parallellogram of the form:



and thus the complex plane can be tesselated by all the parallelograms formed by the period lattice obtained by translating this parallelogram over integer multiples of the two periods. The collection of the following statements are known as *Liouville's theorem.*

The theory of elliptic functions provides a number of general results which we list here without proof:

1. elliptic functions are fully characterised (up to a constant multiplicative factor) by their poles and zeroes, as well as their periods;

2. the sum of the residues with respect to all the poles inside a single parallelogram of the period lattice is zero;
3. there does not exist a nonconstant elliptic function that is regular in a period parallelogram;
4. the number of poles of an elliptic function in a period parallelogram counting multiplicity cannot be less than 2.

The elliptic functions are closely connected to a family of complex algebraic curves, called *elliptic curves*. These are curves which, in appropriate coordinates, can be cast in the form:

$$w^2 = \mathcal{R}(z) \ , \tag{A.2.7}$$

where $\mathcal{R}$ is a polynomial of order $p = 3$ or $p = 4$ in z, i.e.

$$\mathcal{R}(z) = \alpha z^3 + \beta z^2 + \gamma z + \delta \quad \text{or} \quad \mathcal{R}(z) = \alpha z^4 + \beta z^3 + \gamma z^2 + \delta z + \epsilon \ .$$

Although we can consider (A.2.7) for real variables, it is more useful to consider this equation for complex values of w and z. The elliptic functions arise as the functions in terms of which one can naturally parametrise such curves. We can compare this to the case where the order p of the polynomial $\mathcal{R}$ is $p \leq 2$ in which case we call the curve *rational*, and in which case the curve can be parametrised in terms of trigonometric functions. The simplest exanple of the latter is the case of the cicle:

$$w^2 + z^2 = 1 \quad \Leftrightarrow \quad w = \cos(t) \quad , \quad z = \sin(t) \ .$$

The elliptic functions are, thus, closely related to the geometry of the above mentioned elliptic curves. Although this would be the starting point for most treatments, it is not the one adopted in this course. We will take a constructive point of view where we will start with a definition of the functions involved in terms of explicit formulae.

A.3 Theta Functions

We define the theta functions with characteristic $a, b = 0, 1$ and modulus τ as follows:

$$\theta_{ab}(x|\tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi i\tau(n+\frac{a}{2})^2 + 2\pi i(n+\frac{a}{2})(x+\frac{b}{2})\right] \tag{A.3.1}$$

The sum on the r.h.s. being uniformly convergent for all $|x| \leq R$ and all $R > 0$ whenever $\Im\tau > 0$, (i.e. τ is in the Siegel half-plane $\tau \in \mathbb{H}$), the theta function is an entire function of $x \in \mathbb{C}$. It is doubly quasi-periodic with the periodicity relations being given by:

$$\theta_{ab}(x+1|\tau) = e^{\pi i a}\theta_{ab}(x|\tau) \quad , \quad \theta_{ab}(x+\tau|\tau) = e^{-\pi i(\tau+2x+b)}\theta_{ab}(x|\tau) \ . \tag{A.3.2}$$

The resummation $n \leftrightarrow -n$ in (A.3.1) leads to the relation

$$\theta_{ab}(-x|\tau) = \theta_{-a,-b}(x|\tau) = e^{\pi i(\tau a^2 - 2ax)}\theta_{ab}(x - a\tau|\tau) = e^{\pi i ab}\theta_{ab}(x|\tau) \ , \tag{A.3.3}$$

where the last two steps hold for b integer. Thus, it is immediately clear that θ_{00}, θ_{01}, θ_{10} are all even functions of its argument, but that θ_{11} is an odd function:

$$\theta_{00}(-x|\tau) = \theta_{00}(x|\tau) \ , \ \theta_{10}(-x|\tau) = \theta_{10}(x|\tau) \ , \ \theta_{01}(-x|\tau) = \theta_{01}(x|\tau) \ , \ \theta_{11}(-x|\tau) = -\theta_{11}(x|\tau) \ .$$

Fundamental Addition Formulae

We will now use the series (A.3.1) to obtain expressions for the products of two theta functions, namely

$$\begin{aligned}
&\theta_{ab}(x|\tau)\theta_{a'b'}(y|\tau) = \\
&= \sum_{n,m\in\mathbb{Z}} \exp\left\{\pi i\tau[(n+\frac{a}{2})^2+(m+\frac{a'}{2})^2]+2\pi i[(n+\frac{a}{2})(x+\frac{b}{2})+(m+\frac{a'}{2})(y+\frac{b'}{2})\right\} \\
&= \sum_{n,m\in\mathbb{Z}} \exp\left\{\frac{1}{2}\pi i\tau[(n+m+\frac{a+a'}{2})^2+(n-m+\frac{a-a'}{2})^2]\right. \\
&\qquad \left.+\pi i[(n+m+\frac{a+a'}{2})(x+y+\frac{b+b'}{2})+(n-m+\frac{a-a'}{2})(x-y+\frac{b-b'}{2})]\right\}
\end{aligned}$$

This invites the change of summation variables from n, m to $N = n+m$ resp. $M = n-m$. However, since $N+M = 2n \in 2\mathbb{Z}$ and $N-M = 2m \in 2\mathbb{Z}$ implying that either both N and M are even integers, or that they are both odd. Thus these replacements in the double sum lead to

$$\sum_{n,m\in\mathbb{Z}} \rightsquigarrow \sum_{\substack{N,M\in\mathbb{Z}\\ N,M \text{ even}}} + \sum_{\substack{N,M\in\mathbb{Z}\\ N,M \text{ odd}}}$$

Replacing N, M in the first sum by $2N$, $2M$ and in the second sum by $2N+1$, $2M+1$ we get

$$\begin{aligned}
\ldots \quad = \quad & \sum_{N,M\in\mathbb{Z}} \exp\left\{2\pi i\tau[(N+\frac{a+a'}{4})^2+(M+\frac{a-a'}{4})^2]\right. \\
& \qquad \left.+2\pi i[(N+\frac{a+a'}{4})(x+y+\frac{b+b'}{2})+(M+\frac{a-a'}{4})(x-y+\frac{b-b'}{2})]\right\} \quad \text{(A.3.4)} \\
& + \sum_{N,M\in\mathbb{Z}} \exp\left\{2\pi i\tau[(N+\frac{a+a'+2}{4})^2+(M+\frac{a-a'+2}{4})^2]\right. \\
& \qquad \left.+2\pi i[(N+\frac{a+a'+2}{4})(x+y+\frac{b+b'}{2})+(M+\frac{a-a'+2}{4})(x-y+\frac{b-b'}{2})\right\} \\
= \quad & \theta_{AB}(x+y|2\tau)\theta_{A'B'}(x-y|2\tau)+\theta_{A+1,B}(x+y|2\tau)\theta_{A'+1,B'}(x-y|2\tau) \quad \text{(A.3.5)}
\end{aligned}$$

with new characteristics:

$$\begin{aligned}
A &= \frac{a+a'}{2} \quad , \quad B = b+b' \\
A' &= \frac{a-a'}{2} \quad , \quad B' = b-b'
\end{aligned}$$

Clearly for $a, b, a', b' \in \mathbb{Z}_2$ to have integer characteristics we must take either $a = a' = 0$ or $a = a' = 1$, whilst b and b' can take on all values in $\mathbb{Z}_2$. This leads to the eight equations:

$$\begin{aligned}
\theta_{00}(x|\tau)\theta_{00}(y|\tau) &= \theta_{00}(x+y|2\tau)\theta_{00}(x-y|2\tau)+\theta_{10}(x+y|2\tau)\theta_{10}(x-y|2\tau) \\
\theta_{01}(x|\tau)\theta_{00}(y|\tau) &= \theta_{01}(x+y|2\tau)\theta_{01}(x-y|2\tau)+\theta_{11}(x+y|2\tau)\theta_{11}(x-y|2\tau) \\
\theta_{00}(x|\tau)\theta_{01}(y|\tau) &= \theta_{01}(x+y|2\tau)\theta_{01}(x-y|2\tau)-\theta_{11}(x+y|2\tau)\theta_{11}(x-y|2\tau) \\
\theta_{01}(x|\tau)\theta_{01}(y|\tau) &= \theta_{00}(x+y|2\tau)\theta_{00}(x-y|2\tau)-\theta_{10}(x+y|2\tau)\theta_{10}(x-y|2\tau)
\end{aligned}$$

as well as

$$\begin{aligned}
\theta_{10}(x|\tau)\theta_{10}(y|\tau) &= \theta_{10}(x+y|2\tau)\theta_{00}(x-y|2\tau)+\theta_{00}(x+y|2\tau)\theta_{10}(x-y|2\tau)\\
\theta_{11}(x|\tau)\theta_{10}(y|\tau) &= \theta_{11}(x+y|2\tau)\theta_{01}(x-y|2\tau)+\theta_{01}(x+y|2\tau)\theta_{11}(x-y|2\tau)\\
\theta_{10}(x|\tau)\theta_{11}(y|\tau) &= \theta_{11}(x+y|2\tau)\theta_{01}(x-y|2\tau)-\theta_{01}(x+y|2\tau)\theta_{11}(x-y|2\tau)\\
\theta_{11}(x|\tau)\theta_{11}(y|\tau) &= -\theta_{10}(x+y|2\tau)\theta_{00}(x-y|2\tau)+\theta_{00}(x+y|2\tau)\theta_{10}(x-y|2\tau)\ .
\end{aligned} \tag{A.3.6}$$

