Chapter 5

Vectors

5.1 Definition of Vectors

Vectors are quantities which have both magnitude and direction. For example, a displacement from a fixed point A to a fixed point B is a vector because its magnitude is the distance between A and B and the direction is the direction in which it is necessary to move in a straight line to get from A to B. Other examples of vectors are velocity and forces. For example, the gravitational force on an object is a vector because its magnitude is the weight of the object and the direction is downwards.

Vectors are represented in diagrams by directional lines, i.e. lines with arrows indicating the direction (figure 5.1). This type of representation is used whether the vector is an actual displacement or if it represents some other quantity such as a force.

![Figure 5.1: The line shows the vector from point A to point B.]

Vectors are often written as lower case symbols in bold (e.g. $\mathbf{a}$), as lower case symbols underlined (especially in hand-written text, e.g. $\underline{a}$) or in terms of the initial and final (terminal) points with an arrow above indicating the direction (e.g. $\overrightarrow{AB}$).

In contrast to vectors, normal quantities which do not have direction are called scalars. Examples of scalars are temperature and pollutant concentration.
5.2 Equal Vectors

Vectors are equal when they have the same magnitude and direction. So for example in Fig. 5.2, $\overrightarrow{AB} = \overrightarrow{CD}$.

![Figure 5.2: Two equal vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, i.e. $a = b$.](image1)

In the above example, if $\overrightarrow{AB} = a$ and $\overrightarrow{CD} = b$, then we write

$a = b$.

5.3 Negative Vectors

The vector $-\mathbf{a}$ has the same magnitude as the vector $\mathbf{a}$ but has opposite direction. In the example in Fig. 5.3, $b = -a$.

![Figure 5.3: Two vectors, one of which is the negative of the other, i.e. $b = -a$.](image2)
5.4 Addition and Subtraction of Vectors

Vectors are added by joining them “end to end” as shown in Fig. 5.4.

![Figure 5.4: Addition of two vectors $a$ and $b$ by joining them “end to end” to form a triangle.](image)

This method is called the Triangle Law of vector addition. Subtraction of vectors is performed by addition of negative vectors:

$$a - b = a + (-b)$$

as shown in Fig. 5.5.

![Figure 5.5: Subtraction of vector $b$ from vector $a$ by adding $-b$ to $a$.](image)

5.5 Resolving Vectors into Two Perpendicular Components

Just as two vectors can be added to form a combined vector, we can decompose any vector into two perpendicular vectors which, when added together, form the original vector. In Fig. 5.6, the length of the vector $a$ (the length of OA) is $a$. Considering the right-angled triangle OAB, we can see that $OB = a \cos(\theta)$ and $BA = OC = a \sin(\theta)$. So if we know the direction $\theta$ of the vector $a$, we can resolve it into components $a \cos(\theta)$ and $a \sin(\theta)$ in
the $x$ and $y$ directions. Equally, starting from these components, we can reconstruct the original vector because

$$\mathbf{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OB} + \overrightarrow{OC}. \quad (5.1)$$

![Figure 5.6](image)

Figure 5.6: Resolving a vector $\mathbf{a}$ into components $a\cos(\theta)$ and $a\sin(\theta)$ in the $x$ and $y$ directions.

The examples given in this section are two-dimensional, but the results may be extended to three dimensions.

### 5.6 Multiplication of a Vector by a Scalar

The vector $\alpha \mathbf{a}$ is a vector in the direction of $\mathbf{a}$ but with magnitude $\alpha \times$ magnitude of $\mathbf{a}$. As an example, Fig. 5.7 shows the vectors $\mathbf{a}$ and $2\mathbf{a}$.

![Figure 5.7](image)

Figure 5.7: The vectors $\mathbf{a}$ and $2\mathbf{a}$.

### 5.7 Unit Vectors

From the result of section 5.6, it follows that we can always express a vector $\mathbf{a}$ as a vector with magnitude 1 in the same direction as $\mathbf{a}$, multiplied by a scalar equal to the...
magnitude of \( a \). We usually write this as

\[
a = a \hat{a}
\]  

where \( \hat{a} \) has magnitude 1 and \( a \) is the magnitude of \( a \), i.e.

\[
|\hat{a}| = 1 \quad \text{and} \quad |a| = a.
\]

It follows from Eq. (5.2) that

\[
\hat{a} = \frac{a}{|a|} = \frac{a}{|a|}.
\]  

\( \hat{a} \) is called a unit vector. Eq. (5.3) says that we can always construct a unit vector by dividing a vector by its magnitude.

### 5.8 Cartesian Components of a Vector

We know from section 5.5 that we can express any vector as components in the directions of the Cartesian axes \( x \) and \( y \) (more generally in three dimensions, as components along the axes \( x, y \) and \( z \)). From section 5.8, we also know that the vectors along each of the axes can be expressed as scalars multiplied by unit vectors.

