UNIVERSITY OF LEEDS

MATH2420

Multiple Integrals and Vector Calculus

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Course Notes
and
General Information

Vector calculus is the normal language used in applied mathematics for solving problems in two and three dimensions. In ordinary differential and integral calculus, you have already seen how derivatives and integrals interrelate. A derivative can be used as the opposite of an integration; it also occurs in changing variables in an integral. The same interrelation applies in multiple dimensions, but with more richness and variety.

This module starts with a discussion of different coordinate systems in two and three dimensions. The use of Cartesian, plane polar, cylindrical polar and spherical polar coordinates will run through the whole module.

The second section starts with a discussion of vector functions, which are the two- and three-dimensional equivalents of the functions of ordinary calculus. These can be used to describe curves in space. Next we look at functions of several variables: that is, functions of a vector. With these two concepts we can introduce derivatives for fully three-dimensional functions (gradient, divergence and curl).

This brings us to the halfway point of the module, and we will pause to review our new understanding before moving on to multiple-dimensional integrals. Here we extend the familiar idea of integration in one dimension to integration over an area or a volume.

Finally, with the introduction of line and surface integrals we come to the famous integral theorems of Gauss and Stokes. These encompass beautiful relations between line, surface and volume integrals and the vector derivatives studied at the start of this module.

Most real-life problems are not one-dimensional. The amount of heat stored in a piece of metal can be calculated by integrating its temperature in three dimensions; and the diffusion of dye in water is governed by differential equations based on three-dimensional derivatives. This is why a knowledge of vector calculus is essential for further study in many areas of applied mathematics.

Chapter 0

REVIEW

0.1 Calculus

Differentiation

The curve $y = f(x)$ has a slope at point $x = a$ given by the *derivative* of f with respect to x at a:

$$f'(a) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_a = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}. \tag{1}$$

A few particular derivatives are:

$$\begin{aligned} f(x) &= ax^n & f'(x) &= anx^{n-1} \\ f(x) &= e^{\lambda x} & f'(x) &= \lambda e^{\lambda x} \\ f(x) &= u(x)v(x) & f'(x) &= u'(x)v(x) + u(x)v'(x) \\ f(x) &= u(x)/v(x) & f'(x) &= [u'(x)v(x) - u(x)v'(x)]/v^2(x). \end{aligned}$$

Integration

Integration is the opposite of differentiation:

$$\int_a^b f'(x)\,\mathrm{d}x = [f(x)]_a^b = f(b) - f(a) \quad \text{or} \quad \int f'(x)\,\mathrm{d}x = f(x) \tag{2}$$

and it may also be seen as giving the area under a curve – so the integral $\int_a^b f(x)\,\mathrm{d}x$ gives the area under the curve $y = f(x)$ between $x = a$ and $x = b$. Some specific integrals are:

$$\begin{aligned} \int x^n\,\mathrm{d}x &= x^{n+1}/(n+1) \text{ for } n \neq -1 & \int x^{-1}\,\mathrm{d}x &= \ln x \\ \int e^{\lambda x}\,\mathrm{d}x &= e^{\lambda x}/\lambda & \int u(x)v'(x)\,\mathrm{d}x &= [u(x)v(x)] - \int u'(x)v(x)\,\mathrm{d}x \end{aligned}$$

0.2 Lines and Circles

The vector equation of a straight line in three-dimensional space is

$$\underline{x} = \underline{a} + u\underline{b} \text{ with } u \in (-\infty, \infty) \text{ a real scalar.} \tag{3}$$

The equation of a circle of radius r_0 centred about the origin is $x^2 + y^2 = r_0^2$ and if the circle is centred on the point $P(a, b)$ the the equation is $(x - a)^2 + (y - b)^2 = r_0^2$.

0.3 Trigonometry

Recall the functions $\sin x$ and $\cos x$, with the identities:

$$\sin^2 x + \cos^2 x = 1 \quad (4)$$

$$\tan x = \sin x / \cos x \quad (5)$$

and the derivatives

$$(\mathrm{d}/\mathrm{d}x) \sin x = \cos x \quad (6)$$

$$(\mathrm{d}/\mathrm{d}x) \cos x = -\sin x \quad (7)$$

$$(\mathrm{d}/\mathrm{d}x) \tan x = 1 + \tan^2 x. \quad (8)$$

There are memorable values:

x	$\sin x$	$\cos x$
0	0	1
$\pi/2$	1	0
π	0	-1
$3\pi/2$	-1	0
2π	0	1

0.4 Determinant

The determinant of a 3x3 matrix A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \quad (9)$$

0.5 Vector Products

We consider two vectors $\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$. Their *scalar dot product* is given by

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (10)$$

and their *vector cross product* by

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = ([a_2 b_3 - b_2 a_3], [a_3 b_1 - b_3 a_1], [a_1 b_2 - a_2 b_1]). \quad (11)$$

Note that although for the scalar product $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$, for the vector product we have $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$. Recall that

$$\underline{a} \cdot \underline{b} = ||\underline{a}||\,||\underline{b}|| \cos\theta \quad \text{and} \quad ||\underline{a} \times \underline{b}|| = ||\underline{a}||\,||\underline{b}|| \sin\theta \quad (12)$$

where θ is the angle between the vectors $\underline{a}$ and $\underline{b}$.

Chapter 1

COORDINATE SYSTEMS

1.1 Cartesian Coordinates

We are familiar with the rectangular *Cartesian coordinates* x, y in a plane, or x, y, z in space. If the unit vectors in the x, y and z directions are given by $\underline{i}$, $\underline{j}$ and $\underline{k}$ then we can write the position vector $\underline{r}$ of a point as

$$\begin{aligned} \underline{r} &= (x, y, z) && (1.1) \\ &= x\underline{i} + y\underline{j} + z\underline{k}. && (1.2) \end{aligned}$$

These two notations will be used interchangeably in this course.

1.2 Plane Polar Coordinates

In a fixed frame of reference the rectangular coordinates (x, y) are related to the polar coordinates (r, θ) through the relations

$$x = r\cos\theta,\ y = r\sin\theta,\ r > 0,\ \theta \in [0, 2\pi) \tag{1.3}$$

$$r = (x^2 + y^2)^{1/2},\ \theta = \arctan\left(\frac{y}{x}\right). \tag{1.4}$$

1.3 Cylindrical Coordinates

In a fixed frame of reference the Cartesian coordinates (x, y, z) are related to the cylindrical coordinates (r, θ, z) through the relations:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad r > 0, \quad \theta \in [0, 2\pi) \tag{1.5}$$

$$r = (x^2 + y^2)^{1/2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z. \tag{1.6}$$

1.4 Spherical Coordinates

In a fixed frame of reference the Cartesian coordinates (x, y, z) are related to the spherical coordinates (ρ, θ, ϕ) through the relations:

$$x = \rho\sin\theta\cos\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\theta, \tag{1.7}$$

$$\rho > 0, \quad \theta \in [0, \pi), \quad \phi \in [0, 2\pi) \tag{1.8}$$

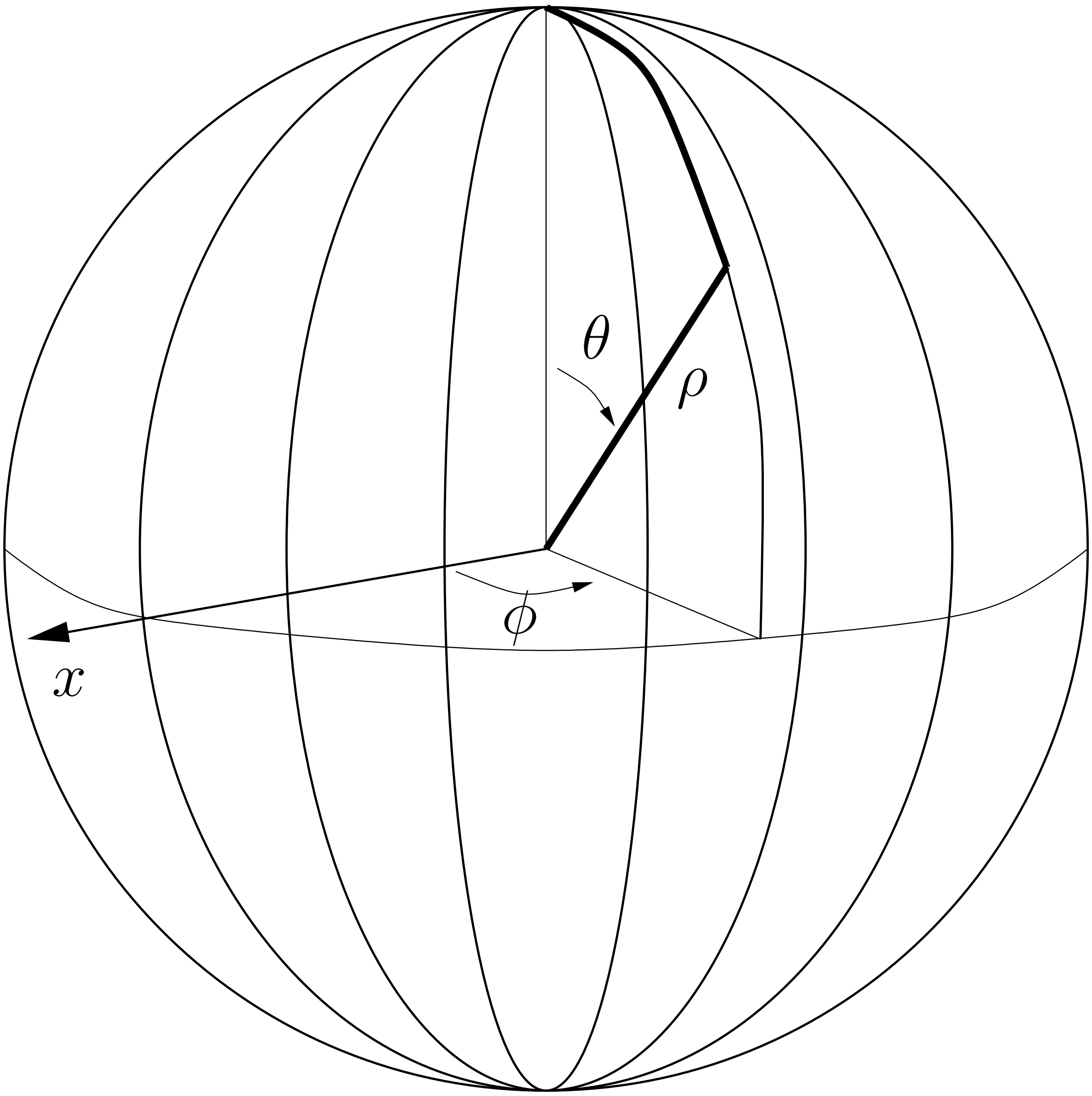


Figure 1.1: Spherical polar coordinates

$$\rho = (x^2 + y^2 + z^2)^{1/2}, \quad \phi = \arctan\left(\frac{y}{x}\right), \quad \theta = \arccos\left[z(x^2 + y^2 + z^2)^{-1/2}\right] \tag{1.9}$$

In the above equations, θ is the latitude or polar angle, and ϕ is the longitude.