These identities will serve as the starting point from which all key identities for elliptic functions will be derived. Fortunately, these relations are not all independent, and neither are the four theta functions θ_{00}, θ_{10}, θ_{01}, θ_{11}. In fact, these functions are related through shifts over *half periods*, i.e. shifts over $\frac{1}{2}$, and $\frac{\tau}{2}$ in the argument. They are given by the relations

$$\theta_{a,b+1}(x|\tau)=\theta_{ab}(x+\frac{1}{2}|\tau) \quad , \quad \theta_{a+1,b}(x|\tau)=e^{\frac{1}{4}\pi i\tau+\pi i(x+\frac{b}{2})}\theta_{ab}(x+\frac{\tau}{2}|\tau) \quad . \tag{A.3.14}$$

Using these relations the fore-last equation in the list can be written as:

$$\begin{vmatrix} \theta_{11}(x|2\tau) & \theta_{11}(y|2\tau) \\ \theta_{01}(x|2\tau) & \theta_{01}(y|2\tau) \end{vmatrix} = \theta_{11}\left(\frac{x+y}{2}-\frac{1}{2}\,|\,\tau\right)\theta_{11}\left(\frac{x-y}{2}\,|\,\tau\right) \tag{A.3.15}$$

which is (A.3.24) for $N=2$, whilst the others can be deduced from (A.3.15) by appropriate shifts in the arument (shifts over τ and 1). Furthermore, taking $y=x+1$ we obtain from (A.3.15) the famous *Landen transform* relations, in particular:

$$\theta_{11}(x|\tau)=2e^{-\pi ix+\frac{1}{2}\pi i\tau}\frac{\theta_{11}(x|2\tau)\theta_{11}(x-\tau|2\tau)}{i\theta_{11}(-\frac{1}{2}|\tau)} \tag{A.3.16}$$

connecting elliptic functions of one set of period to the ones with one of the periods halved.

Below we present a general proof of the identities (A.3.15) and (A.3.28) which allows us to identify the proportionality factor $C_N(\tau)$ in (A.3.24). Below we show that from (A.3.6) all well-known identities between elliptic functions of the same modulus (periods) can be derived, either in the Jacobian representation or in the Weierstrass representation. Also all five-term theta-function relations can be found, such as:

$$\begin{aligned}
&2\theta_{00}(x)\theta_{00}(y)\theta_{00}(z)\theta(w) = \\
&\quad =\theta_{00}(X)\theta_{00}(Y)\theta_{00}(Z)\theta_{00}(W)+\theta_{01}(X)\theta_{01}(Y)\theta_{01}(Z)\theta_{01}(W)\\
&\quad +\theta_{10}(X)\theta_{10}(Y)\theta_{10}(Z)\theta_{10}(W)+\theta_{11}(X)\theta_{11}(Y)\theta_{11}(Z)\theta_{11}(W)
\end{aligned} \tag{A.3.17}$$

in which all theta functions are of the same modulus τ and where we have abbreviated:

$$X=\frac{1}{2}(x+y+z+w) \quad , \quad Y=\frac{1}{2}(x+y-z-w)$$

$$Z=\frac{1}{2}(x-y+z-w) \quad , \quad W=\frac{1}{2}(x-y-z+w)$$

using the oddness of the θ_{11} one actually has a closed-form equation for this function, namely:

$$\theta_{11}(x+y)\theta_{11}(x-y)\theta_{11}(z+w)\theta_{11}(z-w)+\theta_{11}(x+z)\theta_{11}(x-z)\theta_{11}(w+y)\theta_{11}(w-y)$$
$$+\theta_{11}(x+w)\theta_{11}(x-w)\theta_{11}(y+z)\theta_{11}(y-z)=0 \qquad \text{(A.3.18)}$$

The theta function relation (A.3.18) constitutes the key identity which we will use for the constitution of the Weierstrass family of elliptic functions in the next section.

Product Formulae

One of the most important type of identities in combinatorics are the ones that relate infinite sums to infinite products, e.g. the famous Rogers-Ramanujan identities. Many of these identities have an origin in the representation theory of infinite-dimensional Lie algebras. One of the most fundamental of such relations is the *Jacobi triple product relation*, which can be most easily formulated as:

$$\sum_{n\in\mathbb{Z}} q^{n(n-1)/2}(-z)^n = \prod_{j=1}^{\infty}(1-q^{j-1}z)(1-q^j z^{-1})(1-q^j) \quad , \quad |q|<1 \ . \qquad \text{(A.3.19)}$$

Proof: The proof is elementary. If we call the r.h.s. $\vartheta(z;q)$, it is easy to see that it converges uniformly for all z within a finite radius whenever $|q|<1$. By reordering the factors it is clear that the following q-difference equation holds:

$$\vartheta(z;q) = -z\vartheta(qz;q)$$

Thus, if we assume the expansion

$$\vartheta(z;q) = \sum_{n\in\mathbb{Z}} c_n z^n$$

we obtain the recursion relation $c_n = -q^{n-1}c_{n-1}$ for the coefficients, which leads to $c_n = (-1)^n q^{n(n-1)/2}c_0$ upon iteration. The coefficient $c_0 = c_0(q)$ can be determined by considering the series and product for $z = iq^{1/2}$, in which case we get

$$\prod_{j=1}^{\infty}(1+q^{2j-1})(1-q^j) = c_0(q)\sum_{k\in\mathbb{Z}}(-1)^k q^{2k^2} \ ,$$

whilst if we take $z = q^{1/2}$ we get

$$\prod_{j=1}^{\infty}(1-q^{j-1/2})^2(1-q^j) = c_0(q)\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{1}{2}k^2} \ ,$$

a comparison of which leads to the relation

$$\frac{c_0(q^4)}{c_0(q)} = \frac{\prod_{j=1}^{\infty}(1-q^{4j-2})^2(1-q^{4j})}{\prod_{j=1}^{\infty}(1+q^{2j-1})(1-q^j)} = 1$$

where the last equality is a consequence of the surprising relation

$$\prod_{j=1}^{\infty}(1-q^j) = \prod_{k=1}^{\infty}(1-q^{2k-1})(1-q^{2k}) = \prod_{k=1}^{\infty}(1-q^{2k-1})(1+q^k)(1-q^k)$$

$$\Rightarrow \quad \prod_{k=1}^{\infty}(1-q^{2k-1})(1+q^k) = 1 \ .$$

Thus it follows that $c_0(q) = c_0(0) = 1$, since if $q = 0$ the l.h.s. of (A.3.19) reduces to $1 - z$, whilst the r.h.s. only gets contributions from $j = 0, 1$. Thus, we obtain the required identity. Q.E.D.

Eq. (A.3.19) is a confirmation of the fact that $\vartheta(z; q)$ is an entire function of z with simple zeroes at $z = q^j$, $j \in \mathbb{Z}$. We mention that the product

$$\eta(q) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j) \tag{A.3.20}$$

which enters in the coefficient of (A.3.19) plays an important role in combinatorics and is known as the *Dedekind η-function*. A classic result by Euler relates the infinite product in (A.3.20) to the infinite sum $\sum_{k\in\mathbb{Z}}(-1)^k \exp((3k^2 + k)/2)$, whilst its inverse serves as the generating function of partitions.

It is clear that by taking

$$z = e^{2\pi i x} \quad , \quad q = e^{2\pi i \tau}$$

the function $\vartheta(z; q)$ reduces to the θ_{11} function (apart from a factor). Thus, the triple product relation (A.3.19) directly yields the various expressions for the theta functions in terms of infinite products.

