

# GENERALISED MOORE SPECTRA IN A TRIANGULATED CATEGORY

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ABSTRACT. In this paper we consider a construction in an arbitrary triangulated category  $\mathcal{T}$  which resembles the notion of a Moore spectrum in algebraic topology. Namely, given a compact object  $C$  of  $\mathcal{T}$  satisfying some finite tilting assumptions, we obtain a functor which “approximates” objects from the module category of the endomorphism algebra of  $C$  in  $\mathcal{T}$ . This generalises and extends a construction of Jørgensen which appears in [10] in connection with lifts of certain homological functors of derived categories. We show that this new functor is well-behaved with respect to short exact sequences and distinguished triangles, and as a consequence we obtain a new way of embedding a module category in a triangulated category. As an example of the theory, we recover Keller’s canonical embedding of the module category of a path algebra of a quiver with no oriented cycles into its  $u$ -cluster category of  $u \geq 2$ .

## INTRODUCTION

In this paper we discuss the existence of “Moore spectra in a triangulated category. The terminology “Moore spectra” employed in this paper is borrowed from algebraic topology, see [14]. While the notion discussed here does not coincide with its counterpart in algebraic topology its spirit is the same.

In algebraic topology, the notion of spectra can be considered as one of “generalised topological spaces”. In this setting one uses the idea of a Moore spectrum to construct a spectrum with a single (pre-defined) non-vanishing homology group; c.f. the notion of an Eilenberg-MacLane space for homotopy groups. For instance, suppose  $A$  is an abelian group, the Moore spectrum  $MA$  of  $A$  is a spectrum with

$$H(\Sigma^i MA) = \begin{cases} A & i = 0 \\ 0 & i \neq 0, \end{cases}$$

where  $\Sigma$  is the suspension functor in the category of spectra.

Analogously, in this paper we shall consider the following setup: suppose  $C$  is a compact object of a triangulated category  $\mathcal{T}$  which has set indexed coproducts satisfying some finite tilting assumptions (see Setup 3.3 for precise

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conditions). Then consider the endomorphism algebra  $S^{\text{op}} = \text{End}(C^{\text{op}})$  of  $C$ . We are looking for ways to “represent” or “approximate” an  $S^{\text{op}}$ -module in the triangulated category  $\mathcal{T}$ . Compare this with the idea of Moore spectrum in algebraic topology where one starts with given homology groups and “approximates” a representative topological space with given homology.

The idea of “Moore spectra” in triangulated categories in general was first studied by Jørgensen in [10]; Jørgensen’s construction is also in the same spirit as its namesake in algebraic topology. It is used as a tool when considering certain lifts of homological functors between the derived category of the integers  $\mathcal{D}(\mathbb{Z})$  and an arbitrary triangulated category  $\mathcal{T}$ .

However, the notion of Moore spectra in a triangulated category as developed by Jørgensen in [10] holds independent interest, in particular it yields a new technique of embedding an abelian category in a triangulated category and of obtaining a module category from a triangulated category in a nice way. The construction given in this paper is an extension and generalisation of Jørgensen’s construction, though the language and techniques used to prove the theorems are very different.

The outline of this paper is as follows: in the first section we present a brief review of Jørgensen’s construction as it appears in [10]. In Section 3 we proceed with the proof of the main theorem of this paper (Theorem 3.5) which establishes the existence of the Moore spectra functor. The proof is constructive and consists of an induction in which the action of the Moore spectra functor on objects is constructed inductively via projective resolutions of modules and functoriality is obtained by means of a classical representability trick. We then return briefly to Jørgensen’s construction to compare and contrast with the techniques of the general construction before considering some consequences of the main results. Namely, in Section 5 we show that the Moore spectra functor is well behaved, that is it takes short exact sequences to distinguished triangles and that any short exact sequence induces a natural Hom-Ext sequence. The last section contains an example: we show how the Moore spectra functor recovers Keller’s canonical embedding of the module category of a hereditary algebra into its  $u$ -cluster category for  $u \geq 2$ .

Definitions and examples of triangulated categories can be found in [6] and [16]. Background in homological algebra can be found in [7] and [17], and in representation theory and algebra in [1].

## 1. NOTATION AND PRELIMINARIES

Before proceeding with a review of Jørgensen’s construction and beginning the general construction we first take the opportunity to recall some definitions and fix some notation.