We define three unit vectors, written as \( i, j \) and \( k \) as the unit vectors in the \( x, y \) and \( z \) directions respectively. Note that although these are unit vectors, we do not normally use the “hat” notation.

Another useful concept is that of position vector. Note that up to now, we have only discussed vectors in terms of the position of the terminal point relative to the initial point. In simple terms, we can move a vector around on the page and it remains the same vector. But sometimes it is useful to tie down the initial point to the origin of Cartesian axes. The vector may then be used to specify the coordinates of a point relative to the axes. Consider the two-dimensional case in Fig. 5.8.

Note first that \( \overrightarrow{OA} \) is of length \( x \) and therefore the vector \( \overrightarrow{OA} = x \hat{i} \). Similarly, \( \overrightarrow{AP} \) is of length \( y \) and therefore \( \overrightarrow{AP} = y \hat{j} \). Hence, since \( \overrightarrow{r} = \overrightarrow{OA} + \overrightarrow{AP} \)

\[
\overrightarrow{r} = x \hat{i} + y \hat{j}.
\]  

(5.4)

Considering the triangle OAP, we know that

\[
OP^2 = OA^2 + AP^2
\]

and therefore

\[
OP = r = |\overrightarrow{r}| = \sqrt{x^2 + y^2}.
\]  

(5.5)

More generally in three dimensions, if \( P \) is the point \((x, y, z)\), then

\[
\overrightarrow{r} = x \hat{i} + y \hat{j} + z \hat{k}.
\]  

(5.6)

Generalising Eq. (5.5) to three dimensions,

\[
|\overrightarrow{r}| = \sqrt{x^2 + y^2 + z^2}.
\]  

(5.7)
Figure 5.8: Position vector \( \mathbf{r} \) of point \( P \) with coordinates \( (x, y) \).

From the results of section 5.7, we now have a straightforward way of determining a unit vector in the direction of \( \mathbf{r} \):

\[
\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}.
\]

(5.8)

As an example, consider the vector \( \mathbf{r} = 2 \mathbf{i} + 6 \mathbf{j} + 3 \mathbf{k} \).

\[
|\mathbf{r}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7.
\]

Hence

\[
\hat{\mathbf{r}} = \frac{2 \mathbf{i} + 6 \mathbf{j} + 3 \mathbf{k}}{7} = \frac{2}{7} \mathbf{i} + \frac{6}{7} \mathbf{j} + \frac{3}{7} \mathbf{k}.
\]

\[\text{Examples 4, Q9}\]

Addition of vectors in component form is achieved in the obvious way by adding components. So

\[
\mathbf{r}_1 + \mathbf{r}_2 = x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k} + x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k} = (x_1 + x_2) \mathbf{i} + (y_1 + y_2) \mathbf{j} + (z_1 + z_2) \mathbf{k}.
\]

(5.9)

For example

\[
(\mathbf{i} + 4 \mathbf{j} + 7 \mathbf{k}) + (5 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}) = (1 + 5) \mathbf{i} + (4 - 2) \mathbf{j} + (7 + 1) \mathbf{k} = 6 \mathbf{i} + 2 \mathbf{j} + 8 \mathbf{k}.
\]

Similarly

\[
(\mathbf{i} + 4 \mathbf{j} + 7 \mathbf{k}) - (5 \mathbf{i} - 2 \mathbf{j} + \mathbf{k}) = (1 - 5) \mathbf{i} + (4 + 2) \mathbf{j} + (7 - 1) \mathbf{k} = -4 \mathbf{i} + 6 \mathbf{j} + 6 \mathbf{k}.
\]

\[\text{Examples 4, Q10}\]
5.9 Unit vectors and position vectors in polar co-ordinates

In plane polar co-ordinates, the position of a point in space is defined by two co-ordinates \((r, \theta)\), where \(r\) is the distance of the point from the origin and \(\theta\) is the angle between the \(x\)-axis and the straight line from the origin to the point, as shown in Figure 5.9. Instead of the “Cartesian” unit vectors \(\mathbf{i}\) and \(\mathbf{j}\), it is common when using plane polar co-ordinates to define unit vectors which point in the direction of increasing \(r\) and \(\theta\). These vectors are also shown in the diagram, and they are

- \(\hat{\mathbf{r}}\), which points in the direction of increasing \(r\), and
- \(\hat{\theta}\), which points in the direction of increasing \(\theta\).

Note that:

1. These unit vectors change direction with the position of the point we are describing, so that:

\[
\hat{\mathbf{r}} = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta \\
\hat{\theta} = -\mathbf{i} \sin \theta + \mathbf{j} \cos \theta.
\]

This can lead to some confusion when the point moves (see section 5.13 below).