Theta functions of rational characteristic

We introduce now so-called theta function *of rational characteristic*, i.e. when the characteristics (the labels of the theta functions) are given as fractions of a natural number N, namely we introduce:

$$\vartheta_j(x) = \sum_{n\in\mathbb{Z}} \exp\left[N\pi i\tau(n+\frac{1}{2}-\frac{j}{N})^2 + 2\pi i(n+\frac{1}{2}-\frac{j}{N})(x+\frac{1}{2})\right] \tag{A.3.21}$$

for N positive integer and $j = 0, 1, \ldots, N$, which obeys the periodicity conditions

$$\vartheta_j(x+1) = -e^{-2\pi i(j/N)}\vartheta_j(x) \quad , \quad \vartheta_j(x+N\tau) = -e^{-\pi i(N\tau+2x)}\vartheta_j(x) \ . \tag{A.3.22}$$

Collecting such theta function in a matrix of the form

$$\Theta(\boldsymbol{x}) = \begin{pmatrix} \vartheta_0(x_1) & \vartheta_0(x_2) & \ldots & \vartheta_0(x_N) \\ \vartheta_1(x_1) & \vartheta_1(x_2) & \ldots & \vartheta_1(x_N) \\ \vdots & \vdots & & \vdots \\ \vartheta_{N-1}(x_1) & \vartheta_{N-1}(x_2) & \ldots & \vartheta_{N-1}(x_N) \end{pmatrix} \tag{A.3.23}$$

where $\boldsymbol{x} = (x_1, \ldots, x_N)$, the following remarkable identity holds

$$\det(\Theta(\boldsymbol{x})) = C_N(\tau)\theta_{11}\left(\frac{x_1 + \cdots + x_N}{N} - \frac{N-1}{2} \mid \tau\right) \prod_{i<j=1}^{N} \theta_{11}\left(\frac{x_i - x_j}{N} \mid \tau\right) \qquad \text{(A.3.24)}$$

Proof: The proof of eq. (A.3.24) is remarkably simple. Defining $f(\boldsymbol{x}) = f(x_1, \ldots, x_N) = \det(\Theta(\boldsymbol{x}))$, whilst $g(\boldsymbol{x}) = g(x_1, \ldots, x_N)$, denoting the product of theta functions on the r.h.s. of eq. (A.3.24), we use the periodicity conditions (A.3.22) to establish that

$$\begin{aligned} f(\ldots, x_j + N, \ldots) &= (-1)^N f(\ldots, x_j, \ldots)\ , \\ f(\ldots, x_j + N\tau, \ldots) &= -e^{-\pi i(2x_j + N\tau)} f(\ldots, x_j, \ldots) \end{aligned}$$

On the other hand, using the periodicity conditions (A.3.2) for θ_{11} we find exactly the same periodicity conditions for the functions $g(x_1, \ldots, x_N)$ in all its arguments:

$$\begin{aligned} g(\ldots, x_j + N, \ldots) &= (-1)^N g(\ldots, x_j, \ldots)\ , \\ g(\ldots, x_j + N\tau, \ldots) &= -e^{-\pi i(2x_j + N\tau)} g(\ldots, x_j, \ldots) \end{aligned}$$

Noting further that

$$\vartheta_j(x) = e^{\pi i\tau(j^2/N) - 2\pi i(x+\frac{1}{2})j/N}\theta_{11}(x - j\tau|N\tau)\ , \qquad \text{(A.3.25)}$$

it follows that the zeroes of $\vartheta_j(x)$ appear at $x = j\tau \bmod(\Lambda_N)$, where $\Lambda_N = \{n + Nm\tau \mid n, m \in \mathbb{Z}\}$ is the lattice generated by $1, N\tau$. This implies that $\vartheta_0(x)$,...,$\vartheta_{N-1}(x)$ do not have simultaneously a zero, the only zero occurring if $x_i - x_j = 0 \bmod(\Lambda_N)$, for any $i, j = 1, \ldots, N$, i.e. if $(x_i - x_j)/N = 0 \bmod(\Lambda_1)$. Concludingly, both $f(\boldsymbol{s}x)$ and $g(\boldsymbol{s}x)$ are doubly periodic entire functions with the same periodicities in each argument and having the same zeroes, hence by Liouville's theorem they must be proportional to eachother. Q.E.D.

Although strangely enough the fundamental relation (A.3.24) seems not to have appeared in any of the classic textbooks on elliptic functions, (see list of references), expressions in terms of theta functions of one argument as products of theta functions of different moduli do occur in the text (cf. e.g. [Tannery & Molk], nos. 134-140), and these are known as *transformation rules of order N*. They can seen to be simple consequences of the fundamental relation (A.3.24), as follows. Let us take

$$x_j = x + (j-1)\quad ,\quad j = 1, \ldots, N$$

and use the periodicity relations (A.3.22) as well the relation (A.3.25) from which we have that

$$\vartheta_{j+1}(x) = e^{\pi i\tau/N - 2\pi i(x+1/2)/N}\vartheta_j(x-\tau) \ \Rightarrow\ \vartheta_j(x) = e^{\pi i\tau j^2/N - 2\pi i(x+1/2)(j/a)N}\theta_{11}(x - j\tau|N\tau)$$

and using also $\vartheta_{N-j}(x) = \vartheta_j(-x-1)$ we get that

$$C_N(\tau)\left[\prod_{k<l}\theta_{11}\left(\frac{k-l}{N}|\tau\right)\right]\theta_{11}(x|\tau) = \qquad \text{(A.3.26)}$$

$$= \det\left\{e^{\pi i\tau j^2/N - 2\pi i(jk/N)}(-1)^{k+j}\right\}_{j,k=0,\ldots,N-1} e^{-\pi i(N-1)x}\prod_{j=0}^{N-1}\theta_{11}(x - j\tau|N\tau)$$

This equation generalises the Landen transform (A.3.16) to a general Nth order transformation between theta functions of modulus τ and theta functions of modulus $N\tau$.

A further consequence of (A.3.24), which in fact can be considered to be an elliptic analogue of the famous VanderMonde determinant formula, we find using Cramer's rule from linear algebra:

$$\left[\Theta^{-1}(u - N\boldsymbol{x}) \cdot \Theta(u - N\boldsymbol{y})\right]_{ij} = \Psi_\kappa(x_i - y_j)\frac{\prod_{k=1}^{N} \theta_{11}(x_k - y_j)}{\prod_{k \neq i} \theta_{11}(x_k - x_i)} \ , \tag{A.3.27}$$

where $\kappa = u - \left(\sum_{i=1}^{N} x_i\right) - \frac{N-1}{2}$, and

$$\Psi_\kappa(x) = \frac{\theta_{11}(x + \kappa)}{\theta_{11}(x)\theta_{11}(\kappa)} \ ,$$

the θ_{11} functions on the r.h.s. all being of modulus τ. The theta function relation (A.3.18) as well as the relation (A.3.27), the determinant of which leads to the famous Frobenius formula, (A.4.15) below, constitutes the key identities for the Weierstrass family of functions, cf. section 4 below.

The general proof given above of (A.3.24) does not provide us with the proportionality constant C_N. A constructive proof can be given for low values of N, as is clear from the case $N = 2$. Proceeding along similar lines for $N = 3$ we obtain the formula:

$$\theta_{11}\left(\frac{x+y+z}{3} \mid \tau\right)\theta_{11}\left(\frac{x-y}{3} \mid \tau\right)\theta_{11}\left(\frac{x-z}{3} \mid \tau\right)\theta_{11}\left(\frac{y-z}{3} \mid \tau\right) =$$
$$= -e^{\pi i/3}\vartheta_1(0)\begin{vmatrix} \vartheta_0(x) & \vartheta_0(y) & \vartheta_0(z) \\ \vartheta_1(x) & \vartheta_1(y) & \vartheta_1(z) \\ \vartheta_2(x) & \vartheta_2(y) & \vartheta_2(z) \end{vmatrix} \tag{A.3.28}$$

Proof: This case reveals truly the complex combinatorics that is sitting behind eq. (A.3.24). In this case the theta functions of rational characteristic from (A.3.21) we will be using are:

$$\vartheta_0(x) = \sum_{n\in\mathbb{Z}} \exp\left[2\pi i(n+\frac{1}{2})(x+\frac{1}{2}) + 3\pi i\tau(n+\frac{1}{2})^2\right] \tag{A.3.29a}$$

$$\vartheta_1(x) = \sum_{n\in\mathbb{Z}} \exp\left[2\pi i(n+\frac{1}{6})(x+\frac{1}{2}) + 3\pi i\tau(n+\frac{1}{6})^2\right] \tag{A.3.29b}$$

$$\vartheta_2(x) = \sum_{n\in\mathbb{Z}} \exp\left[2\pi i(n-\frac{1}{6})(x+\frac{1}{2}) + 3\pi i\tau(n-\frac{1}{6})^2\right] \tag{A.3.29c}$$