Remark: Spherical coordinates are commonly used in applications where there is a centre of symmetry. The centre of symmetry is then taken as the origin.

Chapter 2

VECTOR CALCULUS

2.1 Vector Functions

A **vector-valued function** $\underline{f}$ is a *vector function* whose components are *single-valued functions* (scalar valued functions).

For example, given three single-valued functions $f_1(t)$, $f_2(t)$, $f_3(t)$ we can form the vector-valued function

$$\underline{f}(t) = (f_1(t), f_2(t), f_3(t)) = f_1(t)\underline{i} + f_2(t)\underline{j} + f_3(t)\underline{k} \tag{2.1}$$

The **magnitude** of the vector-valued function $\underline{f}(t)$ is a scalar-valued function and is defined by

$$||\underline{f}(t)|| = \left[f_1^2(t) + f_2^2(t) + f_3^2(t)\right]^{1/2} \tag{2.2}$$

In general, the graph of the vector function $\underline{f}(t) = f_1(t)\underline{i} + f_2(t)\underline{j} + f_3(t)\underline{k}$ is a curve C, in the sense that, as t varies, the tip of the position vector $\underline{f}(t)$ traces out C. The equations

$$x = f_1(t), y = f_2(t), z = f_3(t) \tag{2.3}$$

corresponding to the components of $\underline{f}$ are the **parametric** equations of C. If one of the components is zero, e.g. $\underline{f}(t) = f_1(t)\underline{i} + f_2(t)\underline{j}$, then C is said to be a *planar curve*, otherwise C is a *space curve*.

2.1.1 Derivative of a vector function

Given the vector function $\underline{f}(t) = (f_1(t), f_2(t), f_3(t))$ the **derivative** of $\underline{f}$ is defined by

$$\underline{f}'(t) = (f_1'(t), f_2'(t), f_3'(t)).$$

Properties of the derivative

- $(\underline{f} + \underline{g})'(t) = \underline{f}'(t) + \underline{g}'(t)$
- $(\alpha\underline{f})'(t) = \alpha\underline{f}'(t)$ where α is a constant scalar
- $(u\underline{f})'(t) = u(t)\underline{f}'(t) + u'(t)\underline{f}(t)$, where u is a scalar function
- $(\underline{f} \cdot \underline{g})'(t) = \underline{f}(t) \cdot \underline{g}'(t) + \underline{f}'(t) \cdot \underline{g}(t)$.
- $(\underline{f} \times \underline{g})'(t) = \underline{f}(t) \times \underline{g}'(t) + \underline{f}'(t) \times \underline{g}(t)$.
- $(\underline{f}(u))'(t) = \underline{f}'(u(t))u'(t)$.

2.1.2 Integral of a vector function

Given the vector function $\underline{f}(t) = (f_1(t), f_2(t), f_3(t))$, the **integral** of $\underline{f}$ is defined by $\int_a^b \underline{f}(t)\,\mathrm{d}t = (\int_a^b f_1(t)\,\mathrm{d}t, \int_a^b f_2(t)\,\mathrm{d}t, \int_a^b f_3(t)\,\mathrm{d}t)$.

Properties of the integral

- $\int_a^b (\underline{f}(t) + \underline{g}(t))\,\mathrm{d}t = \int_a^b \underline{f}(t)\,\mathrm{d}t + \int_a^b \underline{g}(t)\,\mathrm{d}t$
- $\int_a^b (\alpha \underline{f})(t)\,\mathrm{d}t = \alpha \int_a^b \underline{f}(t)\,\mathrm{d}t$, where α is a constant scalar
- $\int_a^b (\underline{c} \cdot \underline{f})(t)\,\mathrm{d}t = \underline{c}(t) \cdot \int_a^b \underline{f}(t)\,\mathrm{d}t$, where $\underline{c}$ is a constant vector.
- $\int_a^b (\underline{c} \times \underline{f})(t) = \underline{c}(t) \times \int_a^b \underline{f}(t)\,\mathrm{d}t$, where $\underline{c}$ is a constant vector.

2.2 Curves

1. The equation of **a straight line** is parametrised by

$$\underline{r}(t) = \underline{r}_0 + t\underline{d}, t \in (-\infty, \infty) \tag{2.4}$$

2. More generally, every vector function

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k} \tag{2.5}$$

parametrises a **curve in space** (or a **curve in plane** if one of the components $x(t)$, $y(t)$ or $z(t)$ is zero). It is also important to understand that a parametrised curve C is an *oriented* curve in the sense that as t increases on some interval of definition I, the tip of the position vector $\underline{r}(t)$ traces out C in a certain direction. For example, the unit circle parametrised by

$$\underline{r}(t) = \cos(t)\underline{i} + \sin(t)\underline{j}, t \in [0, 2\pi)$$

is traversed in the anticlockwise direction, starting at the point (1,0).

2.2.1 Tangent vector, tangent line

For a given curve C parametrised by

$$\underline{r}(t) = (x(t), y(t), z(t)),$$

the derivative vector

$$\underline{r}'(t) = (x'(t), y'(t), z'(t))$$

is called the *tangent vector* to the curve C at the point $P(x(t), y(t), z(t))$, and $\underline{r}'(t)$ points out in the direction of increasing t.

For a given curve C parametrised by $\underline{r}(t) = (x(t), y(t), z(t))$ the *tangent line* at a point t is the vector function

$$\underline{R}(\lambda) = \underline{r}(t) + \lambda\underline{r}'(t), \lambda \in (-\infty, \infty) \tag{2.6}$$

2.2.2 Intersecting curves

Two curves

$$(C_1): \quad \underline{r_1}(t) = x_1(t)\underline{i} + x_2(t)\underline{j} + x_3(t)\underline{k},$$
$$(C_2): \quad \underline{r_2}(u) = y_1(u)\underline{i} + y_2(u)\underline{j} + y_3(u)\underline{k},$$

intersect iff there are numbers t and u for which $\underline{r_1}(t) = \underline{r_2}(u)$. The angle between two intersecting curves (which, by definition, is the angle between the corresponding tangent lines) can be obtained by examining the tangent vectors at the point of intersection.

2.2.3 The unit tangent

For a given curve C parametrised by $\underline{r}(t) = (x(t), y(t), z(t))$, the vector

$$\underline{T} := \frac{\underline{r}'(t)}{||\underline{r}'(t)||} \tag{2.7}$$

is called the *unit tangent vector* to the curve C at the point $P(x(t), y(t), z(t))$. The unit tangent points in the direction of increasing t along the curve and is parallel to the curve.

2.2.4 Reversing the sense of a curve

We make a distinction between the curve

$$\underline{r} = \underline{r}(t), t \in [a, b] \tag{2.8}$$

and the curve

$$\underline{R}(u) = \underline{r}(a + b - u), u \in [a, b]. \tag{2.9}$$

Both vector functions trace out the same set of points, but the order has been *reversed*. Whereas the first curve starts at $\underline{r}(a)$ and ends at $\underline{r}(b)$, the second curve starts at $\underline{r}(b)$ and ends at $\underline{r}(a)$.

Thus, for example, the vector function

$$\underline{r}(t) = \cos(t)\underline{i} + \sin(t)\underline{j}, t \in [0, 2\pi)$$

gives the unit circle traversed anticlockwise while the reversed curve

$$\underline{R}(u) = \cos(2\pi - u)\underline{i} + \sin(2\pi - u)\underline{j}, u \in [0, 2\pi)$$

gives the unit circle traversed clockwise.

Unit tangent

When we reverse the sense of a curve, the unit tangent $\underline{T}$ reverses direction (is multiplied by -1) because it always points in the direction of increasing t or u.

2.3 Arc Length

The length of a continuously differentiable curve (C): $\underline{r} = \underline{r}(t)$, $t \in [a, b]$ is given by

$$L(C) = \int_a^b ||\underline{r}'(t)|| \, \mathrm{d}t. \tag{2.10}$$

Chapter 3

FUNCTIONS OF SEVERAL VARIABLES

3.1 Introduction

Since we live in a three-dimensional world, in applied mathematics we are interested in functions which can vary with any of the three space variables x, y, z and also with time t. For instance, if the function f represents the temperature in this room, then f depends on the location (x, y, z) at which it is measured and also on the time t when it is measured, so f is a function of the independent variables x, y, z and t, i.e. $f(x, y, z, t)$.

3.2 Geometric Interpretation

For a function of two variables, $f(x, y)$, consider (x, y) as defining a point P in the xy-plane. Let the value of $f(x, y)$ be taken as the length PP' drawn parallel to the z-axis (or the height of point P' above the plane). Then as P moves in the xy-plane, P' maps out a *surface* in space whose equation is $z = f(x, y)$.

