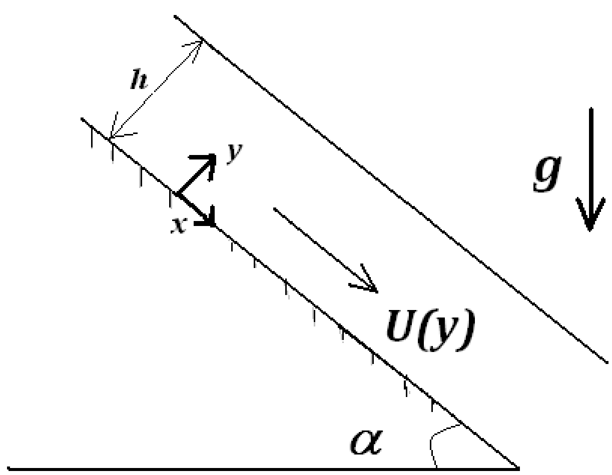
MATH 3375/5376 Hydrodynamic Stability Examples 1

I will discuss this problem sheet in the Examples Class to be held on Friday 14 October 2011. Please have a go at the questions before then. So that I can get an idea of how you are finding the course, could you please hand in your solutions to questions 3 and 6 in the lecture on Thursday 20 October 2011?

1. Viscous incompressible fluid of depth d flows down an inclined plane under the influence of gravity. The lower boundary may be assumed to be no-slip, and the upper free surface to be stress-free. Calculate the velocity as a function of the direction y perpendicular to the plane.



2. Viscous fluid flows steadily along an infinite straight circular cylinder of radius a with velocity $\mathbf{u} = (0, 0, w(r))$ (in cylindrical polar coordinates) driven by a uniform pressure gradient $P = -\partial p/\partial z$. Determine w by using the Navier-Stokes equation and the appropriate boundary conditions.

3. (a) Repeat Question 2 for a tube with annular cross-section, $a \leq r \leq b$.

(b) Now suppose instead that the motion is driven not by an external pressure gradient but by translation of the outer boundary ($r = b$) with velocity V, with the inner boundary ($r = a$) at rest. Determine w in this case.

4. Incompressible fluid occupies the space $0 < y < \infty$ above a plane rigid boundary $y = 0$ that oscillates to and fro in the x-direction with velocity $U \cos \omega t$. Show that, with no applied pressure gradient, the velocity field $\mathbf{u} = (u(y,t), 0, 0)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

By seeking a separable solution of the form

$$u(y,t) = \Re\left(\mathrm{e}^{i\omega t} f(y)\right)$$

and applying the appropriate boundary conditions, determine $u(y,t)$. What can you say about the influence of the oscillating boundary on the bulk of the fluid?

5. Find the steady solutions of the ordinary differential equation

$$\frac{du}{dt} = \lambda u - hu^3.$$

By considering a linear perturbation of these steady solutions, determine their stability. Considering separately the cases of $h > 0$ and $h < 0$, sketch the solutions, with their stability, in the (u, λ) plane.

6. The Eckhaus equation takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial^4 u}{\partial x^4},$$

with boundary conditions $u(x,0,t) = 0$, $u(x,1,t) = 1$ for $-\infty < x < \infty$, $t \geq 0$; R is a positive constant. Consider the linear stability of the basic flow $u = y$ by considering the perturbed solution $u = y + \tilde{u}$ and seeking a normal mode solution of the form

$$\tilde{u}(x,y,t) = \mathrm{e}^{ikx+st} f(y).$$

Show that

$$R(k^2 - k^4 - s) - k^2 = n^2\pi^2 \qquad (n = 1, 2, \ldots).$$

Hence show that the solution $u = y$ is linearly unstable for

$$R > R_c = \frac{\pi^2 + k_c^2}{k_c^2 - k_c^4},$$

where

$$k_c^2 = \pi^2\left(\sqrt{1 + \pi^{-2}} - 1\right).$$