We now investigate the product

$$\theta_{11}\left(\frac{x_1+x_2+x_3}{3}\mid\tau\right)\theta_{11}\left(\frac{x_1-x_2}{3}\mid\tau\right)\theta_{11}\left(\frac{x_1-x_3}{3}\mid\tau\right)\theta_{11}\left(\frac{x_2-x_3}{3}\mid\tau\right)=$$
$$=\sum_{\boldsymbol{n}\in\mathbb{Z}^4}\exp\left\{2\pi i\left[(n_0+\frac{1}{2})(\frac{x_1+x_2+x_3}{3}+\frac{1}{2})+(n_{12}+\frac{1}{2})(\frac{x_{12}}{3}+\frac{1}{2})\right.\right.$$
$$\left.+(n_{13}+\frac{1}{2})(\frac{x_{13}}{3}+\frac{1}{2})+(n_{23}+\frac{1}{2})(\frac{x_{23}}{3}+\frac{1}{2})\right]$$
$$\left.+\pi i\tau\left[(n_0+\frac{1}{2})^2+(n_{12}+\frac{1}{2})^2+(n_{13}+\frac{1}{2})^2+(n_{23}+\frac{1}{2})^2\right]\right\}$$
$$=\sum_{\boldsymbol{n}\in\mathbb{Z}^4}\exp\left\{2\pi i\left[(x_1+\frac{3}{2})(\frac{n_0+n_{12}+n_{13}}{3}+\frac{1}{2})+(x_2+\frac{1}{2})(\frac{n_0-n_{12}+n_{23}}{3}+\frac{1}{6})\right.\right.$$
$$\left.+(x_3-\frac{1}{2})(\frac{n_0-n_{13}-n_{23}}{3}-\frac{1}{6})+\frac{1}{2}(\frac{n_{12}-n_{13}+n_{23}}{3}+\frac{1}{6})\right]$$
$$+\frac{1}{3}\pi i\tau\left[(n_0+n_{12}+n_{13}+\frac{3}{2})^2+(n_0-n_{12}+n_{23}+\frac{1}{2})^2\right.$$
$$\left.\left.+(n_0-n_{13}-n_{23}-\frac{1}{2})^2+(n_{12}-n_{13}+n_{23}+\frac{1}{2})^2\right]\right\}$$

where for notational convenience we use as summation variables the integers n_0, n_{ij}, $(i,j=1,2,3)$, constituting a four-fold sum over the vector of integers $\boldsymbol{n}=(n_0,n_{12},n_{13},n_{23})$, and where we abbreviate $x_{ij}\equiv x_i-x_j$. The last way of rewriting the product invites a change of summation variables, namely

$$\left.\begin{array}{r}N_1=n_0+n_{12}+n_{13}\\N_2=n_0-n_{12}+n_{23}\\N_3=n_0-n_{13}-n_{23}\\N_0=n_{12}-n_{13}+n_{23}\end{array}\right\}\quad\Rightarrow\quad\left\{\begin{array}{l}N_1+N_2+N_3\in3\mathbb{Z}\\N_1-N_2+N_0\in3\mathbb{Z}\\N_1-N_3-N_0\in3\mathbb{Z}\\N_2-N_3+N_0\in3\mathbb{Z}\end{array}\right.$$

which means that the new summation variables actually are represented as:

$$N_i=-\alpha_i+3m_i\quad,\quad m_i\in\mathbb{Z}\quad,\quad\alpha_i\in\mathbb{Z}_3$$

with the α_i are subject to the conditions

$$\begin{array}{ll}\alpha_1+\alpha_2+\alpha_3=0\,(\mathrm{mod}\ 3)\quad,&\alpha_1+\alpha_2+\alpha_0=0\,(\mathrm{mod}\ 3)\\\alpha_1-\alpha_3-\alpha_0=0\,(\mathrm{mod}\ 3)\quad,&\alpha_2-\alpha_3+\alpha_0=0\,(\mathrm{mod}\ 3)\end{array}\qquad(\text{A.3.30})$$

In terms of these new variables α_i,m_i, $(i = 0, 1, 2, 3)$ the sum above can be rewritten as

$$\begin{aligned}
\ldots &= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_3^4}{}' \sum_{\boldsymbol{m}\in\mathbb{Z}^4} \exp\Big\{2\pi i(x_1+\frac{1}{2}+1)(m_1+\frac{1}{2}-\frac{\alpha_1}{3})+3\pi i\tau(m_1+\frac{1}{2}-\frac{\alpha_1}{3})^2 \\
&\qquad 2\pi i(x_2+\frac{1}{2})(m_2+\frac{1}{2}-\frac{\alpha_2+1}{3})+3\pi i\tau(m_2+\frac{1}{2}-\frac{\alpha_2+1}{3})^2 \\
&\qquad 2\pi i(x_3+\frac{1}{2}-1)(m_3+\frac{1}{2}-\frac{\alpha_3+2}{3})+3\pi i\tau(m_3+\frac{1}{3}-\frac{\alpha_3+2}{3})^2 \\
&\qquad 2\pi i(\frac{1}{2})(m_0+\frac{1}{2}-\frac{\alpha_0+1}{3})+3\pi i\tau(m_0+\frac{1}{2}-\frac{\alpha_0+1}{3})^2\Big\} \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{Z}_3^4}{}' \vartheta_{\alpha_1}(x_1+1)\vartheta_{\alpha_2+1}(x_2)\vartheta_{\alpha_3+2}(x_3-1)\vartheta_{\alpha_0+1}(0)
\end{aligned}$$

where the prime in the sum over the α_i denotes that we perform a restricted sum under the condition (A.3.30). Noting that $\vartheta_\alpha(0) = 0$ for $\alpha = 0 \,(\mathrm{mod}\, 3)$ in the present case, in the sum over α_0 we have only contributions from $\alpha_0 = 0, 1$, leading to

$$\begin{aligned}
\ldots = {}& \vartheta_1(0)\,[\vartheta_0(x_1+1)\vartheta_1(x_2)\vartheta_2(x_3-1)+\vartheta_1(x_1+1)\vartheta_2(x_2)\vartheta_0(x_3-1) \\
&\qquad +\vartheta_2(x_1+1)\vartheta_0(x_2)\vartheta_1(x_3-1)] \\
&+\vartheta_2(0)\,[\vartheta_0(x_1+1)\vartheta_2(x_2)\vartheta_1(x_3-1)+\vartheta_1(x_1+1)\vartheta_0(x_2)\vartheta_2(x_3-1) \\
&\qquad +\vartheta_2(x_1+1)\vartheta_1(x_2)\vartheta_0(x_3-1)]
\end{aligned}$$

where use has been made of the condition (A.3.30), and when we make use of the periodicity conditions

$$\vartheta_0(x\pm 1) = -\vartheta_0(x) \quad , \quad \vartheta_1(x\pm 1) = e^{\pm\pi i/3}\vartheta_1(x) \quad , \quad \vartheta_2(x\pm 1) = e^{\mp\pi i/3}\vartheta_2(x)$$

together with the fact that $\vartheta_2(0) = e^{-\pi i/3}\vartheta_1(0)$, an easy calculation demonstrates that we recover eq. (A.3.28). QED.

A.4 The Weierstrass Family of Elliptic Functions

Here, we collect some useful formulae for the Weierstrass elliptic functions. The Weierstrass sigma-function is basically the θ_{11} apart from a multiplicative factor and a scaling in its argument. It is the latter theta function for which we have a closed form addition formula, namely (A.3.18)), so understandibly this function plays the main role in the theory.

Thus, we define the so-called Weierstrass σ-function to be given by

$$\sigma(z|2\omega_1, 2\omega_2) = 2\omega_1 \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\theta_{11}(x|\tau)}{\theta_{11}'(0|\tau)} \quad , \quad \tau = \frac{\omega_2}{\omega_1} \quad , \quad z = 2\omega_1 x\ . \tag{A.4.1}$$

The exponential prefactor and scaling are motivated in order to make the function *automorphic* under the action of the group $SL(2,\mathbb{Z})$, i.e. invariant under linear transformations acting on the period lattice:

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \mapsto \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \tag{A.4.2}$$

The relations between the Weierstrass elliptic functions are given by

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} \quad , \quad \wp(z) = -\zeta'(z) \ , \tag{A.4.3}$$

where $\sigma(z)$ and $\zeta(z)$ are odd functions and $\wp(z)$ is an even function of its argument. We recall also that the $\sigma(z)$ is an entire function, and $\zeta(z)$ is a meromorphic function having simple poles at ω_{kl}, both being quasi-periodic, obeying

$$\zeta(x + 2\omega_{1,2}) = \zeta(x) + 2\eta_{1,2} \quad , \quad \sigma(x + 2\omega_{1,2}) = -\sigma(x)e^{2\eta_{1,2}(x+\omega_{1,2})} \ , \tag{A.4.4}$$

in which

$$\eta_i \equiv \zeta(\omega_i) \quad , \quad i = 1, 2 \ .$$

The relation between periods and the η_i is given by

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2} \ .$$

The function $\wp(z)$ is doubly periodic, and it is only the $\wp$ function that is truly elliptic according to the definition of an elliptic function.