Example: $f(x, y) = 6 - 2x - 3y$

The surface $z = 6 - 2x - 3y$, i.e. $2x + 3y + z = 6$, is a plane
which intersects the x-axis where $y = z = 0$, i.e. $x = 3$;
which intersects the y-axis where $x = z = 0$, i.e. $y = 2$;
which intersects the z-axis where $x = y = 0$, i.e. $z = 6$.

Example: $f(x, y) = y^2 - x^2$

In the plane $x = 0$, there is a *minimum* at $y = 0$; in the plane $y = 0$, there is a *maximum* at $x = 0$. The whole surface is shaped like a horse's saddle; and the picture shows a structure for which $(0, 0)$ is called a *saddle point*.

3.2.1 Plane polar coordinates

Since the variables x and y respresent a point in the plane, we can express that point in plane polar coordinates simply by substituting the definitions:

$$x = r\cos\theta \quad y = r\sin\theta.$$

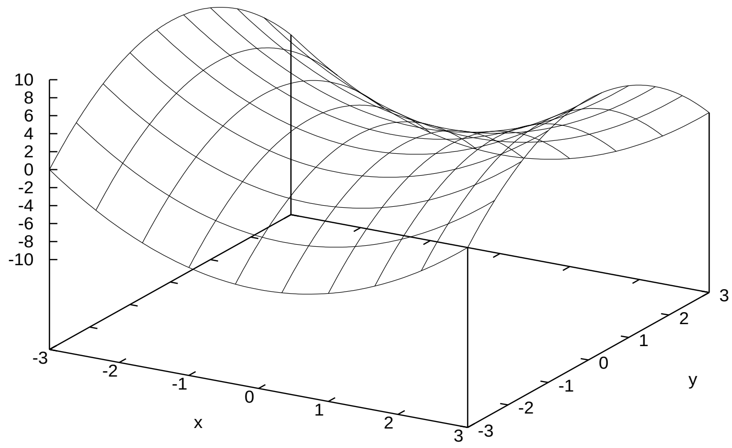


Figure 3.1: The surface $z = x^2 - y^2$

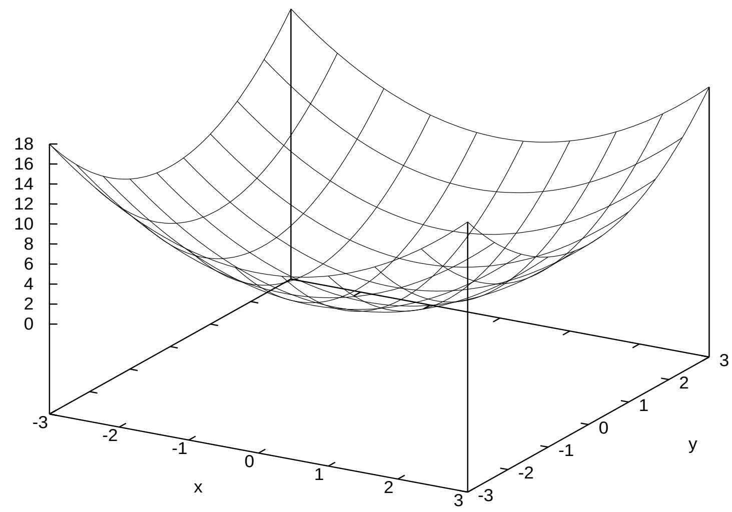


Figure 3.2: The surface $z = x^2 + y^2$

Example: $f(x, y) = x^2 + y^2$

The surface $z = x^2 + y^2$ may be drawn most easily by first converting into plane polar coordinates. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ gives $z = r^2$. The surface is symmetric about the z-axis and its cross-section is a parabola. [Check with the original function and the plane $y = 0$.] Thus the whole surface is a paraboloid (a bowl).

Another way to picture the same surface is to do as map-makers or weather forecasters do and draw *contour lines* (or *level curves*) – produces by taking a section, using a plane $z =$ const. and projecting it onto the xy-plane. For $z = x^2 + y^2$ as above, the contour lines are concentric circles.

3.3 Partial Differentiation

For a function $f(x)$ that depends on a single variable, x, we can form the **ordinary derivative** $\mathrm{d}f/\mathrm{d}x$. For example, if $f(x) = 3x^4 + \sin(x)$ then $\mathrm{d}f/\mathrm{d}x = 12x^3 + \cos(x)$. Similarly, for a function f that depends on **several** variables x, y, ... we can differentiate with respect to each of these variables. This process is called *partial differentiation* (*partial derivatives*). If $f(x, y) = yx^4 + \sin(x)$ then we treat y as a constant as we did 3 for the function of x, and have $\partial f/\partial x = 4yx^3 + \cos(x)$.

To find a **partial derivative** we hold all but one of the independent variables constant and differentiate with respect to that one variable using the ordinary rules for one-variable calculus.

Notation: If f depended on x, y, ... but if only x is allowed to vary, **the partial derivative of f with respect to** x is denoted by $\partial f/\partial x$ or by f_x. (Distinguish carefully between $\mathrm{d}f/\mathrm{d}x$ the **ordinary** derivative with straight d and $\partial f/\partial x$ with curly ∂.) Similarly, we can denote the partial derivative with respect to y by $\partial f/\partial y$ or by f_y, etc.

3.3.1 Second-order partial derivatives

For $f(x, y)$ we can form $\partial f/\partial x$ and $\partial f/\partial y$. Each of these can then be differentiated again with respect to x or y to form the **second-order derivatives**
$\partial/\partial x\,(\partial f/\partial x)$ denoted by $\partial^2 f/\partial x^2$ or f_{xx};
$\partial/\partial y\,(\partial f/\partial x)$ denoted by $\partial^2 f/\partial y\partial x$ or f_{xy};
$\partial/\partial y\,(\partial f/\partial y)$ denoted by $\partial^2 f/\partial y^2$ or f_{yy};
$\partial/\partial x\,(\partial f/\partial y)$ denoted by $\partial^2 f/\partial x\partial y$ or f_{yx}.

f_{xy} and f_{yx} are called **mixed derivatives**.

The concept can be extended, e.g. if f depends on x, y and z we can form $\partial/\partial z\left(\partial^2 f/\partial x^2\right)$, written as $\partial^3 f/\partial z\partial x^2$ or f_{xxz}, etc.

The **Mixed Derivatives Theorem** states that if f_{xy} and f_{yx} are continuous then $f_{xy} = f_{yx}$. Thus to calculate a mixed derivative we can calculate in either order. [Think about calculating $\partial/\partial x\,(\partial f/\partial y)$ if $f(x, y) = xy + 1/(\sin(y^2) + e^y)$.] For third-order derivatives the mixed derivatives theorem gives $f_{xxy} = f_{xyx} = f_{yxx}$.

For a function $f(x)$, the *ordinary* derivative $\mathrm{d}f/\mathrm{d}x$ gives the slope of the tangent to the curve at any point P. For a function $f(x, y)$, the *partial* derivative $\frac{\partial f}{\partial x}$ is evaluated holding y constant, and so gives the slope of the tangent to the surface at P in a plane $y =$ const..

3.4 Summary

- We form partial derivatives with respect to one variable by holding all the other variables constant and differentiating.

- The order of mixed derivatives is not important.
- If $f(x, y)$ represents a surface above the xy plane then $\partial f/\partial x$ is the slope of a section taken in the x direction at a point P and $\partial f/\partial y$ the slope of a section in the y direction. Between them these two tangents define a plane which (we will see later) is the *tangent plane* to the surface at P.

Chapter 4

GRADIENTS

In this chapter we introduce derivatives in multiple dimensions. Although we limit ourselves to differentiating known functions, the real use of these derivatives is in differential equations. When we learnt calculus, there was a big gap between starting to study differentiation and solving our first differential equation. In this chapter we meet three-dimensional derivatives. Because this is a major concept we won't get to any physical applications; but almost no physical problem can be solved without using these tools.

4.1 Gradient and Directional Derivative

For a function of three variables $f(x, y, z)$, the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ measure the rates of change of f along x, y and z directions, respectively. We now ask how we can calculate the rate of change of f in any direction in space. The answer lies in the vector

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k} \tag{4.1}$$

call the *gradient* of f. From its definition, the component of $\underline{\nabla} f$ along $\underline{i}$ is $(\underline{\nabla} f \cdot \underline{i}) = \frac{\partial f}{\partial x}$ = rate of change along $\underline{i}$ and similarly for the y and z directions. But for **any** direction in space we are free to temporarily call it the $\underline{i}$-direction and carry over the above analysis. Thus in general:

For any direction in space defined by a **unit** vector $\underline{u}$ the rate of change of f along $\underline{u}$ is given by $(\underline{\nabla} f \cdot \underline{u})$ and is called the *directional derivative* of f along $\underline{u}$.

4.1.1 Two further properties of the gradient

We look at cases where $\underline{u}$ is parallel or perpendicular to $\underline{\nabla} f$. The change df in f due to a change in the position P by $\underline{dr} = dr\underline{u}$ is given by change in f = (rate of change with distance) $\times$ (distance), i.e.

$$df = (\underline{\nabla} f \cdot \underline{u})dr = \underline{\nabla} f \cdot (\underline{u}dr) = \underline{\nabla} f \cdot \underline{dr} = ||\underline{\nabla} f|| \, ||\underline{dr}|| \cos(\theta) \tag{4.2}$$

where θ is the angle between the vectors $\underline{dr}$ and $\underline{\nabla} f$. From this equation it can be seen that the direction $\underline{dr}$ for which df is a **maximum** is obviously that for which $\cos(\theta) = 1$, or $\theta = 0$, i.e. the direction of $\underline{\nabla} f$. Thus

Property 1. At any point, $\underline{\nabla} f$ points in the direction in which f is increasing most rapidly and its magnitude $||\underline{\nabla} f||$ gives this maximum rate of change.

Again from eq. (4.2), $df = 0$ corresponds to $\theta = \pi/2$, when $\underline{\nabla} f$ and $\underline{dr}$ are perpendicular. But $df = 0$ means that f has not changed - so the displacement $\underline{dr}$ is along the surface f = const.. Thus

Property 2. At any point, $\underline{\nabla} f$ is perpendicular to the surface f = const. through that point.

