

# Rayleigh-Bénard Convection: Summary of §3

## Governing equations and basic state

We shall apply the *Boussinesq approximation*, in which variations in density can be neglected, *except* in the buoyancy term in the equation of motion. Furthermore the density and temperature are assumed to be linearly related; this is a good approximation for a liquid.

Under the Boussinesq approximation the governing equations are

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g} + \nu\nabla^2\mathbf{u}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\rho = \rho_0(1 - \alpha(T - T_0)), \quad (3)$$

$$\frac{DT}{Dt} = \kappa\nabla^2 T. \quad (4)$$

Here  $\mathbf{u}$  is the fluid velocity,  $\rho$  and  $T$  are the density and temperature;  $\rho_0$  and  $T_0$  are representative values of  $\rho$  and  $T$  in the fluid.  $\kappa$  is the *thermal diffusivity*.

We consider a fluid layer confined between two horizontal planes at  $z = 0$  and  $z = d$ . It is straightforward to verify that there is a static ( $\mathbf{u} = 0$ ) basic state with

$$T = T_0 - \beta z, \quad \rho = \rho_0(1 + \alpha\beta z), \quad p = p_0 - g\rho_0(z + \alpha\beta z^2/2). \quad (5)$$

Here  $T_0$ ,  $\rho_0$  and  $p_0$  are the temperature, density and pressure on the lower boundary.

If  $\tilde{\mathbf{u}}$  etc. denote the perturbations away from the basic state then they are governed by the following equations:

$$\frac{D\tilde{\mathbf{u}}}{Dt} = -\frac{1}{\rho_0}\nabla\tilde{p} - \frac{\tilde{\rho}}{\rho_0}g\hat{\mathbf{z}} + \nu\nabla^2\tilde{\mathbf{u}}, \quad (6)$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad (7)$$

$$\tilde{\rho} = -\alpha\rho_0\tilde{T}, \quad (8)$$

$$\frac{D\tilde{T}}{Dt} - \beta\tilde{w} = \kappa\nabla^2\tilde{T}. \quad (9)$$

**[Important note:** At this stage there is no assumption that the perturbation is *small*. Equations (6) – (9) are for a *general* perturbation to the basic state.]

If we substitute for  $\tilde{\rho}$  in terms of  $\tilde{T}$  from (8) into (6) then we eliminate the variable  $\tilde{\rho}$ .

We can also write the equations in dimensionless form by scaling lengths with  $d$ , times with  $d^2/\kappa$ , pressure with  $\rho_0\kappa^2/d^2$  and temperature with  $\beta d$ . Then we obtain the following *dimensionless* equations (with  $\theta$  as the dimensionless temperature):

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + Ra Pr \theta \hat{\mathbf{z}} + Pr \nabla^2 \mathbf{u}, \quad (10)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (11)$$

$$\frac{D\theta}{Dt} - w = \nabla^2 \theta. \quad (12)$$

Equations (10) – (12) are the dimensionless, *nonlinear* governing equations for Rayleigh-Bénard convection. There are just two dimensionless parameters, the Rayleigh number  $Ra$  and the Prandtl number  $Pr$ , defined by

$$Ra = \frac{\alpha \beta g d^4}{\kappa \nu}, \quad Pr = \frac{\nu}{\kappa}.$$

## Boundary conditions

On each boundary we need to impose two conditions for the velocity and one for the temperature, i.e. six boundary conditions in total.

(i) For rigid or free boundaries, we impose no normal flow (impermeability), i.e.  $w = 0$  on  $z = 0$  and  $z = 1$ .

(ii) If the boundaries are, in addition, no-slip, then  $u = v = 0$  on the boundaries. This translates (using  $\nabla \cdot \mathbf{u} = 0$ ) into  $\partial w / \partial z = 0$  on boundaries.

For stress-free boundaries, on the other hand, the tangential stress vanishes at the boundary, i.e.

