MATH 3375 (5376) Hydrodynamic Stability Examples 3

I will discuss this problem sheet in the Examples Class to be held on Thursday 24 November 2011. Please have a go at the questions before then. Could you please hand in your solutions to question 4 in the lecture on Tuesday 29 November 2011?

1. It is sometimes stated that one can observe convection when cooking porridge.

First, what are the appropriate boundary conditions for this problem?

If one takes the values of α and κ for water, namely

$$\alpha = 7 \times 10^{-4} K^{-1} \quad \text{and} \quad \kappa = 1.5 \times 10^{-3} cm^2 s^{-1},$$

what can you say about whether convection will occur (you will need to think what are reasonable values for d and for ΔT, the temperature difference from top to bottom). The critical Rayleigh number for the appropriate boundary conditions is 571, and for comparison the kinematic viscosity of golden syrup is $\nu = 1400\, cm^2 s^{-1}$. (This problem was addressed by the famous geophysicist Sir Harold Jeffreys in 1926.)

2. In lectures we derived the following relation for the growth rate s for Rayleigh-Bénard convection (equation (3.31) in the notes):

$$(\pi^2 + a^2 + s)(\pi^2 + a^2 + \frac{s}{Pr})(\pi^2 + a^2) = Ra\, a^2.$$

Without actually solving this equation, derive approximate expressions for s when the Prandtl number $Pr \gg 1$.

Now go back to the linear perturbation equations for convection:

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + Ra\, Pr\, \theta \hat{\mathbf{z}} + Pr \nabla^2 \mathbf{u},$$
$$\nabla \cdot \mathbf{u} = 0,$$
$$\frac{\partial \theta}{\partial t} - w = \nabla^2 \theta.$$

By supposing $Pr \gg 1$ from the outset, and approximating these equations, re-derive the expressions for s.

3. Verify that the perturbation equations for the axisymmetric centrifugal instabilities ((4.15) – (4.18) in lectures) can be manipulated to give the single equation (4.19).

4. In lectures we considered the axisymmetric ($\partial_\theta \equiv 0$) instabilities of swirling flows, $\mathbf{u} = (0, V(r), 0)$ between two impermeable cylinders at $r = r_1, r_2$. Now consider the complementary problem of *two-dimensional* instabilities; i.e. modes for which $u_z = 0$ and $\partial_z \equiv 0$.

First show that there exists a stream function ψ such that

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}.$$

Show that the perturbation equations can be manipulated to give the following equation for ψ:

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}\right) \nabla^2 \psi - \frac{\mathrm{D}Z}{r}\frac{\partial \psi}{\partial \theta} = 0,$$

where D denotes $\partial/\partial r$, where the basic vorticity is given by

$$Z = \frac{1}{r}\mathrm{D}(rV) = \mathrm{D}_* V, \text{ say,}$$

and where the 2D Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

By taking normal modes of the form $\psi(r, \theta, t) = \phi(r) exp(st + in\theta)$ show that

$$(s + in\Omega)\left(\mathrm{D}_*\mathrm{D} - \frac{n^2}{r^2}\right)\phi - \frac{in}{r}(\mathrm{D}Z)\phi = 0.$$

What are the boundary conditions for ϕ?

Deduce that a necessary condition for instability to two-dimensional perturbations is that the basic vorticity gradient DZ changes sign somewhere in $r_1 < r < r_2$.