4.2 Linear Approximations (Tangents)

Motivation: Many functions arising in applications are difficult to deal with. We thus need ways of *approximating* such functions by others which are easier to handle. The most useful approximations are polynomials. We consider first the single variable case, $f(x)$, which will guide us into the treatment of the two-variable case.

4.2.1 One-variable case (tangent line)

The tangent to the curve $y = f(x)$ at A where $x = a$ has the slope $f'(a)$ and therefore has the equation $y = f'(a)x + \text{const.}$. It passes through A, i.e. for $x = a$, $y = f(a)$, so $\text{const.} = f(a) - f'(a)a$. Thus the equation of the tangent line (the line *parallel* to the curve) is

$$y = f(a) + (x - a)f'(a). \tag{4.3}$$

For points close to A the tangent gives a close approximation to the curve. The approximation

$$f(x) \approx f(a) + (x - a)f'(a) \tag{4.4}$$

is called the *linear approximation* to $f(x)$ near $x = a$.

4.2.2 Two-variables case (tangent plane)

Above we saw that, for a function of one variable, approximating its curve by a tangent line gave a **linear approximation**. For a function of two variables, $f(x, y)$, the corresponding linear approximation arises when we approximate a surface by its tangent plane. Suppose the surface is $z = f(x, y)$. The tangent plane at A is perpendicular to the normal at A, i.e. is perpendicular to the gradient vector at A. Now the surface is $f(x, y) - z = 0$, or $F(x, y, z) = \text{const.} = 0$, whose gradient is $\underline{\nabla} F = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} - \underline{k}$. Thus the tangent plane $\underline{r} \cdot \underline{n} = \text{const.}$ at A is

$$x\left(\frac{\partial f}{\partial x}\right)_A + y\left(\frac{\partial f}{\partial y}\right)_A - z = \text{const.} \tag{4.5}$$

This plane passes through $A(a, b)$ so the constant in eq. (4.5) has the value $a\left(\frac{\partial f}{\partial x}\right)_A + b\left(\frac{\partial f}{\partial y}\right)_A - f(a, b)$ and thus (4.5) becomes

$$z = f(a, b) + (x - a)\left(\frac{\partial f}{\partial x}\right)_A + (y - b)\left(\frac{\partial f}{\partial y}\right)_A. \tag{4.6}$$

Thus the *linear approximation* to the surface close to $A(a, b)$ is given by the tangent plane (4.6), so

$$f(x, y) \approx f(a, b) + (x - a)\left(\frac{\partial f}{\partial x}\right)_A + (y - b)\left(\frac{\partial f}{\partial y}\right)_A \tag{4.7}$$

and we note that we can rewrite this using the gradient $\underline{\nabla} f$ as

$$f(x, y) = f(a, b) + ((x - a), (y - b)) \cdot (\underline{\nabla} f)_A. \tag{4.8}$$

4.3 The Chain Rule

For a function of one variable, $f(x)$, if x also depends on another variable t, then f depends on t and the chain rule gives

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{4.9}$$

We now generalise this result for a function of several variables. For $f(x, y)$, suppose x and y depend on t. Let t increase by Δt and so x and y increase by Δx and Δy. Then from our earlier work on linear approximations we obtain

$$f(t + \Delta t) \approx f(t) + \Delta x \left(\frac{\partial f}{\partial x}\right) + \Delta y \left(\frac{\partial f}{\partial y}\right) \tag{4.10}$$

so if we rearrange and let $\Delta t \to 0$ we obtain the *chain rule* for a function of two variables, which is,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial f}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{4.11}$$

Note that f depends on x and y [so partial derivatives $\partial f/\partial x$, $\partial f/\partial y$] whilst x and y depend on the single variable t [so ordinary derivatives $\mathrm{d}x/\mathrm{d}t$, $\mathrm{d}y/\mathrm{d}t$]. Thus f depends on t and has the derivative $\mathrm{d}f/\mathrm{d}t$ given by the chain rule (4.11).

The chain rule extends directly to functions of three or more variables.

4.3.1 Extended chain rule

For $f(x, y)$ suppose that x and y depend on **two** variables s and t. Then changing either s or t changes x and y, so changes f, i.e. producing $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ according to the *extended chain rule*

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \tag{4.12}$$

4.4 The vector differential operator $\underline{\nabla}$ (grad)

The *vector differential operator* $\underline{\nabla}$ is defined formally by setting

$$\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{4.13}$$

By 'formally' we mean that this is not an ordinary vector but its 'components' are differentiation symbols. As the term 'operator' suggests, $\underline{\nabla}$ is thought of as something that 'operates' on things. What sort of things? Scalar functions and vector functions.

Suppose that $f(x, y, z)$ is a scalar function. Then $\underline{\nabla}$ operates on f as follows:

$$\underline{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right). \tag{4.14}$$

This is the *gradient* of f, which we have just discussed.

Question How does $\underline{\nabla}$ operate on vector functions?

Answer Using either of the vector products we already know – dot or cross.

4.4.1 Divergence

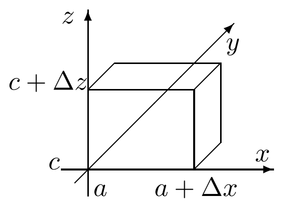
If $\underline{q}(x,y,z) = (q_1(x,y,z), q_2(x,y,z), q_3(x,y,z))$ is a vector function, then by definition

$$div(\underline{q}) = \underline{\nabla} \cdot \underline{q} = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z}. \tag{4.15}$$

This "product", $\underline{\nabla} \cdot \underline{q}$, defined in imitation of the ordinary dot product, is a scalar called the *divergence* of $\underline{q}$.

Physical Meaning

Consider a fluid whose velocity everywhere is given by the vector function $\underline{u}(x,y,z) = (u_1, u_2, u_3)$. Then imagine we have a cubic volume $[a, a+\Delta x] \times [b, b+\Delta y] \times [c, c+\Delta z]$ and we want to know how much fluid is flowing out of it per unit time.



The fluid flowing through one face is the velocity normal to the face multiplied by the area of that face. So for the face on which $x = a$, u_1 gives the velocity pointing in and the face has area $\Delta y \Delta z$, so the flow out is

$$-\Delta y \Delta z u_1(x = a).$$

Looking at the face on which $x = a + \Delta x$, the area of the face is still $\Delta y \Delta z$ but the velocity out is u_1 (since the inside of the cube is on the other side of this face). The contribution from this face is

$$\Delta y \Delta z u_1(x = a + \Delta x).$$

Similarly, the other 4 contributions are

$$\begin{array}{ll} -\Delta x \Delta z u_2(y = b) & \Delta x \Delta z u_2(y = b + \Delta y) \\ -\Delta x \Delta y u_3(z = c) & \Delta x \Delta y u_3(z = c + \Delta z). \end{array}$$

We add together the six contributions to find

$$\begin{aligned} \textit{Rate of flow out} &= \Delta y \Delta z (u_1(x = a + \Delta x) - u_1(x = a)) \\ &\quad + \Delta x \Delta z (u_2(y = b + \Delta y) - u_2(y = b)) \\ &\quad + \Delta x \Delta y (u_3(z = c + \Delta z) - u_3(z = c)) \\ &= \Delta x \Delta y \Delta z [(u_1(x = a + \Delta x) - u_1(x = a))/\Delta x \\ &\quad + (u_2(y = b + \Delta y) - u_2(y = b))/\Delta y \\ &\quad + (u_3(z = c + \Delta z) - u_3(z = c))/\Delta z] \end{aligned}$$

and if we let the Δx etc. terms tend to zero we have

$$\begin{aligned} \textit{Rate of flow out} &= \textit{volume} \times (\partial u_1/\partial x + \partial u_2/\partial y + \partial u_3/\partial z) \\ &= \textit{volume} \times div(\underline{u}). \end{aligned}$$

Thus the divergence represents the flow out of the volume, per unit time, per unit volume. This is where the name comes from – it is the rate at which the fluid is diverging.

4.4.2 Curl

For the vector $\underline{q} = q_1\underline{i} + q_2\underline{j} + q_3\underline{k}$, we also have

$$\begin{aligned} \underline{curl}(\underline{q}) = \underline{\nabla} \times \underline{q} &= det\begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ q_1 & q_2 & q_3 \end{bmatrix} \\ &= \left(\frac{\partial q_3}{\partial y} - \frac{\partial q_2}{\partial z}, \frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial x}, \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} \right). \end{aligned} \tag{4.16}$$

This second "product", $\underline{\nabla} \times \underline{q}$, defined in imitation of the ordinary cross product, is a vector called the *curl* of $\underline{q}$.

Example

Calculate $\underline{\nabla} \times \underline{q}$ for (from above) $\underline{q}(x, y, z) = (x - y, x + y, z)$.

Solution

$$\underline{\nabla} \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x+y & z \end{vmatrix} = \underline{i}(0) + \underline{j}(0) + \underline{k}(1+1) = 2\underline{k}.$$

Physical Meaning

Consider a solid body which is rotating about the z-axis at rate ω, so that each point (x, y, z) inside it has velocity $\underline{u} = (-\omega y, \omega x, 0)$. We can express this as $\underline{u} = \underline{\Omega} \times \underline{r}$, where $\underline{r} = (x, y, z)$ and $\underline{\Omega} = (0, 0, \omega)$ is called the rotation vector. Then the curl of the *velocity field* (the vector field, i.e. three-dimensional vector function $\underline{u}$, defining the velocity) is

$$\begin{aligned} \underline{\nabla} \times \underline{u} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y, & \omega x & 0 \end{vmatrix} \\ &= \underline{k}(\omega + \omega) = 2\omega\underline{k} = (0, 0, 2\omega). \end{aligned}$$

We have shown that for a rotation, the curl is related to the rotation vector. This is true in general – so if we had a fluid velocity which was partly rotating and partly moving in some other way, the curl would extract the rotation part (the *circulation*).

