

### Example of Taylor series: Polynomial

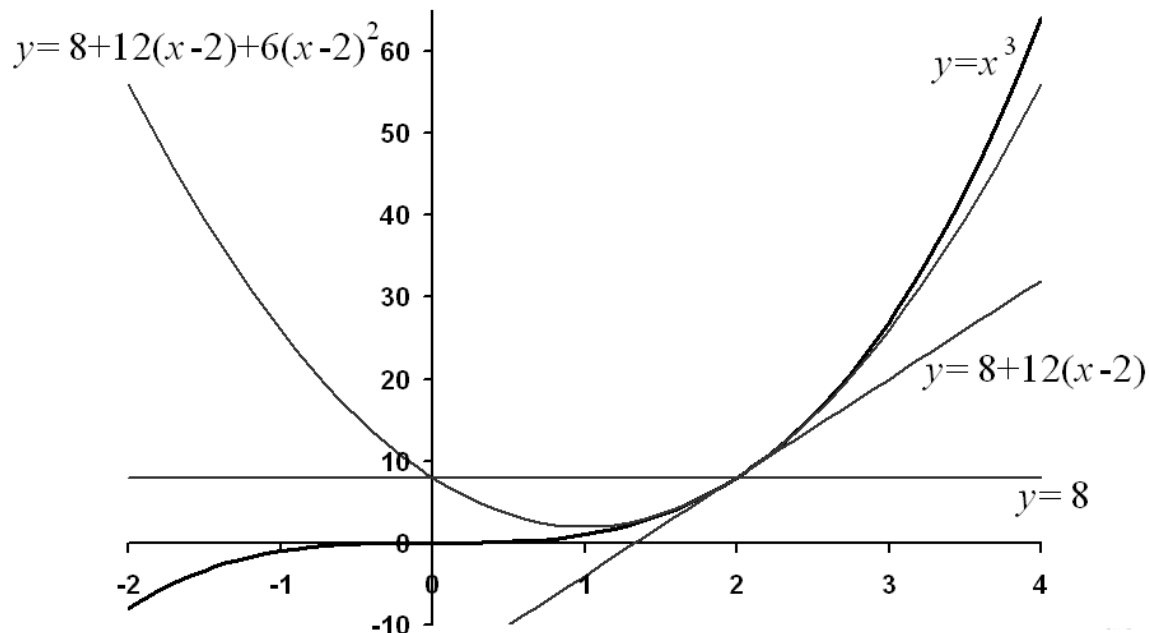
Consider the function  $f(x) = x^3$  and its Taylor series about  $x_0 = 2$ . We can calculate the derivatives:

$$\begin{aligned}f(x) &= x^3 & f(x_0) &= 8 \\f'(x) &= 3x^2 & f'(x_0) &= 12 \\f''(x) &= 6x & f''(x_0) &= 12 \\f'''(x) &= 6 & f'''(x_0) &= 6 \\f^{(4)}(x) &= 0 & f^{(4)}(x_0) &= 0\end{aligned}$$

and all higher derivatives are also zero so the Taylor series terminates:

$$\begin{aligned}f(x) &= 8 + 12(x - 2) + \frac{1}{2!}12(x - 2)^2 + \frac{1}{3!}6(x - 2)^3 \\&= 8 + 12(x - 2) + 6(x - 2)^2 + (x - 2)^3\end{aligned}$$

and these terms give successively better approximations to  $f(x)$  near  $x = 2$ :



### 3.2.3 Maclaurin Series

If  $x_0 = 0$  then the Taylor series is called a *Maclaurin series*. We will illustrate the Maclaurin series for sine, and then give some standard results (derivations in Appendix E).

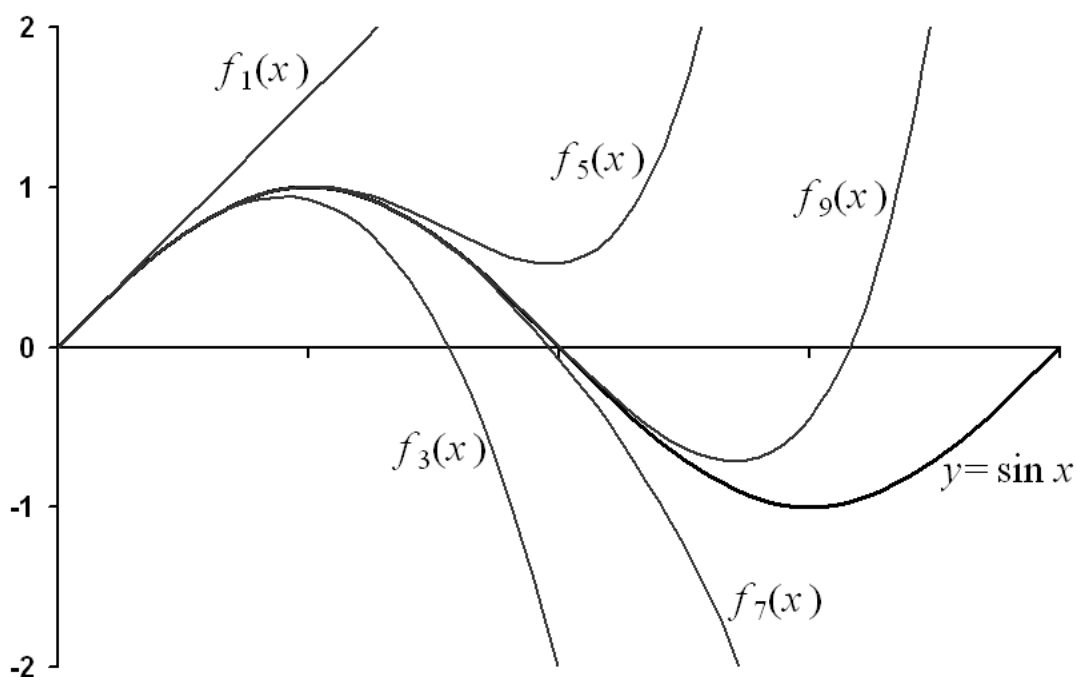
#### The sine function

We look at the derivatives of  $\sin x$  at  $x = 0$ :

$$\begin{aligned}f(0) &= 0 & f'(0) &= 1 \\f''(0) &= 0 & f'''(0) &= -1 \\f^{(4)}(0) &= 0 & f^{(5)}(0) &= 1 \\f^{(6)}(0) &= 0 & f^{(7)}(0) &= -1\end{aligned}$$

and this pattern continues. Thus we can approximate  $\sin x$  by a series of polynomials:

$$\begin{aligned} f_1(x) &= x \\ f_3(x) &= x - \frac{x^3}{3!} \\ f_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ f_7(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ f_9(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \end{aligned}$$



The full Maclaurin series for  $\sin x$  is given by the infinite sum

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

and it converges for all  $x$  (the radius of convergence is infinite).