MATH 3375 (5376) Hydrodynamic Stability Examples 4

I will discuss this problem sheet in an Examples Class to be held in the last week of term (exact slot to be decided). Please have a look at the questions before then.

1. Consider inviscid flows of the form $U(z)\hat{\mathbf{x}}$ between $z = 0$ and $z = 1$. Find examples of flows $U(z)$ such that:
(i) Rayleigh's inflexion point criterion predicts stability;
(ii) Rayleigh's criterion cannot determine stability, but Fjørtoft's theorem guarantees stability;
(iii) neither Rayleigh nor Fjørtoft can guarantee stability.

2. Kelvin-Helmholtz instability in a finite channel. Suppose we have the flow (incompressible, inviscid, no gravity)

$$\mathbf{u} = \begin{cases} U_1\hat{\mathbf{x}} & (-d_1 \le z < 0) \\ U_2\hat{\mathbf{x}} & (0 < z \le d_2). \end{cases}$$

What are the boundary conditions on ϕ at $z = -d_1,\ d_2$? Seek solutions for ϕ separately in the upper and lower regions that satisfy the respective boundary conditions. Then by applying the matching conditions at the interface obtain the following dispersion relation (using the same notation as in lectures):

$$c = \frac{\left((U_2 \coth \alpha d_2 + U_1 \coth \alpha d_1) + i|U_2 - U_1| \left(\coth \alpha d_1 \coth \alpha d_2\right)^{1/2}\right)}{(\coth \alpha d_2 + \coth \alpha d_1)}.$$

What can you say about the stability of the flow?

3. Consider the triangular jet of an inviscid fluid with velocity profile $\mathbf{u} = U(z)\hat{\mathbf{x}}$, with

$$U(z) = \begin{cases} 0 & (z \ge d) \\ U_0(1 - |z|/d) & (0 \le |z| \le d). \end{cases}$$

First consider modes ϕ that are *even* in z. Show that the wave speed c is given by

$$2\alpha^2 c^2 + \alpha(1 - 2\alpha - \exp(-2\alpha))c - (1 - \alpha - (1 + \alpha)\exp(-2\alpha)) = 0,$$

where c has been scaled with U_0 and α has been scaled with $1/d$.

Now consider modes that are *odd* in z. Show that the wave speed c is given by

$$c = \left(1 - \exp(-2\alpha)\right)/2\alpha.$$

What can you say about the stability of the two classes of modes?

4. Show that in the absence of fluid flow in the basic state, stability is guaranteed if $N^2(z) > 0$ *everywhere* in the fluid.

5. Determine the stability of the unbounded vortex sheet

$$\mathbf{u} = \begin{cases} U_1\hat{\mathbf{x}}, & \rho = \rho_1 \quad \text{in } z < 0 \\ U_2\hat{\mathbf{x}}, & \rho = \rho_2 \quad \text{in } z > 0, \end{cases}$$

where U_1, U_2, ρ_1 and ρ_2 are constants, and where gravity acts in the negative z-direction. You will need to solve for ϕ separately in $z < 0$ and $z > 0$ and apply the interface conditions at $z = 0$.