4.4.3 Grad and div in polar coordinates

The forms of grad and div in polar coordinates are not examinable in this module, but you may find them useful in later years. The forms for curl are too unpleasant to be quoted here; do not be fooled into carrying out a simple cross product with the gradient "vector".

Plane polar coordinates

The unit vector in the r direction is $\underline{e}_r$; the unit vector in the θ direction is $\underline{e}_\theta$. The values of these two vectors vary according to location.

We can represent any vector $\underline{v}$ in terms of these two unit vectors:

$$\underline{v} = v_r\underline{e}_r + v_\theta\underline{e}_\theta.$$

Then we have

$$\begin{aligned} \underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta \\ \underline{\nabla}\cdot\underline{v} &= \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \end{aligned}$$

Cylindrical polar coordinates

In a similar fashion to plane polar coordinates, we obtain

$$\begin{aligned} \underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{\partial f}{\partial z}\underline{e}_z \\ \underline{\nabla}\cdot\underline{v} &= \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \end{aligned}$$

Spherical polar coordinates

Here we denote the unit vector in the ρ direction as $\underline{e}_\rho$ and similar for θ and ϕ, and a vector $\underline{v}$ can be written

$$\underline{v} = v_\rho \underline{e}_\rho + v_\theta \underline{e}_\theta + v_\phi \underline{e}_\phi.$$

Grad and div are given by:

$$\begin{aligned} \underline{\nabla} f &= \frac{\partial f}{\partial \rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{1}{\rho\sin\theta}\frac{\partial f}{\partial \phi}\underline{e}_\phi \\ \underline{\nabla}\cdot\underline{v} &= \frac{1}{\rho^2}\frac{\partial (\rho^2 v_\rho)}{\partial \rho} + \frac{1}{\rho\sin\theta}\frac{\partial (\sin\theta v_\theta)}{\partial \theta} + \frac{1}{\rho\sin\theta}\frac{\partial v_\phi}{\partial \phi} \end{aligned}$$

4.4.4 Basic Identities

For vectors we have that $\underline{a}\times\underline{a} = \underline{0}$. Is it true that $\underline{\nabla}\times\underline{\nabla} = \underline{0}$? Define $(\underline{\nabla}\times\underline{\nabla})f$ by $(\underline{\nabla}\times\underline{\nabla})f = \underline{\nabla}\times(\underline{\nabla}f)$.

Theorem (The curl of a gradient is zero)
If $f(x, y, z)$ is a scalar function then

$$\underline{\nabla}\times(\underline{\nabla}f) = \underline{0}.$$

Proof

$$\begin{aligned} \underline{\nabla}\times(\underline{\nabla}f) &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} \\ &= \underline{i}(f_{yz} - f_{zy}) + \underline{j}(f_{zx} - f_{xz}) + \underline{k}(f_{xy} - f_{yx}) \end{aligned}$$

and by the mixed derivatives theorem, $\underline{\nabla}\times(\underline{\nabla}f) = \underline{0}$.

For vectors we have that $\underline{a}\cdot(\underline{a}\times\underline{c}) = 0$. The analogous operator formula $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{q}) = 0$ is also valid.

Theorem (The divergence of a curl is zero)
If $\underline{q}(x, y, z)$ is a vector function then

$$\underline{\nabla}\cdot(\underline{\nabla}\times\underline{q}) = 0.$$

Proof

$$\begin{aligned}\underline{\nabla} \cdot (\underline{\nabla} \times \underline{q}) &= \underline{\nabla} \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ q_1 & q_2 & q_3 \end{vmatrix} \\ &= \underline{\nabla} \cdot \left(\frac{\partial q_2}{\partial z} - \frac{\partial q_3}{\partial y}, \frac{\partial q_3}{\partial x} - \frac{\partial q_1}{\partial z}, \frac{\partial q_1}{\partial y} - \frac{\partial q_2}{\partial x} \right) \\ &= \frac{\partial^2 q_2}{\partial x \partial z} - \frac{\partial^2 q_3}{\partial x \partial y} + \frac{\partial^2 q_3}{\partial y \partial x} - \frac{\partial^2 q_1}{\partial y \partial z} + \frac{\partial^2 q_1}{\partial z \partial y} - \frac{\partial^2 q_2}{\partial z \partial x} \\ &= 0.\end{aligned}$$

The following identities are left for you as an exercise.

Theorem If $f(x, y, z)$ is a scalar function and $\underline{q}(x, y, z)$ is a vector function then show that:

(i) $\underline{\nabla} \cdot (f\underline{q}) = (\underline{\nabla} f) \cdot \underline{q} + f(\underline{\nabla} \cdot q)$

(ii) $\underline{\nabla} \times (f\underline{q}) = (\underline{\nabla} f) \times \underline{q} + f(\underline{\nabla} \times q)$.

4.4.5 The Laplacian

From the operator $\underline{\nabla}$ we can construct other operators. The most important of these is the *Laplacian* (or the *Laplace operator*) $\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$ which operates on scalar functions $f(x, y, z)$ according to the rule:

$$\nabla^2 f = \underline{\nabla} \cdot (\underline{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{4.17}$$

The forms for the Laplacian in different coordinate systems are more complex (cf. the final example of §4.3.1). They are given by:

$$\begin{aligned}\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \text{ in plane polar coordinates} \\ \nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \text{ in cylindrical polar coordinates} \\ \nabla^2 f &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \text{ in spherical polar coordinates}\end{aligned}$$

4.5 Jacobian

The *Jacobian matrix* of the transformation $x = x(s, t)$, $y = y(s, t)$ is the matrix which relates the two sets of partial derivatives:

$$\begin{pmatrix} \partial f/\partial s \\ \partial f/\partial t \end{pmatrix} = \begin{pmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{pmatrix} \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}. \tag{4.18}$$

The *Jacobian* of the transformation is the determinant

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \tag{4.19}$$

of the *Jacobian matrix*.

Question: If $x = x(s, t)$ and $y = y(s, t)$, can we solve for s and t in terms of x and y?

Answer: Yes, provided that the Jacobian $\frac{\partial(x,y)}{\partial(s,t)} \neq 0$. This is the *implicit function theorem.*

4.5.1 Jacobians of the standard coordinate transformations

The following Jacobians may be quoted as standard results:

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= r \text{ for plane polar coordinates} \\
\frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= r \text{ for cylindrical polar coordinates} \\
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \rho^2 \sin\theta \text{ for spherical polar coordinates.}
\end{aligned}$$

Chapter 5

DOUBLE AND TRIPLE INTEGRALS

5.1 Multiple-Integral Notation

Previously ordinary integrals of the form

$$\int_J f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \tag{5.1}$$

where $J = [a, b]$ is an interval on the real line, have been studied. Here we study *double integrals*

$$\int\int_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{5.2}$$

where Ω is some region in the xy-plane, and a little later we will study *triple integrals*

$$\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{5.3}$$

where T is a solid (volume) in the xyz-space.

5.2 Double Integrals

5.2.1 Properties

(1) Area property

$$\int\int_\Omega \mathrm{d}x\,\mathrm{d}y = \text{Area of } \Omega.$$

In particular if Ω is the rectangle $\Omega = [a, b] \times [c, d]$ then $\int\int_\Omega \mathrm{d}x\,\mathrm{d}y = (b - a)(d - c)$.

(2) Linearity

$$\int\int_\Omega [\alpha f(x, y) + \beta g(x, y)]\,\mathrm{d}x\,\mathrm{d}y = \alpha \int\int_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \beta \int\int_\Omega g(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{5.4}$$

where α and β are constants.

(3) Additivity

If Ω is broken up into a finite number of nonoverlapping basic regions $\Omega_1,\ldots,\Omega_n$, then

$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_{\Omega_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \ldots + \int\int_{\Omega_n} f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{5.5}$$

5.2.2 Geometric Interpretation

The double integral over Ω gives the volume of the solid T whose upper boundary is the surface $z = f(x,y)$ and whose lower boundary is the region Ω in the xy-plane:

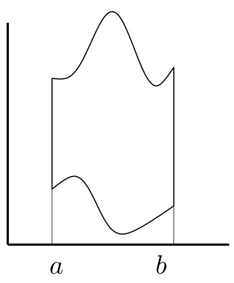
$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \text{volume of } T. \tag{5.6}$$

5.2.3 The Evaluation of Double Integrals by Repeated Integrals

If an ordinary integral $\int_a^b f(x)\,\mathrm{d}x$ proves difficult to evaluate, it is not because of the interval $[a,b]$ but because of the integrand f. Difficulty in evaluating a double integral $\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ can come from two sources: from the integrand f or from the domain Ω. Even such a simple looking integral as $\int\int_{\Omega} 1\,\mathrm{d}x\,\mathrm{d}y$ is difficult to evaluate if Ω is complicated.

In this section we introduce a technique for evaluating double integrals over domains that have special shapes. The key idea is that double integrals over such special domains can be reduced to a pair of ordinary integrals.

Horizontally simple domain



The *projection* of the domain Ω onto the x-axis is a closed interval $[a,b]$ and Ω consists of all points (x,y) with

$$a \le x \le b, \text{ and } \phi_1(x) \le \ y \le \phi_2(x). \tag{5.7}$$

Then

$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x. \tag{5.8}$$

Here we first calculate $\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y$ by integrating $f(x,y)$ with respect to y from $y = \phi_1(x)$ to $y = \phi_2(x)$. The resulting expression is a function of x alone, which we then integrate with respect to x from $x = a$ to $x = b$.

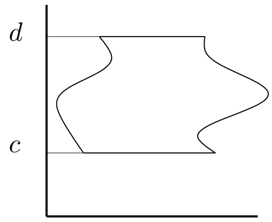
Example

Evaluate $\int\int_{\Omega}(x^4 - 2y)\,\mathrm{d}x\,\mathrm{d}y$, where the domain Ω consists of all points (x,y) with $-1 \le x \le 1$ and $-x^2 \le y \le x^2$.