Throughout this paper, unless stated otherwise,  $\mathcal{T}$  will be a triangulated category with set indexed coproducts.

**1.1. Additive closure and compact objects.** By  $\text{Add}(C)$  we denote the (*infinite*) *additive closure* of  $C$  in  $\mathcal{T}$ , that is the smallest full subcategory of  $\mathcal{T}$  whose objects are direct summands of (possibly infinite) set indexed coproducts of  $C$ .

Recall the following definition from [15]:

**Definition 1.1.** An object  $C$  of  $\mathcal{T}$  is called *compact* if the functor  $\text{Hom}_{\mathcal{T}}(C, -)$  commutes with set indexed coproducts; that is given a set of objects  $\{X_i\}_{i \in I}$  in  $\mathcal{T}$  then there is a canonical isomorphism

$$\coprod_{i \in I} \text{Hom}_{\mathcal{T}}(C, X_i) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}}(C, \coprod_{i \in I} X_i).$$

**1.2. Projective dimension and global dimension.** Let  $R$  be a ring. We shall denote by  $\text{Mod}(R)$  the category of left  $R$ -modules. We shall denote the category of right  $R$ -modules by  $\text{Mod}(R^{\text{op}})$ . We shall refer to a left  $R$ -module simply as an  $R$ -module and a right  $R$ -module simply as an  $R^{\text{op}}$ -module.

Recall that an  $R$ -module  $P$  is called *projective* if the functor  $\text{Hom}_R(P, -) : \text{Mod}(R) \rightarrow \mathbf{Ab}$  is exact, that is the sequence induced by the application of the functor on a short exact sequence is also a short exact sequence.

Let  $A$  be an  $R$ -module. A *projective resolution* of  $A$  consists of chain complex

$$P : \cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

such that  $H_i(P) = 0$  for  $i \geq 1$  and together with an isomorphism  $H_0(P) \xrightarrow{\sim} A$ , where  $H_i(P)$  denotes the  $i^{\text{th}}$ -homology group of the complex  $P$ .

**Definition 1.2.** Let  $A$  be an  $R$ -module. The *projective dimension* of  $A$ , written  $\text{projdim}_R A$ , is the smallest integer  $n$  such that there is a projective resolution of  $A$ ,

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A.$$

We write  $\text{projdim}_R A = n$ . If no finite resolution exists we say  $A$  has infinite projective dimension.

**Definition 1.3.** Let  $R$  be a ring. The *left global dimension* of  $R$  is defined as

$$\text{l.gldim } R := \sup\{\text{projdim}_R M \mid M \in \text{Mod}(R)\}.$$

The *right global dimension* of  $R$ , denoted  $\text{r.gldim } R$ , is defined similarly. Where right or left global dimension can be understood from context we shall simply write  $\text{gldim } R$  and refer to the *global dimension* of  $R$ .

**Definition 1.4.** We define the following full subcategories of  $\text{Mod}(R)$ :

- (a) By  $\text{Proj}(R)$  we denote the full subcategory of  $\text{Mod}(R)$  whose objects are all projective left  $R$ -modules.

(b) By  $\text{Proj}^k(R)$  we denote the full subcategory of  $\text{Mod}(R)$  whose objects are left  $R$ -modules with projective dimension at most  $k$ .

More complete expositions of the concepts defined in this section can be found in [7] and [17].

**1.3. Adjoint functors.** Recall that for two categories  $\mathcal{A}$  and  $\mathcal{B}$  an *adjunction* is a pair of functors  $F$  and  $G$ ,

$$\mathcal{A} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{B},$$

such that there is a natural isomorphism

$$\text{Hom}_{\mathcal{A}}(FB, A) \cong \text{Hom}_{\mathcal{B}}(B, GA).$$

The functor  $F$  is called the *left adjoint* and the functor  $G$  is called the *right adjoint*.  $F$  (resp.  $G$ ) is said to be *left (resp. right) adjoint* to  $G$  (resp.  $F$ ).

Recall that a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called *fully faithful* (or *full and faithful*) if

$$\text{Hom}_{\mathcal{A}}(A, A') \cong \text{Hom}_{\mathcal{B}}(FA, FA')$$

for all objects  $A$  and  $A'$  of  $\mathcal{A}$ .

We shall need the following observation from [13] connecting the notions of fully faithfulness with adjoint functors.

