Centrifugal Instability: Summary of §4

First let us consider the basic state of an *inviscid* fluid. From the Euler equation it can be seen that there is an axisymmetric basic state $\boldsymbol{u} = u_\theta(r)\boldsymbol{e}_\theta$, where the centrifugal force is balanced by the radial pressure gradient:

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

u_θ is related to the angular velocity $\Omega(r)$ by $u_\theta(r) = r\Omega(r)$.

Rayleigh's Criterion

Rayleigh provided a physical argument to determine the stability of a flow $\boldsymbol{u} = u_\theta(r)\boldsymbol{e}_\theta$ to axisymmetric disturbances.

The θ component of the Euler equation is

$$\frac{\mathrm{D}u_\theta}{\mathrm{D}t} + \frac{u_r u_\theta}{r} = 0,$$

which may be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}(ru_\theta) = 0.$$

This may be interpreted physically as a statement of the conservation of angular momentum (ρ is constant, remember).

Following Rayleigh, suppose we interchange the fluid in two rings, of equal masses, at radii r_1 and r_2. So the fluid initially at r_1 is moved to r_2, conserving its angular momentum L_1. Similarly the fluid initially at r_2 is moved to r_1, conserving its angular momentum L_2.

Kinetic energy of ring of fluid $\sim u^2$ and angular momentum $L \sim ru$ (we do not need to bother with factors involving ρ since the density is constant).

Thus:

$$\text{change in KE} \sim \left(\frac{L_1^2}{r_2^2} + \frac{L_2^2}{r_1^2}\right) - \left(\frac{L_1^2}{r_1^2} + \frac{L_2^2}{r_2^2}\right) = (L_2^2 - L_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right).$$

Suppose $r_2 > r_1$. Then the change in KE is positive if $L_2^2 > L_1^2$.

So if L^2 is a monotonically *increasing* function of r then such an interchange of fluid rings requires an *input* of energy. Thus this cannot occur spontaneously — hence stability.

Conversely, if L^2 should decrease somewhere we see that energy will be released. This is *suggestive* of instability, but is not a proof since we do not know if a fluid motion that exchanges these rings will also have to perform work against the rest of the fluid (see the argument later).

So Rayleigh's criterion is a *sufficient* condition for *stability* — or, equivalently, a *necessary* condition for *instability*.

In terms of $\Omega(r)$:

A *sufficient* condition for *stability* is that $\mathrm{d}(r^2\Omega)^2/\mathrm{d}r > 0$ everywhere.

A *necessary* condition for *instability* is that $\mathrm{d}(r^2\Omega)^2/\mathrm{d}r < 0$ somewhere.

Formal Derivation of Rayleigh's Criterion

Consider, again, a steady axisymmetric basic state $\boldsymbol{u} = u_\theta(r)\boldsymbol{e}_\theta = r\Omega(r)\boldsymbol{e}_\theta$. Restrict attention to axisymmetric disturbances, for simplicity. Hence perturb to $\boldsymbol{u} = r\Omega\boldsymbol{e}_\theta + \tilde{\boldsymbol{u}}\ (r, z, t)$ (with a similar form of perturbation for p).

Noting that the basic state is dependent only on r, we seek normal mode solutions of the form

$$\tilde{\boldsymbol{u}}\ (r, z, t) = \hat{\boldsymbol{u}}\ (r)\exp(st + ikz), \qquad p(r, z, t) = \hat{p}(r)\exp(st + ikz).$$

Substituting into the governing equations and retaining only linear terms gives the following set of equations:

$$s\hat{u}_r - 2\Omega\hat{u}_\theta = -\frac{1}{\rho}\frac{\mathrm{d}\hat{p}}{\mathrm{d}r},$$

$$s\hat{u}_\theta + \hat{u}_r\frac{\mathrm{d}}{\mathrm{d}r}(r\Omega) + \Omega\hat{u}_r = 0,$$

$$s\hat{u}_z = -\frac{1}{\rho}ik\hat{p},$$

$$\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}r} + \frac{\hat{u}_r}{r} + ik\hat{u}_z = 0.$$

These can be manipulated to give an equation just for $\hat{u}_r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right) - k^2\hat{u}_r = \frac{k^2}{s^2}\Phi(r)\hat{u}_r, \qquad (*)$$

where

$$\Phi(r) \equiv \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}(r^4\Omega^2)$$

is the *Rayleigh discriminant*.