Solution

$$\begin{aligned}\int\int_{\Omega}(x^4 - 2y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{x=-1}^{x=1}\int_{y=-x^2}^{y=x^2}(x^4 - 2y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=-1}^{x=1}[x^4y - y^2]_{y=-x^2}^{y=x^2}\,\mathrm{d}x \\ &= \int_{x=-1}^{x=1} 2x^6\,\mathrm{d}x = [2x^7/7]_{x=-1}^{x=1} = 4/7\end{aligned}$$

Vertically simple domain



The *projection* of the domain Ω onto the y-axis is a closed interval $[c, d]$ and Ω consists of all points (x, y) with

$$c \leq y \leq d, \text{ and } \psi_1(y) \leq x \leq \psi_2(y). \tag{5.9}$$

Then

$$\int\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y. \tag{5.10}$$

Here we first calculate $\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$ by integrating $f(x,y)$ with respect to x from $x = \psi_1(y)$ to $x = \psi_2(y)$. The resulting expression is a function of y alone, which we then integrate with respect to y from $y = c$ to $y = d$.

The integrals in the right-hand sides of formulae (5.8) and (5.10) are called *repeated integrals.*

Remark 1

Sometimes a domain can be expressed both as a horizontally simple domain: $a \leq x \leq b$, $\phi_1(x) \leq y \leq \phi_2(x)$, and as a vertically simple domain: $c \leq y \leq d$, $\psi_1(y) \leq x \leq \psi_2(y)$. Then

$$\int\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y. \tag{5.11}$$

Therefore we can, at least in theory, perform the integration in either order. However, there are situations where one order is preferable over the other.

Remark 2

Finally, if Ω, the domain of integration, is neither horizontally nor vertically simple, then it is usually possible to break it up into a finite number of domains, say $\Omega_1, \ldots, \Omega_n$, each of which is either horizontally or vertically simple. Then we can use the additivity property given by eq. (5.5).

5.2.4 Evaluating Double Integrals Using Polar Coordinates

Let Ω be a domain formed with all points (x, y) that have polar coordinates (r, θ) in the set

$$\Gamma : \alpha \leq \theta \leq \beta,\ \rho_1(\theta) \leq r \leq \rho_2(\theta) \tag{5.12}$$

where $\beta \leq \alpha + 2\pi$. Then

$$\begin{aligned} \int\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int\int_\Gamma f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned} \tag{5.13}$$

5.3 Triple Integrals

5.3.1 Properties

(1) Volume property

$$\int\int\int_T \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \text{Volume of } T.$$

In particular if T is the box $T = [a, b] \times [c, d] \times [e, f]$ then $\int\int\int_T \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = (b-a)(d-c)(f-e)$.

(2) Linearity

$$\int\int\int_T [\alpha f(x,y,z) + \beta g(x,y,z)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \alpha \int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \beta \int\int\int_T g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad (5.14)$$

where α and β are constants.

(3) Additivity If T is broken up into a finite number of nonoverlapping basic regions $T_1, \ldots, T_n$, then

$$\int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int\int_{T_1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \ldots + \int\int\int_{T_n} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \quad (5.15)$$

5.3.2 The Evaluation of Triple Integrals by Repeated Integrals

Let T be a solid whose projection onto the xy-plane is labelled Ω_{xy}. Then the solid T is the set of all points (x, y, z) satisfying

$$(x,y) \in \Omega_{xy}, \chi_1(x,y) \le z \le \chi_2(x,y). \quad (5.16)$$

The triple integral over T can be evaluated by setting

$$\int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_{\Omega_{xy}} \left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y. \quad (5.17)$$

In eq. (5.17) we can evaluate the integration with respect to z first and then evaluate the double integral over the domain Ω_{xy} as studied for double integrals. In particular if Ω_{xy} is horizontally simple, say

$$a \le x \le b, \quad \phi_1(x) \le \ y \le \phi_2(x). \quad (5.18)$$

then the solid T itself is the set of all points (x, y, z) such that

$$a \le x \le b, \quad \phi_1(x) \le \ y \le \phi_2(x), \quad \chi_1(x,y) \le z \le \chi_2(x,y) \quad (5.19)$$

and the triple integral over T can be expressed by three ordinary integrals as:

$$\int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}y \right] \mathrm{d}x. \quad (5.20)$$

Here we first integrate with z [from $z = \chi_1(x, y)$ to $z = \chi_2(x, y)$], then with respect to y [from $y = \phi_1(x)$ to $y = \phi_2(x)$], and finally with respect to x [from $x = a$ to $x = b$].

There is nothing special about this order of integration. Other orders of integration are possible and in some cases more convenient. Suppose for example that the projection of T onto the xz-plane is a domain Ω_{xz} of the form

$$z_1 \leq z \leq z_2, \quad \phi_1(z) \leq x \leq \phi_2(z). \tag{5.21}$$

If T is the set of all (x, y, z) with

$$z_1 \leq z \leq z_2, \quad \phi_1(z) \leq \ x \leq \phi_2(z), \quad \psi_1(x, z) \leq y \leq \psi_2(x, z) \tag{5.22}$$

then

$$\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z_1}^{z_2} \left[\int_{\phi_1(z)}^{\phi_2(z)} \left(\int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x, y, z)\,\mathrm{d}y\right) \mathrm{d}x\right] \mathrm{d}z. \tag{5.23}$$

In this case we integrate first with respect to y, then with respect to x, and finally with respect to z. Still four other orders of integration are possible.

5.3.3 Evaluating Triple Integrals Using Cylindrical Coordinates

Let T be a solid whose projection onto the xy-plane is labelled Ω_{xy}. Then the solid T is the set of all points (x, y, z) satisfying

$$(x, y) \in \Omega_{xy}, \chi_1(x, y) \leq z \leq \chi_2(x, y). \tag{5.24}$$

The domain Ω_{xy} has polar coordinates in some set $\Omega_{r\theta}$ and then the solid T in cylindrical coordinates is some solid S satisfying

$$(r, \theta) \in \Omega_{r\theta}, \quad \chi_1(r\cos(\theta), r\sin(\theta)) \leq z \leq \chi_2(r\cos(\theta), r\sin(\theta)). \tag{5.25}$$

Then

$$\begin{aligned}\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int\int_{\Omega_{xy}} \left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x, y, z)\,\mathrm{d}z\right) \mathrm{d}x\,\mathrm{d}y \\ &= \int\int_{\Omega_{r\theta}} \left(\int_{\chi_1(r\cos(\theta),r\sin(\theta))}^{\chi_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\,\mathrm{d}z\right) r\,\mathrm{d}r\,\mathrm{d}\theta = \\ &\int\int\int_S f(r\cos(\theta), r\sin(\theta), z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.\end{aligned} \tag{5.26}$$

5.3.4 Evaluating Triple Integrals Using Spherical Coordinates

Let T be a solid in xyz-space with spherical coordinates in the solid S of $\rho\theta\phi$-space. Then

$$\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int\int_S f(\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)\,\rho^2\sin\theta \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{5.27}$$

5.4 Jacobians and changing variables in multiple integration

During the course of the last few sections you have met several formulae for changing variables in multiple integration: to polar coordinates, to cylindrical coordinates, to spherical coordinates. The purpose of this section is to bring some unity to that material and provide a general description for other changes of variable.

5.4.1 Change of variables for double integrals

Consider the change of variables $x = x(u, v)$ and $y = y(u, v)$, which maps the points (u, v) of some domain Γ into the points (x, y) of some other domain Ω. Then

$$\text{The area of } \Omega = \int\int_{\Gamma} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\,\mathrm{d}v. \tag{5.28}$$

Suppose now that we want to integrate some function $f(x, y)$ over Ω. If this proves difficult to do directly, then we can change variables (x, y) to (u, v) and try to integrate over Γ instead. Then

$$\int\int_{\Omega} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_{\Gamma} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\,\mathrm{d}v. \tag{5.29}$$

5.4.2 Change of variables for triple integrals

Consider the change of variables $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ which maps the points (u, v, w) of some solid S into the points (x, y, z) of some other solid T. Then

$$\text{The volume of } T = \int\int\int_{S} \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w. \tag{5.30}$$

Suppose now that we want to integrate some function $f(x, y, z)$ over T. If this proves difficult to do directly, then we can change variables (x, y, z) to (u, v, w) and try to integrate over S instead. Then

$$\int\int\int_{T} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int\int_{S} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w. \tag{5.31}$$

Referring back to equations (5.26) and (5.27), and the Jacobians given at the end of §4.5, we can verify that this formula is correct for a change from Cartesian to cylindrical coordinates (Jacobian is r) and for a change from Cartesian to spherical coordinates (Jacobian is $\rho^2 \sin\theta$).

Chapter 6

LINE INTEGRALS AND SURFACE INTEGRALS

In this chapter we will study integration along curves and integration along surfaces. At the heart of this subject lie three great theorems: *Green's theorem*, *Gauss's theorem* (commonly known as the *divergence theorem*) and *Stokes's theorem*. All of these are ultimately based on the *fundamental theorem of integral calculus*, and all can be cast in the same general form: *An integral over a region S = An integral over the boundary of S.*

6.1 Line integrals

Let $\underline{h}(x, y, z) = (h_1(x, y, z), h_2(x, y, z), h_3(x, y, z))$ be a vector function that is continuous over a smooth curve C parametrised by $C : \underline{r}(u) = (x(u), y(u), z(u))$ with $u \in [a, b]$. The *line integral* of $\underline{h}$ over C is the number

$$\int_C \underline{h}(\underline{r}) \cdot \mathrm{d}\underline{r} = \int_a^b [\underline{h}(\underline{r}(u)) \cdot \underline{r}'(u)] \, \mathrm{d}u. \tag{6.1}$$

Although we stated this definition in terms of three-dimensional vectorial functions $\underline{h}(x, y, z)$ and curves in space $\underline{r}(u) = (x(u), y(u), z(u))$, it also includes the two-dimensional case: $\underline{h}(x, y)$ and plane curves $\underline{r}(u) = (x(u), y(u))$.

