MATH 3375 (5376) Hydrodynamic Stability Examples 2

I will discuss this problem sheet in the Examples Class to be held on Thursday 3 November 2011. Please have a go at the questions before then. Could you please hand in your solutions to question 4 in the lecture on Thursday 10 November 2011?

1. The following set of two ordinary differential equations can be thought of as a 'toy' model for the instability of Rayleigh-Bénard convection:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{\nu}{d^2}w + \alpha g\theta, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{\kappa}{d^2}\theta + \beta w.$$

Show that the trivial solution $w = \theta = 0$ is unstable if $Ra > 1$, where $Ra = \alpha\beta g d^4/\kappa\nu$.

2. In lectures we obtained the following equations governing the perturbations (not necessarily small) in Rayleigh-Bénard convection (labelled (3.3) - (3.6) in lectures):

$$\begin{aligned}
\frac{\mathrm{D}\tilde{\mathbf{u}}}{\mathrm{D}t} &= -\frac{1}{\rho_0}\nabla\tilde{p} - \frac{\tilde{\rho}}{\rho_0}g\hat{\mathbf{z}} + \nu\nabla^2\tilde{\mathbf{u}}, \\
\nabla\cdot\tilde{\mathbf{u}} &= 0, \\
\tilde{\rho} &= -\alpha\rho_0\tilde{T}, \\
\frac{\mathrm{D}\tilde{T}}{\mathrm{D}t} - \beta\tilde{w} &= \kappa\nabla^2\tilde{T}.
\end{aligned}$$

By eliminating the density, and rescaling as outlined in lectures, obtain the following set of non-dimensional governing equations:

$$\begin{aligned}
\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= -\nabla p + Ra\,Pr\,\theta\hat{\mathbf{z}} + Pr\nabla^2\mathbf{u}, \\
\nabla\cdot\mathbf{u} &= 0, \\
\frac{\mathrm{D}\theta}{\mathrm{D}t} - w &= \nabla^2\theta,
\end{aligned}$$

where the Rayleigh number Ra and the Prandtl number Pr are defined by

$$Ra = \frac{\alpha\beta g d^4}{\kappa\nu}, \qquad Pr = \frac{\nu}{\kappa}.$$

3. Verify that

$$f(x,y) = \cos\left[\frac{a}{2}\left(\sqrt{3}x + y\right)\right] + \cos\left[\frac{a}{2}\left(\sqrt{3}x - y\right)\right] + \cos ay$$

is a solution of the equation

$$\nabla_H^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = -a^2 f.$$

Verify that f is invariant under rotations of $60°$ about the z-axis. [This is probably done most easily by transforming from (x,y) to (r,θ).] What pattern would you see in an experiment if the vertical velocity took the form $w(z)f(x,y)$?

4. In a porous medium the frictional term takes a different form to the Newtonian viscous term. It can be modelled as being proportional to the velocity itself — known as Darcy's law. Using this law it is possible to study convection in a porous medium — so-called Rayleigh-Darcy convection.

Consider the simplified case of two dimensional motions $\mathbf{u} = (u(x,z,t), 0, w(x,z,t))$. Then since, as in Rayleigh-Bénard convection, $\nabla \cdot \mathbf{u} = 0$, we can introduce a stream function ψ such that

$$u = -\frac{\partial \psi}{\partial z}, \qquad w = \frac{\partial \psi}{\partial x}.$$

The nonlinear equations describing Rayleigh-Darcy convection take the form

$$\begin{aligned} \nabla^2 \psi &= R\frac{\partial \theta}{\partial x}, \\ \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= \nabla^2 \theta, \end{aligned}$$

where R is a modified Rayleigh number.

Show that there is a static basic state with $\psi = 0$ and $\theta = -z$.

Suppose the boundaries at $z = 0$ and $z = 1$ are impermeable and perfectly thermally conducting. Write down the conditions satisfied by ψ and θ on the boundary.

Now consider linear perturbations to the static basic state, $\theta = -z + \tilde{\theta}$, etc. Seek normal mode solutions of the form

$$\tilde{\theta}(x,z,t) = \hat{\theta}(z)\exp(ikx + st), \qquad \text{etc.}$$

Find s and hence show that there is instability if $R > 4\pi^2$.