2. The position vector of the point \((r, \theta)\) is given simply by

\[
r = r \hat{\mathbf{r}} \tag{5.10}
\]

Figure 5.9:
5.10 The Scalar Product of Two Vectors

The *scalar product* of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is written as \( \mathbf{a} \cdot \mathbf{b} \) and is defined as

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)
\]  

(5.11)

where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \) (see Fig. 5.10). Note that the result of this product is a *scalar* — hence the name. The scalar product is sometimes called the *dot product*.

![Figure 5.10: Angle \( \theta \) used in the definition of the scalar product of vectors \( \mathbf{a} \) and \( \mathbf{b} \).](image)

Note that the angle between any vector and itself is 0 and hence

\[
\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}| |\mathbf{a}| \cos(0) = |\mathbf{a}|^2.
\]  

(5.12)

Note also that if \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular, then

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\frac{\pi}{2}) = 0
\]

because \( \cos(\frac{\pi}{2}) = 0 \). Special cases of these are

\[
\begin{align*}
i \cdot i &= j \cdot j = k \cdot k &= 1 \\
\text{and} & \quad i \cdot j = i \cdot k = j \cdot k &= 0
\end{align*}
\]  

(5.13) (5.14)

Two useful results concerning the scalar product are:

\[
\begin{align*}
\mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} & \text{(5.15)} \\
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} & \text{(5.16)}
\end{align*}
\]

As an example, consider

\[
(i + 4j + 7k) \cdot (5i - 2j + k) = 5i \cdot i - 2i \cdot j + i \cdot k + (4)(5)i \cdot i + (4)(-2)j \cdot j + 4j \cdot k + (7)(5)k \cdot i + (7)(-2)k \cdot j + 7k \cdot k
\]

\[
= 5 \cdot 0 + 0 + 0 - 8 + 0 + 0 - 0 + 7
\]

\[
= 4.
\]

Examples 4, Q11
Note also that
\[ |\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}| = \sqrt{1 + 16 + 49} = \sqrt{66} \]
and \[ |5\mathbf{i} - 2\mathbf{j} + \mathbf{k}| = \sqrt{25 + 4 + 1} = \sqrt{30} \]

Therefore from the definition of the scalar product
\[ (\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}).(5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 4 = \sqrt{66}\sqrt{30} \cos(\theta) \]
where \( \theta \) is the angle between the vectors \( \mathbf{i} + 4\mathbf{j} + 7\mathbf{k} \) and \( 5\mathbf{i} - 2\mathbf{j} + \mathbf{k} \). Hence
\[ \cos(\theta) = \frac{4}{\sqrt{66}\sqrt{30}} \approx 0.0899. \]

Therefore \( \theta \approx 1.489 \) radians or \( \theta \approx 84.84^\circ \).

**Examples 4, Q12**

### 5.11 The Vector Product of Two Vectors

The vector product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is defined as
\[ \mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin(\theta)\mathbf{n} \]  \hspace{1cm} (5.17)

where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \) and \( \mathbf{n} \) is a unit vector perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \). Since the two vectors \( \mathbf{a} \) and \( \mathbf{b} \) may be thought of as defining a plane, there must be two possible unit vectors which are perpendicular to this plane (for example, if the plane is horizontal, then the vector could be “up” or “down”). The correct direction of \( \mathbf{n} \) is given by the *right-hand screw rule*. Fig. 5.11. shows how \( \mathbf{n} \) is determined by this rule.

![Figure 5.11: A right-handed screw is aligned along the direction perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \). If the screw is then rotated in the direction such that \( \mathbf{a} \) rotates towards \( \mathbf{b} \), then the screw will advance in the direction \( \mathbf{n} \).](image)

Note that the result of the vector product is *another vector*. The vector product is sometimes called the *cross product*.

Since the angle between any vector and itself is 0, it follows that
\[ \mathbf{a} \times \mathbf{a} = |\mathbf{a}||\mathbf{a}| \sin(0)\mathbf{n} = 0. \]  \hspace{1cm} (5.18)
If \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular, then
\[
\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \left( \frac{\pi}{2} \right) \mathbf{\hat{n}} = |\mathbf{a}||\mathbf{b}| \mathbf{\hat{n}}.
\]
because \( \sin \left( \frac{\pi}{2} \right) = 1 \). If the order of \( \mathbf{a} \) and \( \mathbf{b} \) is reversed, then the direction of rotation will be reversed and therefore the direction of the unit vector \( \mathbf{\hat{n}} \) will be reversed. This means that
\[
\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.
\] (5.19)

Special cases of the above are
\[
\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \quad (5.20)
\]
and
\[
\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}. \quad (5.21)
\]

Fig. 5.12 shows the order of \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) in eqn 5.21, making the results easy to remember. From Eq. (5.21) and (5.19),
\[
\begin{align*}
\mathbf{j} \times \mathbf{i} &= -\mathbf{k}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \quad (5.22)
\end{align*}
\]

It is not necessary to remember these as they can easily be obtained from Eq. (5.21) by reversing the order of multiplication.