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \quad \text{on the boundaries.}$$

Combined with (i) this becomes

$$\frac{\partial u}{\partial z} = 0 \quad \text{and} \quad \frac{\partial v}{\partial z} = 0 \quad \text{on the boundaries.}$$

Using  $\nabla \cdot \mathbf{u} = 0$  this can be written as

$$\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{on the boundaries.}$$

(iii) If the temperature is fixed on a boundary (i.e. the boundary is a perfect conductor of heat) then the temperature perturbation  $\theta$  must vanish on the boundary.

Alternatively, if the *flux* of heat is fixed (insulating boundary condition), then  $\partial\theta/\partial z$  must vanish on the boundary.

### Linear analysis

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + Ra Pr \theta \hat{\mathbf{z}} + Pr \nabla^2 \mathbf{u}, \quad (13)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (14)$$

$$\frac{\partial \theta}{\partial t} - w = \nabla^2 \theta. \quad (15)$$

Eliminate the pressure by taking the curl of (15). If we define  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  then:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = Ra Pr \nabla \theta \times \hat{\mathbf{z}} + Pr \nabla^2 \boldsymbol{\omega}. \quad (16)$$

Now take the curl again, using

$$\nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u}, \quad \text{using (14)}. \quad (17)$$

Then the curl of (16) becomes

$$\frac{\partial \nabla^2 \mathbf{u}}{\partial t} = Ra Pr \left( \nabla^2 \theta \hat{\mathbf{z}} - \frac{\partial}{\partial z} (\nabla \theta) \right) + Pr \nabla^4 \mathbf{u}. \quad (18)$$

The  $z$ -component of this gives

$$\frac{\partial \nabla^2 w}{\partial t} = Ra Pr \nabla_H^2 \theta + Pr \nabla^4 w, \quad (19)$$

where

$$w = \mathbf{u} \cdot \hat{\mathbf{z}} \quad \text{and} \quad \nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

On eliminating  $\theta$  between (15) and (19) we obtain the following equation for  $w$ :

$$\left( \frac{\partial}{\partial t} - \nabla^2 \right) \left( \frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 w = Ra \nabla_H^2 (\partial t - \nabla^2) \theta = Ra \nabla_H^2 w. \quad (20)$$

This is now an equation purely in terms of  $w$ . We shall impose boundary conditions at  $z = 0$  and  $z = 1$ .

Seek normal mode solutions

$$\theta = \hat{\theta}(z)f(x, y) \exp(st), \quad w = \hat{w}(z)f(x, y) \exp(st).$$

On substituting into (15) and re-arranging:

$$\frac{s\hat{\theta} + \hat{w} - \hat{\theta}''}{\hat{\theta}} = \frac{\nabla_H^2 f}{f}.$$

LHS = function of  $z$ , RHS = function of  $x$  and  $y$ . So LHS = RHS = constant =  $-a^2$ , say. So  $f$  satisfies:

$$\nabla_H^2 f + a^2 f = 0. \quad (21)$$

Equation (19) becomes

$$\left( \frac{d^2}{dz^2} - a^2 - s \right) \left( \frac{d^2}{dz^2} - a^2 - \frac{s}{Pr} \right) \left( \frac{d^2}{dz^2} - a^2 \right) \hat{w} = -a^2 Ra \hat{w}. \quad (22)$$

### Principle of Exchange of Stabilities

We are able to prove a rather general result — namely that instability sets in via an exchange of stabilities (i.e.  $s$  passes through zero) — irrespective of the particular choice of boundary conditions.