**Lemma 1.5** ([13], Theorem IV.3.1). *Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories and suppose*

$$\mathcal{A} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{B},$$

*is an adjunction with  $F$  left adjoint to  $G$ . Then the unit of the adjunction is an isomorphism if and only if  $F$  is fully faithful.*

The reader is directed to [13] for a more complete description of adjoint functors.

We are now ready to review Jørgensen's construction.

## 2. JØRGENSEN'S CONSTRUCTION

Let  $R$  be a ring. Recall that a triangulated category  $\mathcal{T}$  is called  *$R$ -linear* if for any two objects  $X$  and  $Y$  of  $\mathcal{T}$  the Hom-space  $\text{Hom}_{\mathcal{T}}(X, Y)$  is an  $R$ -module and the composition of morphisms is  $R$ -bilinear. A functor  $F : \mathcal{T} \rightarrow \mathcal{T}'$  of  $R$ -linear triangulated categories is said to be  *$R$ -linear* if  $F(r\alpha) = rF(\alpha)$  for all morphisms  $\alpha$  of  $\mathcal{T}$  and all elements  $r \in R$ ; see [10, Definition 1.2].

In [10, Section 4], Jørgensen considers the following setup.

**Setup 2.1** ([10], Setups 4.1 and 4.12). *Let  $R$  be a principal ideal domain, let  $\mathcal{T}$  be an  $R$ -linear triangulated category with set indexed coproducts and let  $C$  be a compact object of  $\mathcal{T}$  which satisfies the following assumptions:*

- (1)  $\mathrm{Hom}_{\mathcal{T}}(C, C)$  is a flat  $R$ -module;
- (2)  $\mathrm{Hom}_{\mathcal{T}}(C, \Sigma^{-1}C) = 0$ ;
- (3)  $\mathrm{Hom}_{\mathcal{T}}(C, \Sigma^{-2}C) = 0$ .

The idea in [10] is to construct the best possible approximation of an  $R$ -module  $A$  in  $\mathcal{T}$ . This approximation of  $A$  in  $\mathcal{T}$  is denoted by  $M(A)$  and is called the *Moore spectrum* of  $A$  in  $\mathcal{T}$ . Jørgensen's construction depends on which object,  $C$ , of  $\mathcal{T}$  is employed as the Moore spectrum of the ring itself.

Jørgensen introduces an auxilliary category  $\mathcal{M}$ , an analogue of which we shall also introduce in the general construction. Below is the definition of  $\mathcal{M}$  in Jørgensen's setting; see [10, Definition 4.3].

**Definition 2.2.** Let  $\mathcal{M}$  be the full subcategory of  $\mathcal{T}$  consisting of all objects  $M$  of  $\mathcal{T}$  which occur in distinguished triangles of the form

$$C \otimes F_1 \xrightarrow{1_C \otimes f} C \otimes F_0 \longrightarrow M \longrightarrow \Sigma C \otimes F_1,$$

when  $A$  is an  $R$ -module with free resolution

$$0 \longrightarrow F_1 \xrightarrow{f} F_0 \longrightarrow A \longrightarrow 0.$$

Note that in this definition, the tensor product is not the "usual" tensor product but is in fact a bifunctor  $-\otimes- : \mathcal{T} \times \mathrm{Free}(R) \rightarrow \mathcal{T}$  which is  $R$ -linear, preserves set indexed coproducts and has  $X \otimes R \cong X$  for each  $X$  in  $\mathcal{T}$ . Here,  $\mathrm{Free}(R)$  denotes the full subcategory of  $\mathrm{Mod}(R)$  consisting of all free  $R$ -modules. Indeed, the construction of  $-\otimes-$ , for each  $X$  in  $\mathcal{T}$  and  $F$  in  $\mathrm{Free}(R)$ , identifies  $X \otimes F$  with a coproduct  $\coprod_{I_F} X$ , where  $I_F$  is an indexing set for a basis of  $F$ . For full details of the construction see [10, Construction 1.4 and Lemma 1.5]. Jørgensen then obtains the following theorem.