As a consequence of the product formula for the theta functions we obtain the following product formula foe the Weierstrass sigma-function:

$$\sigma(z) = z \prod_{(k,\ell)\neq(0,0)} (1 - \frac{z}{\omega_{k\ell}}) \exp\left[\frac{z}{\omega_{k\ell}} + \frac{1}{2}(\frac{z}{\omega_{k\ell}})^2\right] \ , \tag{A.4.5a}$$

with $\omega_{kl} = 2k\omega_1 + 2\ell\omega_2$. Taking the logarithmic derivatives we obtain from (A.4.5a) the following double-sum expansions for the Weierstrass ζ- and $\wp$ functions:

$$\zeta(z; 2\omega_1, 2\omega_2) = \frac{1}{z} + \sum_{(k,\ell)\neq(0,0)} \left[\frac{1}{z + \omega_{k,\ell}} - \frac{1}{\omega_{k,\ell}} + \frac{z}{\omega_{k,\ell}^2}\right] \ , \tag{A.4.5b}$$

$$\wp(z; 2\omega_1, 2\omega_2) = \frac{1}{z^2} + \sum_{(k,\ell)\neq(0,0)} \left[\frac{1}{(z + \omega_{k,\ell})^2} - \frac{1}{\omega_{k,\ell}^2}\right] \ . \tag{A.4.5c}$$

From a computational point of view, the most important property of these elliptic functions is the existence of a number of functional relations, the most fundamental being the three-term relation for the σ-function

$$\begin{aligned} &\sigma(x + y)\sigma(x - y)\sigma(a + b)\sigma(a - b) = \\ &= \sigma(x + a)\sigma(x - a)\sigma(y + b)\sigma(y - b) - \sigma(x + b)\sigma(x - b)\sigma(y + a)\sigma(y - a) \ , \end{aligned} \tag{A.4.6}$$

which is a direct consequence of the similar relation for the θ_{11} functions (A.3.18). From this functional equation one can derive by differentiation:

$$\zeta(x) + \zeta(y) + \zeta(z) - \zeta(x + y + z) = \frac{\sigma(x + y)\sigma(x + z)\sigma(y + z)}{\sigma(x)\sigma(y)\sigma(z)\sigma(x + y + z)} \ . \tag{A.4.7}$$

In fact, all identities for the Weiertrass functions as functions of their arguments can be derived from these functional relations.

Now it will be convenient to express the addition formulae in terms of a function $\Phi_\kappa(x)$ (with κ some complex number) given by

$$\Phi_\kappa(x) \equiv \frac{\sigma(x+\kappa)}{\sigma(x)\sigma(\kappa)} \ , \tag{A.4.8}$$

which is similar to the function Ψ used in the previous section[3].

Eq. (A.4.6) can also be cast into the following form

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa)+\zeta(x)+\zeta(y)-\zeta(\kappa+x+y)\right] \ , \tag{A.4.9}$$

The following three-term relation for $\sigma(x)$ is a consequence of (A.4.7) and this equation can be cast into the following convenient form

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\Phi_\lambda(y-x) \ , \tag{A.4.10}$$

which is obtained from the elliptic analogue of the partial fraction expansion, i.e. eq. (A.4.9).

From (A.4.7) various other addition formulae can be derived by simple manipulations, taking into account that these are functional identities valid for all values of the arguments (away from the singularities of the functions). Thus, by differentiation we can derive:

$$\zeta(x+y)-\zeta(x)-\zeta(y) = \frac{1}{2}\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)} \ . \tag{A.4.11}$$

Furthermore, we have the relation

$$\wp(x)+\wp(y)+\wp(x+y) = \frac{1}{4}\left(\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right)^2 \ . \tag{A.4.12}$$

Exercise: Derive eqs. (A.4.11) and (A.4.12).

Eq. (A.4.12) is the well-known addition formula for the Weierstrass $\wp$ function. Finally, we get the following remarkable addition formula:

$$\frac{\sigma(x-y)\sigma(x+y)}{\sigma^2(x)\sigma^2(y)} = \wp(y)-\wp(x) \ , \tag{A.4.13}$$

and the fundamental 3-term relation form σ (A.4.6) can be easily seen to arise as the consistency condition for the functional relation (A.4.13), by rewriting the trivial identity

$$(\wp(x)-\wp(y))(\wp(a)-\wp(b)) = (\wp(x)-\wp(a))(\wp(y)-\wp(b)). - (\wp(x)-\wp(b))(\wp(y)-\wp(a))$$

in terms of σ-functions. Thus, we have come full circle: from the three-term relation for σ we have derived a sequel of addition formulae at the end of which we recover the three-term relation itself!

[3] In the literature sometimes a different form for $\Phi_k(x)$ is introduced, namely

$$\Phi(x;\kappa) := \frac{\sigma(x-\kappa)}{\sigma(x)\,\sigma(\kappa)}\, e^{\zeta(\kappa)x} \ ,$$

which is oftern referred to as the *Lamé function*. We prefer to work with the form given in (A.4.8).

Higher degree addition formulae

We finish the list of addition formulae by mentioning a number of higher order addition formulae, which are most conveniently expressed in terms of the function Φ. The bilinear relation (A.4.10) for the function Φ can be generlised to multilinear equations by simple iteration. Thus we obtain e.g. the trilinear relation:

$$\Phi_\kappa(x)\Phi_\lambda(y)\Phi_\mu(z) = \Phi_{\kappa+\lambda+\mu}(x)\Phi_\lambda(y-x)\Phi_\mu(z-x) + \Phi_\kappa(x-y)\Phi_{\kappa+\lambda+\mu}(y)\Phi_\mu(z-y) + \Phi_\kappa(x-z)\Phi_\lambda(y-z)\Phi_{\kappa+\lambda+\mu}(z) . \tag{A.4.14a}$$

The latter relations can be iterated to higher degree relations, the simplest of which is of the form which in the limit $z \to y$ yields the relation

$$\Phi_\kappa(x)\Phi_\lambda(y)\Phi_\mu(y) - \Phi_{\kappa+\lambda+\mu}(x)\Phi_\lambda(y-x)\Phi_\mu(y-x) = = \Phi_\kappa(x-y)\Phi_{\kappa+\lambda+\mu}(y)\left[\zeta(\lambda)+\zeta(\mu)+\zeta(y)-\zeta(\kappa+\lambda+\mu+y)+\zeta(\kappa+x-y)-\zeta(x-y)\right] , \tag{A.4.14b}$$

which can be conveniently rewritten as

$$\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y) = = \frac{\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_{\kappa+\lambda+\mu}(y)-\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_{\kappa+\lambda+\mu}(x)}{\Phi_{\kappa+\lambda+\mu}(x+y)\left(\wp(x)-\wp(y)\right)} . \tag{A.4.14c}$$

In the double limit $y,z \to x$ yields the relation

$$\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x) = \frac{1}{2}\Phi_{\kappa+\lambda+\mu}(x)\Big\{ \left[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)-\zeta(\kappa+\lambda+\mu+x)\right]^2 - \left[\wp(\kappa)+\wp(\lambda)+\wp(\mu)+\wp(\kappa+\lambda+\mu+x)-\wp(x)\right] \Big\}. \tag{A.4.14d}$$

In a similar fashion we can derive increasingly higher degree multilinear relations (all of which can be easily proven by induction). These relations all follow from general elliptic determinant type relations, mostly due to Frobenius and Stickelberger, which we will explore now.

Elliptic Determinantal Identities

Possibly the most important and surprising identity in the whole theory is the elliptic version of the well-known Cauchy determinantal identity, which reads

$$\det\left(\Phi_\kappa(x_i-y_j)\right) = \Phi_\kappa(\Sigma)\sigma(\Sigma)\frac{\prod_{k<\ell}\sigma(x_k-x_\ell)\sigma(y_\ell-y_k)}{\prod_{k,\ell}\sigma(x_k-y_\ell)} \quad , \quad \text{where } \Sigma \equiv \sum_i (x_i-y_i) , \tag{A.4.15}$$

and which is due to Frobenius, [Frobenius,1882].

Proof: A proof of the elliptic Cauchy formula (A.4.15) can be given by induction using the following general block determinantal formula:

$$\left|\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{b} \\ \hline \boldsymbol{c}^t & d \end{array}\right| = d^{1-N}\det\left\{d\boldsymbol{A}-\boldsymbol{b}\,\boldsymbol{c}^t\right\} , \tag{A.4.16}$$

where $\boldsymbol{A}$ is an $N \times N$ matrix and where $\boldsymbol{b}$ and $\boldsymbol{c}$ a N-component column- and row vector respectively, and where $d \neq 0$. Applying this formula to a $(N+1)\times(N+1)$ matrix of the Frobenius form, namely

$$\det\left(\Phi_\kappa(x_i - y_j)\right)_{i,j=1,\dots,N+1}$$

and exploiting the addition formula (A.4.9) and (A.4.10) the induction step can be verified. Furthermore for $N = 2$, the determinantal formula is readily verified using the same equations. ■

Corollary: The elliptic Cauchy determinantal formula (A.4.15) is closely related to another identity found by Frobenius and Stickelberger, [Frobenius & Stickelberger,1880], which is of the form

$$\begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & (\zeta(x_i - y_j)) & \\ 1 & & & \end{vmatrix} = -\sigma(\Sigma)\frac{\prod_{k<\ell}\sigma(x_k - x_\ell)\sigma(y_\ell - y_k)}{\prod_{k,\ell}\sigma(x_k - y_\ell)} \ . \tag{A.4.17}$$

Exercise: By taking the limit $\kappa \to 0$ in (A.4.15) derive the formula (A.4.17).