If the curve C is not smooth but is made up of a finite number of adjoining smooth pieces $C_1, \ldots, C_n$, i.e. it is *piecewise smooth*, then we define the integral over C as the sum of the integrals over C_i for $i = 1, \ldots, n$, that is $\int_C = \int_{C_1} + \cdots + \int_{C_n}$. All polygonal paths are piecewise smooth.

When we integrate over a parametrised curve, we integrate in the direction determined by the parametrisation. If we integrate in the opposite direction, our answer is altered by a factor of -1, that is $\int_{-C} = -\int_C$.

6.1.1 Another notation for line integrals

If $\underline{h}(x, y, z) = (h_1(x, y, z), h_2(x, y, z), h_3(x, y, z))$ then the line integral over a curve C can be written as

$$\int_C \underline{h}(\underline{r}) \cdot \mathrm{d}\underline{r} = \int_C \{h_1(x, y, z) \, \mathrm{d}x + h_2(x, y, z) \, \mathrm{d}y + h_3(x, y, z) \, \mathrm{d}z\}. \tag{6.2}$$

6.2 The Fundamental Theorem for Line Integrals

In general, if we integrate a vector function $\underline{h}$ from one point to another, the value of the line integral depends on the path chosen. There is, however, an important exception. If the vector function $\underline{h}$ is a *gradient*, i.e. there exists a scalar function f such that $\underline{h} = \underline{\nabla} f$, then the value of the line integral depends only on the endpoints of the path and not on the path itself. The details are spelled out in the following theorem.

Theorem

Let C, parametrised by $\underline{r} = \underline{r}(u)$ with $u \in [a, b]$, be a piecewise smooth curve that begins at $\underline{\alpha} = \underline{r}(a)$ and ends at $\underline{\beta} = \underline{r}(b)$. Then if the vector function $\underline{h}$ is a gradient, i.e. $\underline{h} = \underline{\nabla} f$, we have

$$\int_C \underline{h}(\underline{r}) \cdot \underline{\mathrm{d}r} = \int_C \underline{\nabla} f(\underline{r}) \cdot \underline{\mathrm{d}r} = f(\underline{\beta}) - f(\underline{\alpha}). \tag{6.3}$$

NOTE: It is important to see that this result is an extension of the fundamental theorem of integral calculus: $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.

Corollary

If the curve C is closed, i.e. $\underline{\alpha} = \underline{\beta}$, then $f(\underline{\alpha}) = f(\underline{\beta})$ and $\int_C \underline{\nabla} f(\underline{r}) \cdot \underline{\mathrm{d}r} = 0$.

6.3 Line integrals with respect to arc length

Suppose that f is a scalar function continuous on a piecewise smooth curve C parametrised by $\underline{r} = \underline{r}(u)$ with $u \in [a, b]$. If $s(u)$ is the length of the curve from the tip of $\underline{r}(a)$ to the tip of $\underline{r}(u)$, then, as we have seen in section 2.3, $s'(u) = ||\underline{r}'(u)||$. The integral of f over C *with respect to arc length* s is defined by setting

$$\int_C f(\underline{r})\,\mathrm{d}s = \int_a^b f(\underline{r}(u)) s'(u)\,\mathrm{d}u. \tag{6.4}$$

6.4 Green's Theorem

If $P(x, y)$ and $Q(x, y)$ are scalar functions defined over a domain Ω with piecewise smooth closed boundary C, then

$$\int\int_\Omega \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right] \mathrm{d}x\,\mathrm{d}y = \oint_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y \tag{6.5}$$

where the integral on the right is a line integral over C taken in the anticlockwise direction.

Remark As indicated, the symbol $\oint$ is used to denote the line integral over a simple closed curve C taken in the anticlockwise direction.

6.5 Parametrised Surfaces; Surface Area

We have seen that a space curve C can be parametrised by a vector function $\underline{r} = \underline{r}(u)$ where u ranges over some interval I of the u-axis. In an analogous manner, we can parametrise a surface S in space by a vector function $\underline{r} = \underline{r}(u, v)$ where (u, v) ranges over some domain Ω of the uv-plane.

Example (The graph of a function)

The graph of a function $y = f(x)$, $x \in [a, b]$ can be parametrised by setting $\underline{r}(u) = (u, f(u))$, $u \in [a, b]$.

Similarly, the graph of a function $z = f(x, y)$, $(x, y) \in \Omega$ can be parametrised by setting $\underline{r}(u, v) = (u, v, f(u, v))$, $(u, v) \in \Omega$.

Example (A plane)

If two vectors $\underline{a}$ and $\underline{b}$ are not parallel, then the set of all combinations $u\underline{a} + v\underline{b}$ generates a plane P_0 that passes through the origin. We can parametrise this plane by setting $\underline{r}(u, v) = u\underline{a} + v\underline{b}$, u, v real numbers.

The plane P that is parallel to P_0 and passes through the tip of a vector $\underline{c}$ can be parametrised by setting $\underline{r}(u, v) = u\underline{a} + v\underline{b} + \underline{c}$, u, v real numbers.

Example (A sphere)

The sphere of radius a centred at the origin can be parametrised by setting

$$\underline{r}(u, v) = (a\sin(u)\cos(v), a\sin(u)\sin(v), a\cos(u)), \quad (u, v) \in [0, \pi] \times [0, 2\pi). \tag{6.6}$$

6.5.1 The fundamental vector product

Let S be a surface parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$. The cross product

$$\underline{N} = \underline{r}'_u \times \underline{r}'_v \tag{6.7}$$

is called the *fundamental vector product* of the surface S.

The vector $\underline{N}(u, v)$ is perpendicular to the surface S at the point with position vector $\underline{r}(u, v)$ and, if different from zero, can be taken as the normal to the surface S at that point.

Example

For the plane $\underline{r}(u, v) = u\underline{a} + v\underline{b} + \underline{c}$, the vector $\underline{a} \times \underline{b}$ is normal to the plane.

Example

The fundamental vector product for a sphere is parallel to the radius vector $\underline{r}(u, v)$. (Using the parametrisation given above, $\underline{N} = a\sin(u)\underline{r}$.)

6.5.2 The area of a parametrised surface

The area of a surface S parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$, is given by

$$\text{Area of } S = \int\int_S ||\underline{N}(u, v)||\, \mathrm{d}u\, \mathrm{d}v. \tag{6.8}$$

Example (The surface area of a sphere)

Using the parametrisation given by equation (6.6), we had $\underline{N} = a\sin(u)\underline{r}$ so $||\underline{N}|| = a^2\sin(u)$ and the area is

$$\int\int_S ||\underline{N}(u, v)||\, \mathrm{d}u\, \mathrm{d}v = a^2 \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sin(u)\, \mathrm{d}u\, \mathrm{d}v = 2\pi a^2[-\cos(v)]_0^{\pi} = 4\pi a^2.$$

Example (The area of a plane domain)

A plane domain may be parametrised as $\underline{r} = (u, v, 0)$ for $(u, v) \in \Omega$. Then $\underline{r}'_u = (1, 0, 0)$ and $\underline{r}'_v = (0, 1, 0)$ and so the fundamental vector product is $\underline{N} = (0, 0, 1)$ which has magnitude 1.

$$\int\int_\Omega 1\, \mathrm{d}u\, \mathrm{d}v = \text{Area of } \Omega.$$

6.5.3 The area of a surface $z = f(x, y)$

Let the surface S be the graph of the function $z = f(x, y)$ with $(x, y) \in \Omega$. Then

$$\text{Area of } S = \int\!\!\int_\Omega \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right]^{1/2} \mathrm{d}x\,\mathrm{d}y. \tag{6.9}$$

In this case the parametrisation of S is $\underline{r}(u, v) = (u, v, f(u, v))$, $(u, v) \in \Omega$ and so $\underline{N} = (-f_x, -f_y, 1)$. The unit vector $\underline{n} = \underline{N}/||\underline{N}||$ is called the *upper unit normal*.

6.6 Surface Integrals

Let $H(x, y, z)$ be a scalar function, continuous over a surface S parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$. The *surface integral* of H over S is the number

$$\int\!\!\int_S H(x, y, z)\,\mathrm{d}\sigma = \int\!\!\int_\Omega H(\underline{r}(u, v))||\underline{N}(u, v)||\,\mathrm{d}u\,\mathrm{d}v. \tag{6.10}$$

Taking $H \equiv 1$ and referring back to eq. (6.8) we get

$$\int\!\!\int_S \mathrm{d}\sigma = \text{Area of } S. \tag{6.11}$$

6.6.1 Flux of a vector function

Let $\underline{q}(x, y, z)$ be a vector function that is continuous over a smooth surface S parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$. The *flux* of $\underline{q}$ across S in the direction of the unit normal $\underline{n}$ to the surface S is the number

$$\int\!\!\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma \tag{6.12}$$

which can be calculated as

$$\int\!\!\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma = \int\!\!\int_\Omega \underline{q}(\underline{r}(u, v)) \cdot \underline{n}\,||\underline{N}||\,\mathrm{d}u\,\mathrm{d}v = \int\!\!\int_\Omega \underline{q}(\underline{r}(u, v)) \cdot \underline{N}\,\mathrm{d}u\,\mathrm{d}v. \tag{6.13}$$

Proposition

If S is the graph of a function $z = f(x, y)$ with $(x, y) \in \Omega$ and $\underline{n}$ is the upper unit normal, then the flux of the vector function $\underline{q} = (q_1(x, y, z), q_2(x, y, z), q_3(x, y, z))$ across S in the direction of $\underline{n}$ is

$$\int\!\!\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma = \int\!\!\int_\Omega (-q_1 f_x - q_2 f_y + q_3)\,\mathrm{d}x\,\mathrm{d}y. \tag{6.14}$$

Proof

We can parametrise the surface by $\underline{r} = (u, v, f(u, v))$ with $(u, v) \in \Omega$. Then the fundamental vector product is

$$\underline{N} = \underline{r}'_u \times \underline{r}'_v = (1, 0, f_u) \times (0, 1, f_v) = (-f_u, -f_v, 1)$$

and we have

$$\begin{aligned} \int\!\!\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma &= \int\!\!\int_\Omega (\underline{q} \cdot \underline{N})\,\mathrm{d}u\,\mathrm{d}v \\ &= \int\!\!\int_\Omega (-q_1 f_u - q_2 f_v + q_3)\,\mathrm{d}u\,\mathrm{d}v = \int\!\!\int_\Omega (-q_1 f_x - q_2 f_y + q_3)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

where we have simply changed the names of the variables at the end.