**Theorem 2.3** ([10], Proposition 4.7 and Theorem 4.9). *Under the hypotheses of Setup 2.1, the functor*

$$\mathrm{Hom}_{\mathcal{T}}(C, -) : \mathcal{M} \rightarrow \mathrm{Mod}(R)$$

*has a left adjoint*

$$M : \mathrm{Mod}(R) \rightarrow \mathcal{M}.$$

*If  $M$  is viewed as a functor  $M : \mathrm{Mod}(R) \rightarrow \mathcal{T}$  by composition with the inclusion functor  $i : \mathcal{M} \hookrightarrow \mathcal{T}$ , then  $M$  is an  $R$ -linear functor, it has  $M(R) \cong C$  and it preserves set indexed coproducts.*

Jørgensen then continues to prove that the functor  $M$  constructed above is well behaved with respect to short exact sequences in  $\mathbf{Mod}(R)$  and distinguished triangles in  $\mathcal{T}$  as well as under the functor  $\mathbf{Ext}(-, -)$ .

Note that, in the proof of Theorem 2.3, the assumption that  $\mathbf{Hom}_{\mathcal{T}}(C, C)$  is flat as an  $R$ -module is required for proving the injectivity of a certain map which is used in the construction, see [10, Lemma 4.5].

The main result of this paper generalises Theorem 2.3 to arbitrary triangulated categories and dispenses with the requirement that  $R$  be a principal ideal domain. We are also able to prove that the generalised Moore spectra functor  $M$  is well behaved with respect to short exact sequences and distinguished triangles as well as under the functor  $\mathbf{Ext}(-, -)$ . The next section concerns the construction of generalised Moore spectra.

### 3. THE GENERAL CONSTRUCTION

The starting point of the general construction is the following generalisation of a well-known result.

**Proposition 3.1.** *Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts and suppose  $C$  is a compact object of  $\mathcal{T}$ . Let  $S = \mathbf{End}_{\mathcal{T}}(C)$ . Then the functor*

$$\mathbf{Hom}_{\mathcal{T}}(C, -) : \mathbf{Add}(C) \rightarrow \mathbf{Proj}(S^{\mathrm{op}})$$

*is an equivalence of categories.*

**Proof:** See [2, Proposition II.2.1] for example. The compactness of  $C$  can be used to pass to the infinite additive closure and infinitely generated projective modules.  $\square$

**Remark 3.2.** The fact that  $\mathbf{Hom}_{\mathcal{T}}(C, -) : \mathbf{Add}(C) \rightarrow \mathbf{Proj}(S^{\mathrm{op}})$  is an equivalence of categories means that it is part of an adjunction which is an equivalence of categories:

$$\mathbf{Add}(C) \begin{array}{c} \xleftarrow{M_0} \\ \xrightarrow{\mathbf{Hom}_{\mathcal{T}}(C, -)} \end{array} \mathbf{Proj}(S^{\mathrm{op}}).$$

In particular, the unit of this adjunction is an isomorphism; see Lemma 1.5.

Throughout this paper we shall use the following setup (c.f. Setup 2.1).

**Setup 3.3.** Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts and suppose  $C$  is an object of  $\mathcal{T}$  satisfying the following assumptions:

- (1)  $C$  is a compact object of  $\mathcal{T}$ ;
- (2) Its endomorphism algebra  $S^{\mathrm{op}} = \mathbf{End}_{\mathcal{T}}(C)^{\mathrm{op}}$  has finite global dimension  $n$ ; and,
- (3) We have  $\mathbf{Hom}_{\mathcal{T}}(C, \Sigma^i C) = \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{-i} C) = 0$  for  $i = 1, \dots, n + 1$ .

In [10], an auxilliary category  $\mathcal{M}$ , which is a certain full subcategory of  $\mathcal{T}$ , is introduced; see Definition 2.2. We define auxilliary categories  $\mathcal{M}_k$  for  $k \in \mathbb{N} \cup \{0\}$  with a view to arriving at an analogous definition of the auxilliary category  $\mathcal{M}$ .

**Definition 3.4.** We shall define full subcategories  $\mathcal{M}_k$  of  $\mathcal{T}$  as follows:

$$\mathcal{M}_k := \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(C, \Sigma^{-i} X) = 0 \text{ for } i = 1, \dots, k\}$$

with the convention that  $\mathcal{M}_0 = \mathcal{T}$ .