Equation (15) becomes (on dropping the hats):

$$\left( \frac{d^2}{dz^2} - a^2 - s \right) \theta = -w.$$

Multiply through by  $\theta^*$  and integrate from  $z = 0$  to  $z = 1$ . After integrating by parts and applying the temperature boundary condition (either one) we get:

$$I_0 + I_1 = \int_0^1 w\theta^* dz,$$

where

$$I_0 = \int_0^1 |\theta|^2 dz, \quad I_1 = \int_0^1 \left( \left| \frac{d\theta}{dz} \right|^2 + a^2 |\theta|^2 \right) dz.$$

Equation (19) becomes

$$\left(\frac{d^2}{dz^2} - a^2\right) \left(\frac{d^2}{dz^2} - a^2 - \frac{s}{Pr}\right) w = a^2 Ra \theta.$$

Similarly, multiply through by  $w^*$  and integrate by parts, making use of the boundary conditions on  $w$ , to obtain:

$$J_2 + s \frac{J_1}{Pr} = a^2 Ra \int_0^1 w^* \theta dz,$$

where

$$J_1 = \int_0^1 \left( \left| \frac{dw}{dz} \right|^2 + a^2 |w|^2 \right) dz, \quad J_2 = \int_0^1 \left( \left| \frac{d^2 w}{dz^2} \right|^2 + 2a^2 \left| \frac{dw}{dz} \right|^2 + a^4 |w|^2 \right) dz.$$

Important point:  $I_0, I_1, J_1, J_2$  all real and positive.

Write  $s = \sigma + i\omega$  and take the real and imaginary parts. It then follows that if  $Ra < 0$  then  $\sigma < 0$ , i.e. stability. And if  $Ra > 0$  then  $\omega = 0$ . So instability must set in as  $s$  passes through zero, i.e. via an exchange of stabilities.

### A Specific Choice of Boundary Conditions

The detailed analysis is simplest for the case of impermeable, stress-free, fixed temperature boundary conditions — though these may certainly not be the most relevant for an experiment, for example. In this case (again dropping hats) we have

$$w = \frac{d^2 w}{dz^2} = \theta = 0 \quad \text{on } z = 0, 1.$$

From (19) one can write  $\theta$  as a sum of the form

$$\theta = ( \quad ) \frac{d^4 w}{dz^4} + ( \quad ) \frac{d^2 w}{dz^2} + ( \quad ) w.$$

So  $\theta = 0$  on the boundary, together with  $w = 0$  and  $d^2 w/dz^2 = 0$  implies that  $d^4 w/dz^4 = 0$  on  $z = 0, 1$ .

So we need to solve (22) subject to

$$w = \frac{d^2 w}{dz^2} = \frac{d^4 w}{dz^4} = 0 \quad \text{on } z = 0, 1.$$

With these boundary conditions it is easy to see that the solution must take the form:

$$w(z) \sim \sin n\pi z.$$

All boundary conditions are then satisfied. Substitution into (22) then leads to the dispersion relation

$$(n^2\pi^2 + a^2 + s)(n^2\pi^2 + a^2 + \frac{s}{Pr})(n^2\pi^2 + a^2) = Ra a^2.$$

Quadratic for  $s$ . Instability for the  $n$ th mode occurs when  $s$  passes through zero. The critical Rayleigh number for the  $n$ th mode is therefore

$$Ra = \frac{(n^2\pi^2 + a^2)^3}{a^2}.$$

Clearly the  $n = 1$  mode is the most readily destabilised — i.e. it becomes unstable for the smallest value of  $Ra$ . Then minimising  $Ra$  over  $a^2$  shows that the most readily destabilised mode has  $a^2 = \pi^2/2$  and that the corresponding critical Rayleigh number is

$$Ra = \frac{27\pi^4}{4} \approx 657.5$$

### Horizontal Planform

All we know about the horizontal dependence  $f(x, y)$  is that  $f$  satisfies the equation

$$\nabla_H^2 f = -a^2 f. \tag{23}$$

There are a number of possibilities for  $f(x, y)$ . For example,

- (i)  $f(x) = \sin ax$ . These are two-dimensional roll solutions.
- (ii)  $f(x, y) = \sin ax/\sqrt{2} + \sin ay/\sqrt{2}$ . This has a square planform.

Equation (23) also supports triangular planforms, hexagonal planforms, and also more exotic patterns.

Linear theory cannot distinguish between these various possibilities. All are equally likely. Of course, in an experiment, a specific planform *will* be preferred and one will see, for example, hexagonal cells. The planform though is determined by initial conditions and by nonlinear terms.