6.7 The Divergence (Gauss) Theorem

Recall that if $P(x,y)$ and $Q(x,y)$ are scalar functions defined over a domain Ω with piecewise smooth closed boundary C, then Green's theorem (section 6.4) allowed us to express a double integral over Ω as a line integral over C:

$$\int\int_{\Omega}\left[\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)\right]\mathrm{d}x\,\mathrm{d}y=\oint_C P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y. \tag{6.15}$$

This formula can be rewritten in vector terms (using $\underline{q}=(Q,-P)$) to give the *divergence theorem in two dimensions* as follows:

The divergence theorem in two dimensions

Let Ω be a two-dimensional domain bounded by a piecewise smooth closed curve C. Then for any (continuously differentiable) vector function $\underline{q}(x,y)$ we have that

$$\int\int_{\Omega}(\underline{\nabla}\cdot\underline{q})\,\mathrm{d}x\,\mathrm{d}y=\oint_C(\underline{q}\cdot\underline{n})\,\mathrm{d}s \tag{6.16}$$

where $\underline{n}$ is the outer unit normal and the integral on the right is taken with respect to arc length.

We can now give the three-dimensional analogue of the divergence (Gauss) theorem.

The divergence theorem in three dimensions

Let T be a three-dimensional solid bounded by a piecewise smooth closed surface S. Then for any (continuously differentiable) vector function $\underline{q}(x,y,z)$ we have that

$$\int\int\int_T(\underline{\nabla}\cdot\underline{q})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int\int_S(\underline{q}\cdot\underline{n})\,\mathrm{d}\sigma \tag{6.17}$$

where $\underline{n}$ is the outer unit normal.

6.7.1 Divergence as outward flux per unit volume

In eq. (6.17), the right-hand side $\int\int_S(\underline{q}\cdot\underline{n})\,\mathrm{d}\sigma$ represents the $\underline{q}$ across S in the direction of $\underline{n}$. In this sense, from eq. (6.17) we can say that *the divergence is the outward flux per unit volume*, as we discussed in section 4.4.1.

Points $(x,y,z)\in T$ for which

- $\underline{\nabla}\cdot\underline{q}(x,y,z)<0$ are called *sinks*.
- $\underline{\nabla}\cdot\underline{q}(x,y,z)>0$ are called *sources*.
- If $\underline{\nabla}\cdot\underline{q}(x,y,z)\equiv 0$ then $\underline{q}$ is called *solenoidal*.

6.8 Stokes's Theorem

We return to Green's theorem (section 6.4):

$$\int\int_{\Omega}\left[\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)\right]\mathrm{d}x\,\mathrm{d}y=\oint_C P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y. \tag{6.18}$$

and this time setting $\underline{q} = (P, Q, R)$ a vector function, we have

$$(\underline{\nabla} \times \underline{q}) \cdot \underline{k} = det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \cdot \underline{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \tag{6.19}$$

Thus in vector terms Green's theorem can be written as

$$\int \int_{\Omega} [(\underline{\nabla} \times \underline{q}) \cdot \underline{k}] \, \mathrm{d}x \, \mathrm{d}y = \oint_C \underline{q}(\underline{r}) \cdot \underline{\mathrm{d}r}. \tag{6.20}$$

Since any plane can be coordinatised as the xy-plane, this result can be phrased in the following theorem

Stokes's theorem

Let S be a smooth surface with smooth bounding curve C. Then for any (continuously differentiable) vectorial function $\underline{q}(x, y, z)$ we have

$$\int \int_{S} [(\underline{\nabla} \times \underline{q}) \cdot \underline{n}] \, \mathrm{d}\sigma = \oint_C \underline{q}(\underline{r}) \cdot \underline{\mathrm{d}r} \tag{6.21}$$

where $\underline{n}$ is a unit normal that varies continuously on S, and the line integral $\oint_C$ is taken in the positive sense with respect to $\underline{n}$.

6.8.1 The normal component of $\underline{\nabla} \times \underline{q}$ as circulation per unit area; Irrotational flow

Interpret the vector function $\underline{q}(x, y, z)$ as the velocity of a fluid. In eq. (6.21), the right-hand side line integral $\oint_C \underline{q}(\underline{r}) \cdot \underline{\mathrm{d}r}$ is called the *circulation* of $\underline{q}$ around the curve C. In this sense, from eq. (6.21), we can say that $\underline{\nabla} \times \underline{q}$ in the direction $\underline{n}$ is the circulation of $\underline{q}$ per unit area, which relates to the rotation of the fluid as discussed in section 4.4.2.

If $\underline{\nabla} \times \underline{q} \equiv \underline{0}$ then there is no circulation and $\underline{q}$ is called *irrotational*, i.e. the fluid has no rotational tendency.

Chapter 7

EXAMPLE SHEETS

There are 6 example sheets, five of which are to be handed in on the timetable below:

Sheet 1 Friday 5th October

Sheet 2 Friday 19th October

Sheet 3 Friday 2nd November

Sheet 4 Friday 16th November

Sheet 5 Friday 30th November

Sheet 6 Not handed in.

Chapter 8

GENERAL INFORMATION

Lecturer: Prof. F. W. Nijhoff
Department of Applied Mathematics
Room 9.20a, telephone ext. 35120
e-mail: nijhoff@maths.leeds.ac.uk

Lectures and example classes There will be 33 hours total of lectures and classes during 11 weeks.

- Lectures on Tuesdays 3-4pm in RSLT 8;
- Classes on Mondays 4-5pm in RSLT 6;
- Lectures on Fridays 1-2pm in RSLT 6.

webpage The MATH2420 webpage conatining most of the material can be found under the URL:
http://www.maths.leeds.ac.uk/%7Efrank/math2420.html
or follow the link under the Lecturer's staff page:
http://www.maths.leeds.ac.uk/ frank
and click on **Teaching**. This page will contain most of the material, but be aware that some material may be updated during the term.

Example sheets Every two weeks you are expected to hand in the solutions to an example sheet. Your work will be marked to monitor progress on the first 5 sheets, but worked solutions to the last sheet (on the last section of the course) will be handed out to help your revision over the Christmas vacation. The completed sheets will be handed in at the Friday lectures, on the dates given on page 34. The schedule will be as follows:

Expl. sheet	Handout date	Due date
# 1	25/9	5/10
# 2	5/10	19/10
# 3	19/10	2/11
# 4	2/11	16/11
# 5	16/11	30/11
# 6	27/11	not marked

Marking The exercises will be marked by postgraduate students (the top mark being 5 for each sheet). They count for up to 15% of your course marks.

What I expect from you

- Attend lectures and examples classes
- Take notes during lectures (this document is only a summary)
- Attempt the examples on your own
- Hand in example sheets on time
- Ask questions as they occur to you
- Turn off your mobile phone!
- be considerate to your fellow students (do not hold disruptive conversations during lectures)

Booklist

1. M. R. Spiegel, Theory and Problems of Vector Analysis, McGraw-Hill.
2. E. Kreyszig, Advanced Engineering Mathematics, Wiley.
3. P. V. O'Neil, Advanced Engineering Mathematics, PWS-Kent Publishing.
4. A. C. Bajpai, Advanced Engineering Mathematics, Wiley, 1977.
5. C. R. Wylie and L. C. Barrett, Advanced Engineering Mathematics, McGraw-Hill.
6. P. C. Matthews, Vector Calculus, 1998.

Appendix A

GREEK ALPHABET

Letter name	Lower case	Capital
Alpha	α	A
Beta	β	B
Gamma	γ	Γ
Delta	δ	Δ
Epsilon	ϵ	E
Zeta	ζ	Z
Eta	η	H
Theta	θ	Θ
Iota	ι	I
Kappa	κ	K
Lambda	λ	Λ
Mu	μ	M
Nu	ν	N
Xi	ξ	Ξ
Omicron	o	O
Pi	π	Π
Rho	ρ	P
Sigma	σ	Σ
Tau	τ	T
Upsilon	υ	Υ
Phi	ϕ	Φ
Chi	χ	X
Psi	ψ	Ψ
Omega	ω	Ω

Appendix B

$\underline{\nabla}$ IN POLAR COORDINATES

You need not memorise these. In each coordinate system we use the unit vectors pointing along the directions of increasing coordinates: thus $\underline{e}_r$ is the unit vector in the r direction and so on. Any vector can be written in terms of a set of unit vectors, e.g. in spherical polars

$$\underline{v} = v_\rho \underline{e}_\rho + v_\theta \underline{e}_\theta + v_\phi \underline{e}_\phi .$$

Plane polar coordinates

$$\begin{aligned}
\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta \\
\underline{\nabla} \cdot \underline{v} &= \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}
\end{aligned}$$

Cylindrical polar coordinates

$$\begin{aligned}
\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{\partial f}{\partial z}\underline{e}_z \\
\underline{\nabla} \cdot \underline{v} &= \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Spherical polar coordinates

$$\begin{aligned}
\underline{\nabla} f &= \frac{\partial f}{\partial \rho}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\underline{e}_\phi \\
\underline{\nabla} \cdot \underline{v} &= \frac{1}{\rho^2}\frac{\partial (\rho^2 v_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta}\frac{\partial (\sin\theta\, v_\theta)}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
\nabla^2 f &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.
\end{aligned}$$

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