From (A.4.15), by expanding along one of its rows or columns, an elliptic form of the Lagrange interpolation formula is obtained in the form

$$\prod_{i=1}^N \frac{\sigma(\xi - x_i)}{\sigma(\xi - y_i)} = \sum_{i=1}^N \Phi_{-\Sigma}(\xi - y_i)\frac{\prod_{j=1}^N \sigma(y_i - x_j)}{\prod_{\substack{j=1\\ j\neq i}}^N \sigma(y_i - y_j)} \ , \tag{A.4.18}$$

for $\Sigma \neq 0$, where

$$\Sigma \equiv \sum_{i=1}^N (x_i - y_i) \ . \tag{A.4.19}$$

When $\Sigma = 0$ we recover the following formula

$$\prod_{i=1}^N \frac{\sigma(\xi - x_i)}{\sigma(\xi - y_i)} = \sum_{i=1}^N \left[\zeta(\xi - y_i) - \zeta(x - y_i)\right]\frac{\prod_{j=1}^N \sigma(y_i - x_j)}{\prod_{\substack{j=1\\ j\neq i}}^N \sigma(y_i - y_j)} \ , \tag{A.4.20}$$

in which x denotes any one of the zeroes x_i. Note that in this case the left hand side is a meromorphic function on the elliptic curve as a consequence of Abel's theorem. Using (A.4.18) it can be easily verified that eq. (A.4.20) is independent of the choice of x. In fact, this follows from the key property that

$$\sum_{i=1}^N \frac{\prod_{j=1}^N \sigma(y_i - x_j)}{\prod_{\substack{j=1\\ j\neq i}}^N \sigma(y_i - y_j)} = 0, \tag{A.4.21}$$

whenever $\sum_i (x_i - y_i) = 0$. This latter relation (A.4.21) is nothing else than a rewriting of (A.4.18).

Finally, we give the expression for the inverse of the elliptic Cauchy matrix, namely

$$\left[(\Phi_\kappa(x_\cdot - y_\cdot))^{-1}\right]_{ij} = \Phi_{\kappa+\Sigma}(y_i - x_j)\frac{P(y_i)Q(x_j)}{Q_1(y_i)P_1(x_j)} \ , \tag{A.4.22}$$

(with Σ as before), in terms of the elliptic polynomials

$$P(\xi) = \prod_{k=1}^{N} \sigma(\xi - x_k) \quad , \quad Q(\xi) = \prod_{k=1}^{N} \sigma(\xi - y_k) \ ,$$

and

$$P_1(x_j) = \prod_{k\neq j} \sigma(x_j - x_k) \, , \qquad Q_1(y_i) = \prod_{k\neq i} \sigma(y_i - y_k) \, . \tag{A.4.23}$$

Equation (A.4.22) can be derived using (A.4.18) and (A.4.20).

Finally, there are a number of alternative determinantal formulae that can be expressed in terms of the $\wp$ function rather than in terms of the function Φ, and we give them without proof. Most celebrated is the Frobenius-Stickelberger formula:

$$\begin{vmatrix} 1 & \wp(x_0) & \wp'(x_0) & \cdots & \cdots & \wp^{(n-1)}(x_0) \\ 1 & \wp(x_1) & \wp'(x_1) & \cdots & \cdots & \wp^{(n-1)}(x_1) \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots & \cdots & \wp^{(n-1)}(x_n) \end{vmatrix} =$$

$$= \quad (-1)^{\frac{1}{2}n(n-1)} 1!2!\cdots n! \, \frac{\sigma(x_0 + x_1 + \cdots + x_n)\prod_{i<j=0}^{n} \sigma(x_i - x_j)}{\sigma^{n+1}(x_0)\,\sigma^{n+1}(x_1)\cdots\sigma^{n+1}(x_n)} \ . \tag{A.4.24}$$

Denoting the Frobenius-Stickelberger *matrix* by $\mathcal{P}(x_0, x_1, \ldots, x_n) = \mathcal{P}(\boldsymbol{x})$, given by the matrix associated with the left-hand side of (A.4.24), we have the following factorisation formula:

$$\left[\mathcal{P}(\boldsymbol{x})\cdot\mathcal{P}(\boldsymbol{y})^{-1}\right]_{i,j} = \frac{1}{\sigma^{n+1}(x_i)}\,\Phi_\Sigma(x_i - y_j)\sigma^{n+1}(y_j)\frac{\prod_{l=0}^{n}\sigma(x_i - y_l)}{\prod_{l\neq j}\sigma(y_j - y_l)} \ , \tag{A.4.25}$$

in which $\Sigma \equiv \Sigma_{l=0}^{n}(x_l - y_l)$. Thus, we can derive the determinantal formula (A.4.15) from (A.4.24) and vice versa.

Landen Transforms

A number of transformation properties of the Weierstrass functions relate functions of different periods together. The famous *Landen transforms* relate elliptic functions of a given period to those of twice (or vice versa half) the period. They are obtained from the fundamental relation (A.3.16) for the θ functions. Translating them to the Weierstrass family by using the correspondence (A.4.2) we obtain the following relations

$$\sigma(z) \quad = \quad e^{-\frac{1}{2}(e_1+\bar{e}_2)z^2+\bar{\eta}_2 z}\bar{\sigma}(z)\frac{\bar{\sigma}(z-2\omega_2)}{\bar{\sigma}(-2\omega_2)} \tag{A.4.26a}$$

$$\eta(z) \quad = \quad -(e_1+\bar{e}_2)z + \bar{\eta}_2 + \bar{\zeta}(z) + \bar{\zeta}(z-2\omega_2) \tag{A.4.26b}$$

$$\wp(z) \quad = \quad e_1 + \bar{e}_2 + \bar{\wp}(z) + \bar{\wp}(z-2\omega_2) \tag{A.4.26c}$$

in which $\sigma(z) = \sigma(z|2\omega_1, 2\omega_2)$, $\bar{\sigma}(z) = \sigma(z|2\omega_1, 4\omega_2)$, and $\zeta(z)$, $\wp(z)$ the corresponding Weierstrass functions with periods $2\omega_1, 2\omega_2$, respectively $\bar{\zeta}(z)$ and $\bar{\wp}(z)$ the corresponding Weierstrass functions with periods $2\omega_1, 4\omega_2$.

A.5 Half-period functions and the elliptic curve

Since the Weierstrass functions are periodic or quasi-periodic with primitive period For convenience let us introduce as a third half period $\omega_3 = -\omega_1 - \omega_2$, for which $\eta_3 \equiv \zeta(\omega_3) = -\eta_1 - \eta_2$. We then have also the relations:

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2} \quad \Rightarrow \quad \eta_2\omega_3 - \eta_3\omega_2 = \frac{\pi i}{2} \quad , \quad \eta_3\omega_1 - \eta_1\omega_3 = \frac{\pi i}{2} \ .$$

Introducing now the three functions:

$$W_i(x) \equiv \Phi_{\omega_i}(x) e^{-\eta_i x} \quad , \quad i = 1, 2, 3 \tag{A.5.1}$$

we can derive several relations from the addition formulae (A.4.9) and (A.4.10) for Φ as well using the periodicity conditions (A.4.4).

The following properties hold for the functions W_i:

1. The $W_i(x)$ are periodic with period $2\omega_i$, but picking up a minus sign w.r.t. the other periods:
$$W_i(x + 2\omega_i) = W_i(x) \quad , \quad W_i(x + 2\omega_j) = -W_i(x) \quad , \quad i, j, k = 1, 2, 3 \ \text{cyclic.} \tag{A.5.2}$$
2. The W_i are *odd* functions w.r.t. their argument:
$$W_i(-x) = -W_i(x) \ . \tag{A.5.3}$$
3. The addition formula:
$$W_i(x)W_j(z) + W_j(y)W_k(x) + W_k(z)W_i(y) = 0 \quad , \quad x + y + z = 0 \ . \tag{A.5.4}$$
follows by applying (A.4.10) together with the periodicity relations (A.4.4).
4. The following relations are a consequence of (A.4.13) together with the periodicity (A.4.4)
$$W_i^2(x) = \wp(x) - e_i \quad , \quad e_i \equiv \wp(\omega_i) \ , \tag{A.5.5}$$
which means that for fixed constants e_1, e_2, e_3 the (W_1, W_2, W_3) are coordinates on the intersection of three complex cylinders.
5. From (A.4.9) we obtain the relation
$$\frac{W_i(x)W_j(x)}{W_k(x)} = \eta_k + \zeta(x) - \zeta(x + \omega_k) = -\frac{1}{2}\frac{\wp'(x)}{\wp(x) - e_k} \quad , \quad i, j, k = 1, 2, 3 \quad \text{cyclic} \ . \tag{A.5.6}$$
6. Using the periodicity (A.4.4) we have
$$W_i(x)W_i(x + \omega_i) = -\frac{e^{\eta_i\omega_i}}{\sigma^2(\omega_i)} \ . \tag{A.5.7}$$

Exercise: Verify, by direct computation, the properties 1-6 of the functions W_i, using the addition formulae and periodicity relations of the Weierstrass functions.