To obtain information on the growth rate s we follow a strategy similar to that used in §3 when we proved the principle of exchange of stabilities in convection. Multiply equation $(*)$ by $r\hat{u}_r^*$ and integrate from the inner to outer boundary, i.e. from $r = r_1$ to $r = r_2$. Use integration by parts and the impermeability boundary condition ($\hat{u}_r = 0$ on $r = r_1, r_2$) to obtain

$$\frac{k^2}{s^2}\int_{r_1}^{r_2} r\Phi(r)|\hat{u}_r|^2\,\mathrm{d}r = -\int_{r_1}^{r_2}\frac{1}{r}|\frac{d}{dr}r\hat{u}_r|^2\,\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r.$$

and hence

$$\frac{k^2}{s^2}\int_{r_1}^{r_2} r\Phi(r)|\hat{u}_r|^2\,\mathrm{d}r < 0.$$

Thus if $\Phi(r) > 0$ for $r_1 < r < r_2$ we must have $s^2 < 0$. Thus s will be imaginary, implying stability. So a *sufficient* condition for *stability* is that $\Phi(r) > 0$ everywhere in $r_1 < r < r_2$. (Note that for this problem s^2 is real, so the 'most stable' one can be is for the eigenvalues s to be imaginary.)

Alternatively, a *necessary* condition for *instability* is that $\Phi(r) < 0$ somewhere in the fluid.

Note: It turns out that having $\Phi(r) < 0$ somewhere in the fluid is actually also a sufficient condition for instability (proved by Synge in 1933) — but to show this requires additional analysis to that shown above.

The Stability of Couette Flow

Recall that in the absence of viscosity we can have a basic state for $u_\theta(r) = r\Omega(r)$ of arbitrary r-dependence. Let us now consider the stability of a specific flow, so-called Couette flow, which is the steady, r-dependent flow that occurs for viscous fluids between two rotating cylinders. We shall though, as above, treat the perturbations as inviscid. Suppose the inner (outer) cylinder has radius r_1 (r_2) and angular velocity Ω_1 (Ω_2).

Then Couette flow takes the form (derived in §1.5)

$$u_\theta = Ar + \frac{B}{r}, \quad \text{with } A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \qquad B = \frac{(\Omega_1 - \Omega_2)r_1^2 r_2^2}{r_2^2 - r_1^2}.$$

Thus $\Omega(r) = A + B/r^2$.

It can be easily shown that

$$\Phi(r) = 4A^2\left(1 + \frac{B}{Ar^2}\right) = 4A^2\left(1 - \frac{(1-\mu)}{(\eta^2-\mu)}\frac{r_1^2}{r^2}\right),$$

where we have defined $\mu = \Omega_2/\Omega_1$ and $\eta = r_1/r_2$ ($\eta < 1$).

We are interested in seeing whether $\Phi < 0$ anywhere in $r_1 < r < r_2$. Since $\Phi(r)$ is a monotonic function then we need only consider the end values, $r = r_1$ and $r = r_2$.

$$\Phi(r_1) = 4A^2\frac{(\eta^2-1)}{(\eta^2-\mu)}, \qquad \Phi(r_2) = 4A^2\mu\frac{(\eta^2-1)}{(\eta^2-\mu)}.$$

If the cylinders are rotating in the same direction then $\mu > 0$. Then

$$\Phi(r) > 0 \ \forall r \in (r_1, r_2) \quad \text{if} \quad \mu > \eta^2 \quad \text{and} \tag{1}$$
$$\Phi(r) < 0 \ \forall r \in (r_1, r_2) \quad \text{if} \quad \mu < \eta^2. \tag{2}$$

If the cylinders are counter-rotating ($\mu < 0$) then $\Phi(r_1) < 0$ and $\Phi(r_2) > 0$. So Φ is negative somewhere in the flow — this is sufficient for instability using Synge's result.

In the $\Omega_2\,\Omega_1$ plane this leads to the following stability diagram.