![Figure 5.12: Aid to memorising the order of \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) in Eq. (5.21).](image)

As an example consider
\[
(\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}) \times (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 5(\mathbf{i} \times \mathbf{i}) - 2(\mathbf{i} \times \mathbf{j}) + (\mathbf{i} \times \mathbf{k}) + 20(\mathbf{j} \times \mathbf{i}) - 8(\mathbf{j} \times \mathbf{j}) + 4(\mathbf{j} \times \mathbf{k}) + 35(\mathbf{k} \times \mathbf{i}) - 14(\mathbf{k} \times \mathbf{j}) + 7(\mathbf{k} \times \mathbf{k})
\]
\[
= 0 - 2\mathbf{k} - \mathbf{j} - 20\mathbf{k} + 0 + 4\mathbf{i} + 35\mathbf{j} + 14\mathbf{i} + 0
\]
\[
= 18\mathbf{i} + 34\mathbf{j} - 22\mathbf{k}.
\]

There is a quick way of doing this, which will be demonstrated in class!

Examples 4, Q13

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5.12 Vector Functions

A vector may be a function of a scalar. An obvious example is the position vector $\mathbf{r}$ of a moving object, which is a function of time. In this case we would denote the position vector by $\mathbf{r}(t)$ where $t$ is time. Fig. 5.13 shows the position $P$ of an object as it moves along the trajectory indicated by the dashed line.

![Figure 5.13: Position vector $\mathbf{r}(t)$ of an object moving along the trajectory shown by the dashed line.](image)

5.13 Differentiation of Vectors

The idea of vector functions introduced in the previous section introduces the possibility of differentiation of vectors. In the case of a position vector $\mathbf{r}(t)$ which is a function of time $t$, the derivative of $\mathbf{r}(t)$ with respect to $t$ is called the velocity. We define the derivative of a vector function in the same way as for scalar functions:

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}. \quad (5.23)$$

This is shown graphically in Fig. 5.14. The object is at point $P$ at time $t$ and at point $Q$ at time $t + \delta t$. It is clear that as $\delta t \to 0$, point $Q$ moves towards point $P$ and the direction of $\delta \mathbf{r}$ becomes parallel to the tangent to the trajectory at the point $P$. Hence $d\mathbf{r}/dt$ is parallel to the tangent to the trajectory at $P$.

Examples 4, Q14

A special case of differentiation of vectors concerns unit vectors. Consider the unit position vector $\hat{\mathbf{r}}(t)$. As this is a unit vector, its magnitude must always be 1 and so only the direction can change. The situation is shown in Fig. 5.15, where the unit vector $\hat{\mathbf{r}}(t)$ changes its direction by an angle $\delta \theta$ in time $\delta t$. From this figure we see that

$$\frac{d\hat{\mathbf{r}}}{dt} = \lim_{\delta t \to 0} \frac{\delta \hat{\mathbf{r}}}{\delta t}.$$
Figure 5.14: Change in position vector $\delta \mathbf{r}$ during a small time interval $\delta t$ for a moving object.

But the magnitude of $\delta \mathbf{r}$ is $|\mathbf{r}| \delta \theta = \delta \theta$ because $\mathbf{r}$ is a unit vector. In the limit as $\delta t \to 0$, $\delta \theta \to 0$ and from the figure it is clear that the direction of $\delta \mathbf{r}$ becomes that of the unit vector $\hat{\theta}$. Hence

$$\delta \mathbf{r} = \delta \theta \hat{\theta}.$$ 

Therefore

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \theta \hat{\theta}}{\delta t} = \frac{d\theta}{dt} \hat{\theta} \quad (5.24)$$

We can use the above result to differentiate the more general position vector $\mathbf{r}$. For this we use the fact that we can write $\mathbf{r}$ in the form

$$\mathbf{r} = r \hat{\mathbf{r}}$$

where $r = |\mathbf{r}|$. Differentiating this we get

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt} (r \hat{\mathbf{r}}) = dr \hat{\mathbf{r}} + r \frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\theta} \quad (5.25)$$

Hence the velocity has components $dr/dt$ in the radial ($\hat{\mathbf{r}}$) direction and $r d\theta/dt$ in the tangential ($\hat{\theta}$) direction.
Figure 5.15: Change to the unit vector $\hat{r}$ in time $\delta t$. Note that because $\hat{r}$ is a unit vector, its magnitude must remain 1 and so its terminal point can only move on a circular trajectory.