We are now in a position to derive the most important property, nemaly the equation for the elliptic curve. In fact, combining (A.5.5) and (A.5.6), we get

$$W_1(x)W_2(x)W_3(x) = -\frac{1}{2}\wp'(x) \ , \tag{A.5.8}$$

and taking the square of this relation, using again (A.5.5), we obtain:

$$(\wp'(x))^2 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3) \ . \tag{A.5.9}$$

Using the fact that

$$e_1 + e_2 + e_3 = 0 \ ,$$

and taking as (complex) coordinates $(z, w) = (\wp(x), \wp'(x)$, we can cast (A.5.9) in the form:

$$w^2 = \mathcal{R}(z) = 4z^3 - g_2 z - g_3 \ , \tag{A.5.10}$$

which is the standard form of the so-called *Weierstrass curve*, which is one of the standard forms of an elliptic curve. The constants g_2, g_3 are given in terms of the e_i through the formulae:

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) \quad , \quad g_3 = 4e_1e_2e_3 \ , \tag{A.5.11}$$

and are called the *moduli of the curve.* The roots e_i, $i = 1, 2, 3$, of the cubic $\mathcal{R}(z)$ are called the *branch points* of the curve, and they are expressed through the formulae $e_i = \wp(\omega_i)$ in terms of the Weierstrass elliptic function.

We finish this account on Weierstrass functions by mentioning the relations:

$$\wp'(x) = -\frac{\sigma(2x)}{\sigma^4(x)} \quad , \quad \frac{1}{2}\frac{\wp''(x)}{\wp'(x)} = \zeta(2x) - 2\zeta(x) \ , \tag{A.5.12}$$

as well as the so-called duplication formulae:

$$\sigma(2x) = 2\frac{\sigma(x)\,\sigma(x+\omega_1)\,\sigma(x+\omega_2)\,\sigma(x+\omega_3)}{\sigma(\omega_1)\,\sigma(\omega_2)\,\sigma(\omega_3)} \ , \tag{A.5.13a}$$

$$2\zeta(2x) = \zeta(x) + \zeta(x+\omega_1) + \zeta(x+\omega_2) + \zeta(x+\omega_3) \ , \tag{A.5.13b}$$

$$4\wp(2x) = \wp(x) + \wp(x+\omega_1) + \wp(x+\omega_2) + \wp(x+\omega_3) \ . \tag{A.5.13c}$$

Exercise: Derive eqs. (A.5.12) from (A.4.13) by setting $y = x + \epsilon$ and taking the limit $\epsilon \to 0$, using $\sigma(\epsilon) \sim \epsilon$ in this limit. Subsequently, derive (A.5.13a) by using (A.5.8) and the others by differentiation.

Finally, we mention that by eliminating the derivatives from (A.4.12) using the relation of the elliptic curve (A.5.9), we obtain the purely "discrete" addition formula for the $\wp$ function, namely

$$\left(XY + XZ + YZ + \frac{g_2}{4}\right)^2 - (4XYZ - g_3)(X + Y + Z) = 0 \tag{A.5.14}$$

in which

$$X = \wp(x) \quad , \quad Y = \wp(y) \quad , \quad Z = \wp(z) \quad \text{such that} \quad x + y + z = 0 \ .$$

The *triquadratic* relation (A.5.14) can be viewed as a discretization of the formula for the elliptic curve: if we set $z = \varepsilon$ and take the limit $\varepsilon \to 0$, the dominant contribution in the relation (A.5.14) goes over into the defining equation for the Weierstrass curve in standard form.

A.6 Jacobi Elliptic Functions

The half-period functions of the latter chapter are up to a scaling equivalent to the other famous class of elliptic functions: the Jacobi class. In fact, by setting

$$u = \sqrt{e_1 - e_3}\, x \quad , \quad k^2 \equiv \frac{e_2 - e_3}{e_1 - e_3} \ , \tag{A.6.1}$$

we can introduce the functions

$$\mathrm{sn}(u;k) = \frac{\sqrt{e_1 - e_3}}{W_3(x)} \quad , \quad \mathrm{cn}(u;k) = \frac{W_1(x)}{W_3(x)} \quad , \quad \mathrm{dn}(u;k) = \frac{W_2(x)}{W_3(x)} \ . \tag{A.6.2}$$

Albeit this was not the way in which these functions, were introduced initially, in the context of the previous section it is a convenient way to make the connection with the Weierstrass class. In fact, we can rewrite (A.6.2) as

$$\mathrm{sn}(u;k) = \frac{\sqrt{e_1 - e_3}}{\sqrt{\wp(x) - e_3}} \quad , \quad \mathrm{cn}(u;k) = \sqrt{\frac{\wp(x) - e_1}{\wp(x) - e_3}} \quad , \quad \mathrm{dn}(u;k) = \sqrt{\frac{\wp(x) - e_2}{\wp(x) - e_3}} \ . \tag{A.6.3}$$

These functions are periodic with periods $4K$ and $4iK'$ in the argument u, where K and K' (the real and imaginary quarter periods) are given by

$$K = \sqrt{e_1 - e_3}\, \omega_1 \quad , \quad iK' = \sqrt{e_1 - e_3}\, \omega_2 \ ,$$

whereas the parameter k is called the modulus of the Jacobi elliptic functions. From the properties of the functions W_i, or in fact from (A.6.3) it follows that

$$\mathrm{sn}^2(u;k) + \mathrm{cn}^2(u;k) = 1 \quad , \quad k^2\mathrm{sn}^2(u;k) + \mathrm{dn}(u;k)^2 = 1 \ . \tag{A.6.4}$$

Various properties can be asserted for the Jacobi functions directly from the definitions above. In fact, it follows that sn is an *odd* function of u, whilst cn and dn are *even* functions:

$$\mathrm{sn}(u;k) = -\mathrm{sn}(u;k) \quad , \quad \mathrm{cn}(-u;k) = \mathrm{cn}(u;k) \quad , \quad \mathrm{dn}(-u;k) = \mathrm{dn}(u;k) \ , \tag{A.6.5}$$

whilst

$$\mathrm{sn}(0;k) = 0 \quad , \quad \mathrm{cn}(0;k) = \mathrm{dn}(0;k) = 1 \ , \tag{A.6.6}$$

and they allow the following series expansions:

$$\begin{aligned}
\mathrm{sn}(u;k) &= u-(1+k^2)\frac{u^3}{3!}+(1+14k^2+k^4)\frac{u^5}{5!}-\cdots && \text{(A.6.7a)}\\
\mathrm{cn}(u;k) &= 1-\frac{u^2}{2!}+(1+4k^2)\frac{u^4}{4!}-(1+44k^2+16k^4)\frac{u^6}{6!}+\cdots && \text{(A.6.7b)}\\
\mathrm{dn}(u;k) &= 1-k^2\frac{u^2}{2!}+k^2(4+k^2)\frac{u^4}{4!}-k^2(16+44k^2+k^4)\frac{u^6}{6!}+\cdots\ , && \text{(A.6.7c)}
\end{aligned}$$

Remark: It is illustrative to compare the expansions (A.6.7) with the series expansions for the trigonometric functions $\sin(u)$, $\cos(u)$, (including the constant function 1) by setting $k=0$.

From now on we shall consider the parameter k to be fixed and simply write $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ for the Jacobi functions, suppressing the second argument.

The (quasi-)periodicity properties for these functions follow equally from the properties 1 and 5 of the $W_i(x)$ as listed in the previous section, namely

$$\begin{aligned}
&\mathrm{sn}(2K-u)=\mathrm{sn}(u)\quad,\quad \mathrm{cn}(2K-u)=-\mathrm{cn}(u)\quad,\quad \mathrm{dn}(2K-u)=\mathrm{dn}(u)\ ,\\
&\mathrm{sn}(2iK'-u)=-\mathrm{sn}(u)\quad,\quad \mathrm{cn}(2iK'-u)=-\mathrm{cn}(u)\quad,\quad \mathrm{dn}(2iK'-u)=-\mathrm{dn}(u)\ ,
\end{aligned}$$

In fact, sn has periods $4K$ and $2iK'$, cn has periods $4K$ and $2K+2iK'$, whilst dn has periods $2K$ and $4iK'$. The zeroes of sn are located at $u=2mK+2niK'$, of cn at $u=(2m+1)K+2niK'$, whilst of dn they are at $u=(2m+1)K+2n+1)iK'$, for $n,m\in\mathbb{Z}$. Poles of all three functions are located at $u=2mK+(2n+1)iK'$, $n,m\in\mathbb{Z}$.

