Essential Points of Hydrodynamic Stability: Review of §2

Any flow that is observed either in nature or in an experiment (and indeed also in a numerical experiment) must both satisfy the governing equations *and* be stable.

This is of course true in a wider sense, outside fluid mechanics. For example, a pencil balanced on its point satisfies the equations of statics — with the weight of the pencil balanced by an upward reaction from the table — but it is impossible to realise this state since it is unstable. The slightest perturbation causes the pencil to fall.

So for any stability problem there are two stages. The first is to identify a solution of the governing equations ("the basic state"); the second is to consider perturbations to this solution.

The Basic State

This will typically be a geometrically simple, steady flow. For example, for flow in a channel we may consider a basic state $\boldsymbol{u} = U(y)\hat{\boldsymbol{x}}$; for flow in a pipe we can examine a basic state $\boldsymbol{u} = U(r)\hat{\boldsymbol{z}}$; and for convection we can consider a static basic state with a linear temperature gradient.

Stability theory can be pursued for more complicated basic states — for example, one might be able to find time-dependent, wave-like solutions. The subsequent stability analysis will though be technically more difficult.

Stability

The basic idea of stability theory is to determine whether perturbations to a given basic state grow or decay. The former would lead to instability, the latter to stability.

A crucial point is the size of the perturbations. If a basic state can be shown to be stable to *all* perturbations, of whatever size, then we say that the solution is globally stable. Such results are quite hard to achieve.

Most stability theory — and just about everything we shall cover in this course — concerns the study of small (formally infinitesimal) perturbations. If the perturbations are small then it is legitimate to neglect nonlinear products of perturbations, leading to a set of linear perturbation equations. Perturbations will then, typically, grow or decay exponentially. If there is at least one growing perturbation then the system is linearly unstable; if *all* infinitesimal perturbations decay then the system is linearly stable.

It should be pointed out that in an idealised system in which dissipation is neglected (not strictly physical) then linear perturbations will either grow or will oscillate; they cannot decay in the absence of any damping mechanism. Here, stable solutions therefore are oscillatory in time, not decaying.

Linear stability theory is a very useful tool in fluid dynamics. It does though have its limitations. According to linear theory, an unstable perturbation will grow exponentially with time, without bound. Obviously this is unphysical. At some stage it will no longer be valid to neglect nonlinear products of perturbations and these neglected nonlinear terms will have to be reinstated.

An Example

Consider axial flow along an infinitely long pipe of circular cross-section. A simple basic state solution (see Example Sheet 1) is $\boldsymbol{u} = \boldsymbol{u}_0 = U(r)\hat{\boldsymbol{z}}$ driven by a constant axial pressure gradient, $p = Gz$ (cylindrical polar coordinates). $U(r)$ is readily calculated.

We then perturb $\boldsymbol{u}$ and p as

$$\boldsymbol{u} = U(r)\hat{\boldsymbol{z}} + \tilde{\boldsymbol{u}}, \qquad p = Gz + \tilde{p}.$$

We then substitute these expressions into the governing equations (here the Navier-Stokes equation and $\nabla \cdot \boldsymbol{u} = 0$) and retain only the linear terms. This gives, quite generally,

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + \boldsymbol{u}_0 \cdot \nabla \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{u}_0 = -\frac{1}{\rho}\nabla \tilde{p} + \nu \nabla^2 \tilde{\boldsymbol{u}}, \qquad \nabla \cdot \tilde{\boldsymbol{u}} = 0. \qquad (*)$$

Note that the basic state has dropped out (this must always happen) and that the nonlinear term $\tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}}$ is neglected as it is second order in the small perturbation.

In any such stability analysis we end up with a linear system of equations, for which there are well-developed methods of solution.

For this particular problem of pipe flow, the basic state is dependent only on r (the pressure depends linearly on z, but it is only the pressure gradient that matters, and this is constant) — and independent of θ, z and t.

We may thus develop a *normal mode* solution of the form

$$\tilde{\boldsymbol{u}}(r, \theta, z, t) = \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}k \sum_{n=-\infty}^{\infty} \int \mathrm{d}s\, \hat{\boldsymbol{u}}(r, k, n, s) \exp(st + i(kz + n\theta)) \qquad (**)$$

(similarly for $\tilde{p}$). This may be regarded as a Laplace transform with respect to t, a Fourier series with respect to θ and a Fourier transform with respect to z.

Since we are dealing with a linear system, each mode *separately* must obey equation (*), so in reality we do not need to deal with expressions as complicated as (**). Note that we cannot specify the form of the r-dependence. Since the basic state is r-dependent then we will obtain an ordinary differential equation for $\hat{\boldsymbol{u}}$ for each mode. The fluid boundary conditions for $\hat{\boldsymbol{u}}$ will need to be satisfied; this will only be possible for certain values of s, the *eigenvalues* of the problem. If *any* mode has an eigenvalue s with positive real part then the system is (linearly) unstable.

Suppose instead that we had managed to identify a steady basic state that was more complicated, depending on r and θ, but not on z or t. Then our normal mode solution for $\tilde{\boldsymbol{u}}$ would take the form

$$\tilde{\boldsymbol{u}}(r, \theta, z, t) = \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}k \int \mathrm{d}s\, \hat{\boldsymbol{u}}(r, k, n, s) \exp(st + i(kz + n\theta)).$$

We would then obtain *partial* differential equations for $\hat{\boldsymbol{u}}$, involving partial derivatives in r and θ. The general ideas are the same, but technically the problem is more complicated.