Fluid Dynamics Revision

A concise review of some of the most important elements that you will have met (or maybe are currently meeting) in MATH 2620 or MATH 3501.

1 Basic Ideas

For a fluid with velocity $\boldsymbol{u}(x, y, z, t) = (u, v, w)$ (in Cartesian coordinates) we can define three important means of visualising the flow:

(i) *Streamlines* are an instantaneous snapshot of the flow at a given time t. They are defined by

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w}.$$

Here t is held constant.

(ii) *Particle paths* show the path taken by a particular blob of fluid as it is carried around by the flow. They are defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = w.$$

i.e. three (coupled) ordinary differential equations need to be solved. The blob in question is identified by its position at a given time; e.g. $(x, y, z) = (x_0, y_0, z_0)$ at time $t = t_0$. Often this is specified as an initial condition, i.e. $t_0 = 0$.

(iii) *Streaklines* are the lines formed by releasing dye into the fluid at a given point for a certain length of time.

Streamlines, particle paths and streaklines are the same for *steady* flows, i.e. flows that do not depend on time. For a general time-dependent flow though they are all different.

2 Forces

There are two types of forces that can act on a fluid:

(i) *Long range forces* (or body forces). The most familiar and important of these is the force due to gravity.

(ii) *Short range* (internal) forces between adjacent fluid elements. In a frictionless (inviscid) fluid the force across a surface element of area δS and normal $\mathbf{n}$ is

$$p\mathbf{n}\delta S,$$

where $p(x, y, z, t)$ is a scalar function, independent of the normal $\mathbf{n}$, called the *pressure.*

In reality there are no such things as inviscid fluids, and it is important therefore to consider the nature of the viscous forces. We shall do this shortly.

3 The equation of motion

Applying Newton's 2nd law of motion to a blob of fluid leads to the equation of motion

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) = -\nabla p + \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{F}$ represents the sum of all the body forces. The operator $\mathrm{D}/\mathrm{D}t$ denotes the rate of change of a quantity *whilst following the fluid* (sometimes referred to as the material, convective or Lagrangian derivative). Equation (1) is known as *Euler's equation.*

The most significant feature of this equation is that the acceleration consists of two terms. The first, which is to be expected, is the time derivative of the velocity; the second, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ term, is characteristic of fluid dynamics and represents the fact that a fluid element can change its velocity by moving within the fluid, even though the fluid velocity itself may not change in time. A simple example of this is being carried down on a raft by a river that suddenly narrows; the velocity increases and the raft accelerates.

The material derivative of a function $F(\boldsymbol{x}(t), t)$ can be derived from the chain rule for differentiation, recognising that the position $\boldsymbol{x}$ of a fluid particle changes with time. Thus, in Cartesian coordinates,

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + \boldsymbol{u}\cdot\nabla F.$$

Note that $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is a *vector*. In Cartesian coordinates it takes the form

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$
$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\mathbf{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\mathbf{j} + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\mathbf{k}.$$

In other common coordinate systems (e.g. cylindrical or spherical polar coordinates) the $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ term is more complicated. We shall look at this again when we need it.

4 Conservation of mass

The principle of conservation of mass leads to the following equation (sometimes referred to as the continuity equation):

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0,$$

where ρ is the fluid density. For *incompressible* flows, for which ρ is constant, this reduces simply to

$$\nabla\cdot\boldsymbol{u} = 0.$$

5 Boundary conditions

For an ideal (inviscid) fluid the normal component of the velocity must vanish on a rigid impermeable boundary, i.e. $\boldsymbol{u}\cdot\mathbf{n} = 0$.

At a free surface the pressure is assumed constant (and equal to the atmospheric pressure of the air above).

6 Special classes of flow

The *vorticity* $\boldsymbol{\omega}$ of a fluid flow is defined as the curl of the velocity field, i.e.

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}.$$

Flows for which the vorticity vanishes are said to be *irrotational*. It is then possible to write the velocity as the gradient of a scalar, i.e.

$$\boldsymbol{u} = \nabla \phi,$$

where ϕ is known as the *velocity potential*.

If, furthermore, the fluid is incompressible ($\nabla \cdot \boldsymbol{u} = 0$) then all of the fluid dynamics reduces to the solution of Laplace's equation,

$$\nabla^2 \phi = 0,$$

subject to appropriate boundary conditions for ϕ.

7 Some Important Results

Kelvin's circulation theorem

Consider an inviscid, incompressible fluid of constant density under the action of a conservative body force $\boldsymbol{g} = -\nabla\chi$. Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds. Then the circulation

$$\Gamma = \int_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{s}$$

is independent of time.

Bernoulli's theorem for steady flow

For the steady flow of an ideal (inviscid and incompressible) fluid under the action of gravity (with $\boldsymbol{g} = -\nabla\chi$), H is constant along streamlines, where H is defined by:

$$H = \frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + \chi. \tag{2}$$

Bernoulli's theorem for steady irrotational flow

If an ideal fluid is in steady irrotational flow then H (as defined in (2)) is constant throughout the fluid.

8 Real Fluids

So far you will have met only *inviscid* fluids — i.e. fluids with no viscosity or "stickiness". In reality there are no such fluids; all fluids have viscosity. That said, there are certain circumstances for which the inviscid assumption is a good approximation — e.g. water waves and flow past streamlined bodies (aerofoils). Conversely there are many situations (e.g. flow past bluff bodies) in which the inviscid solution is not at all representative of the realistic viscous solution, even when the viscosity is, in some sense, small.