Differential relations follow from the differential relations for the $W_i(x)$. For instance, from

$$\frac{d}{dx}\ln W_3(x)=\zeta(x+\omega_3)-\zeta(x)-\eta_3=-\frac{W_1(x)W_2(x)}{W_3(x)}\qquad\Rightarrow$$
$$\Rightarrow\qquad \frac{d}{du}\ln\,\mathrm{sn}(u)=\frac{1}{\sqrt{e_1-e_3}}W_3(x)\,\mathrm{cn}(u)\,\mathrm{dn}(u)$$

and hence we obtain the differential equation for sn, and similarly for the others, namely

$$\frac{d}{du}\mathrm{sn}(u)=\mathrm{cn}(u)\,\mathrm{dn}(u)\quad,\quad \frac{d}{du}\mathrm{cn}(u)=-\mathrm{sn}(u)\mathrm{dn}(u)\quad,\quad \frac{d}{du}\mathrm{dn}(u)=-k^2\mathrm{sn}(u)\mathrm{cn}(u)\ . \tag{A.6.8}$$

We note that from the first differential relations (A.6.8), eliminating cn and dn, the sn function obeys the Jacobi differential equation

$$\left(\frac{ds}{du}\right)^2=\left(1-s^2\right)\left(1-k^2\,s^2\right)\ , \tag{A.6.9}$$

which is related to the *Jacobi curve*:

$$w^2=\mathcal{R}(z)=(1-z^2)(1-k^2z^2)\ . \tag{A.6.10}$$

Thus, alternatively we could have introduced the sn function by defining it through the inversion of the *elliptic integral*

$$u = \int_0^{\mathrm{sn}(u)} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \ , \tag{A.6.11}$$

which by the change of integration variables: $s = \sin\theta$, can also be written in the form:

$$u = \int_0^{\mathrm{am}(u)} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \ , \tag{A.6.12}$$

in which the upper integration limit is called the *amplitude function*. In terms of this function we have

$$\mathrm{sn}(u) = \sin(\mathrm{am}(u)) \ .$$

Clearly, if $k = 0$ we have $\mathrm{am}(u) = u$, and the sn function reduces to the usual sin function.

The quarter periods K and iK' can be obtained from the *complete elliptic integral of the first kind*:

$$K = \int_0^1 \left(1-s^2\right)^{-1/2} \left(1-k^2\,s^2\right)^{-1/2} ds, \tag{A.6.13}$$

with K' obtained from (A.6.13) by replacing k by $k' \equiv \sqrt{1-k^2}$. Furthermore, we have the *complete elliptic integrals of the second kind*:

$$E = \int_0^1 \left(1-s^2\right)^{-1/2} \left(1-k^2\,s^2\right)^{1/2} ds, \tag{A.6.14}$$

and similarly E' obtained by replacing k by k'. The relation between these quantities is given by

$$EK' + E'K - KK' = \frac{1}{2}\pi \ ,$$

paralleling the relation between η, η', ω and ω' in the Weierstrass case.

Addition formulae follow from (A.5.4), from which we can infer

$$\mathrm{cn}(v)\,\mathrm{sn}(u+v) = \mathrm{sn}(u)\,\mathrm{dn}(v) + \mathrm{dn}(u)\,\mathrm{sn}(v)\,\mathrm{cn}(u+v) \ , \tag{A.6.15a}$$
$$\mathrm{dn}(u)\,\mathrm{sn}(u+v) = \mathrm{cn}(u)\,\mathrm{sn}(v) + \mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(u+v) \ , \tag{A.6.15b}$$
$$\mathrm{sn}(u)\,\mathrm{cn}(u+v) + \mathrm{sn}(v)\,\mathrm{dn}(u+v) = \mathrm{cn}(u)\,\mathrm{dn}(v)\,\mathrm{sn}(u+v) \ . \tag{A.6.15c}$$

By supplementing these relations with the ones with u and v interchanged, one obtains several linear systems, from which $\mathrm{sn}(u+v)$, $\mathrm{cn}(u+v)$ and $\mathrm{dn}(u+v)$ can be solved in terms of the Jacobi functions with single arguments u or v, leading after some manipulation to the following addition formulae:

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v) + \mathrm{sn}(v)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{1-k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \ , \tag{A.6.16a}$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}(u)\,\mathrm{cn}(v) - \mathrm{sn}(u)\,\mathrm{dn}(u)\,\mathrm{sn}(v)\,\mathrm{dn}(v)}{1-k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \ , \tag{A.6.16b}$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}(u)\,\mathrm{dn}(v) - k^2\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{sn}(v)\,\mathrm{cn}(v)}{1-k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \ . \tag{A.6.16c}$$

Exercise: Derive the addition formula (A.6.16a) from the integral representation (A.6.11).

Proof: We will follow [Akhiezer,1980], §28, or [Whittaker/Watson,1927/88], p. 495, making use of the differential equation (A.6.9), from which we obtain immediately the following second order differential equation

$$\frac{d^2 s}{du^2} = 2\,k^2\,s^3 - (1+k^2)s\ . \tag{A.6.17}$$

Essentially, the proof of eq. (A.6.16a) is due to Euler (1756,1757), who considered a differential equation in the form:

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0\ ,$$

where X and Y are quartic polynomials of x, y respectively, leading to an integral of the form

$$\int \frac{dx}{\sqrt{X}} + \int \frac{dy}{\sqrt{Y}} = C\ ,$$

with C constant. In the present context this corresponds to considering the arguments u and v to varify while keeping $u+v$ constant, say equal to c, i.e., implying that

$$\frac{dv}{du} = -1\ .$$

Now writing for the sake of brevity

$$s_1 := \mathrm{sn}(u), \quad s_2 := \mathrm{sn}(v)$$

and denoting by $\dot{}$ the derivative w.r.t. u

$$\dot{s}_1 := \frac{d}{du}\mathrm{sn}(u), \quad \dot{s}_2 = \frac{d}{du}\mathrm{sn}(v) = -\frac{d}{dv}\mathrm{sn}(v)\ ,$$

we have from (A.6.17)

$$\ddot{s}_1\,s_2 - s_1\,\ddot{s}_2 = 2\,k^2\,s_1\,s_2\,\left(s_1^2 - s_2^2\right). \tag{A.6.18}$$

However, the right side can be re-expressed by using eq. (A.6.9):

$$\begin{aligned}
\dot{s_1}^2\,s_2^2 - s_1^2\,\dot{s_2}^2 &= {s_2}^2\left(1-s_1^2\right)\left(1-k^2\,s_1^2\right) - {s_1}^2\left(1-s_2^2\right)\left(1-k^2\,s_2^2\right)\\
&= s_2^2 - \left(1+k^2\right)s_2^2\,s_1^2 + k^2\,s_2^2\,s_1^4 - s_1^2 + (1+k)^2\,s_1^2\,s_2^2 - k^2\,s_1^2\,s_2^4\\
&= \left(s_1^2 - s_2^2\right)\,\left(-1 + k^2\,s_1^2\,s_2^2\right)
\end{aligned}$$

We use this result to replace $\left(s_1^2 - s_2^2\right)$ on the right side of Equation (A.6.18). Notice that the left side of Equation (A.6.18) is the derivative of $\dot{s}_1\,s_2 - s_1\,\dot{s}_2$. Putting these two results together, we can rewrite Equation (A.6.18) as

$$\begin{aligned}
\frac{\left(\dot{s}_1\,s_2 - s_1\,\dot{s}_2\right)^{\cdot}}{\dot{s}_1\,s_2 - s_1\,\dot{s}_2} &= \frac{2\,k^2\,s_1\,s_2\left(\dot{s}_1\,s_2 + s_1\,\dot{s}_2\right)}{k^2\,s_1^2\,s_2^2 - 1}\\
&= \frac{\left(1 - k^2\,s_1^2\,s_2^2\right)^{\cdot}}{1 - k^2\,s_1^2\,s_2^2}
\end{aligned}$$

Since both sides are derivatives with respect to u, we can integrate to get

$$\frac{\dot{s}_1\,s_2 - s_1\,\dot{s}_2}{1 - k^2\,s_1^2\,s_2^2} = -A \tag{A.6.19}$$

where A is a constant of integration. But this is a second integral of the first order equation $du/dv = -1$. Therefore, its integration constant must be a function of the first constant c which we already obtained earlier, that is, $A = f(u+v)$. Letting $v \to 0$, we find from Equation (A.6.19) that

$$f(u) = s_1 = \mathrm{sn}(u).$$

■

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