**Theorem 3.5.** *Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts. Suppose  $C$  is an object of  $\mathcal{T}$  satisfying the assumptions of Setup 3.3. Let  $\mathcal{M} = \mathcal{M}_{n+1}$ . Then, the functor*

$$\text{Hom}_{\mathcal{T}}(C, -) : \mathcal{M} \rightarrow \text{Mod}(S^{\text{op}})$$

has a left adjoint

$$M : \text{Mod}(S^{\text{op}}) \rightarrow \mathcal{M}.$$

Moreover, the functor  $M$  is a full embedding of the module category  $\text{Mod}(S^{\text{op}})$  into the full subcategory  $\mathcal{M}$  of  $\mathcal{T}$ .

The proof of Theorem 3.5 consists of a large induction in which several related claims need to be proved. There are two analogies which are useful for viewing this construction. The first is to view the induction as a kind of “bootstrapping” procedure. The second analogy is to consider the construction of a building: here, Theorem 3.5 is the desired building and the related claims in the induction hypotheses are the scaffolding which is required to build it.

For ease of exposition, we break down the proof into several steps and constructions. At times we will make assumptions which will be later justified in the section devoted to the scaffolding. Before we start the induction we explicitly state all the induction hypotheses.

### 3.1. The induction hypotheses and base step.

**Hypotheses 3.6.** Under the assumptions of Setup 3.3, for  $k \geq 0$  we have:

- (1) There exists a functor  $M_k : \text{Proj}^k(S^{\text{op}}) \rightarrow \mathcal{M}_k$ .
- (2) Let  $\mathcal{N}_k = \{X \in \mathcal{T} \mid X \cong M_k(A) \text{ for some } A \in \text{Proj}^k(S^{\text{op}})\}$ . We have  $\mathcal{N}_k \subseteq \mathcal{M}_k$  so that  $\mathcal{N}_k$  will become the essential image of  $M_k$  in  $\mathcal{M}_k$ . We also have a natural isomorphism

$$\text{Hom}_{\text{Mod}(S^{\text{op}})}(A, \text{Hom}_{\mathcal{T}}(C, X)) \simeq \text{Hom}_{\mathcal{T}}(M_k(A), X)$$

which is natural for  $A \in \text{Proj}^k(S^{\text{op}})$  and  $X \in \mathcal{M}_k$ . This restricts to an adjoint pair

$$(3.1) \quad \mathcal{N}_k \begin{array}{c} \xleftarrow{M_k} \\ \xrightarrow{\text{Hom}_{\mathcal{T}}(C, -)} \end{array} \text{Proj}^k(S^{\text{op}}),$$

where the left adjoint is written above the right adjoint.

- (3) The unit of the adjunction in (3.1),  $\eta : A \mapsto \mathbf{Hom}_{\mathcal{T}}(C, M_k(A))$ , is an isomorphism.
- (4)  $\mathbf{Hom}_{\mathcal{T}}(C, \Sigma M_k(A)) = \cdots = \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{n+1-k} M_k(A)) = 0$  for all  $A \in \mathbf{Proj}^k(S^{\text{op}})$ .
- (5)  $\mathbf{Hom}_{\mathcal{T}}(M_k(A), \Sigma^{-1} X) = 0$  for all  $A \in \mathbf{Proj}^k(S^{\text{op}})$  and  $X \in \mathcal{M}_{k+1}$ .
- (6)  $M_k(P) \in \mathbf{Add}(C)$  for any  $P \in \mathbf{Proj}(S^{\text{op}})$ .

In condition (2) of Hypotheses 3.6 we take, as a convention,  $\mathcal{N}_0 = \mathbf{Add}(C)$ .

**Remark 3.7.** Note that the fact that  $\mathcal{N}_k \subseteq \mathcal{M}_k$  means we have the following:

$$(3.2) \quad \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{-1} M_k(A)) = \cdots = \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{-k} M_k(A)) = 0$$

for all  $A \in \mathbf{Proj}^k(S^{\text{op}})$ .

One can now think of the first three induction hypotheses as being the building which we are constructing in the proof of the theorem and the last three conditions as the scaffolding which we require to complete the construction.

A useful tool in the proof will be Lemma 1.5 which relates the full fidelity of an adjoint functor with the unit or counit of the adjunction being an isomorphism. Before embarking on the construction we first prove the base step of the induction.

**Lemma 3.8.** *Under the assumptions of Setup 3.3, Hypotheses 3.6 are true for  $k = 0$ .*

**Proof:** By convention, we have  $\mathcal{N}_0 = \mathbf{Add}(C)$ . From Proposition 3.1, we have an adjoint equivalence of categories:

$$\mathbf{Add}(C) = \mathcal{N}_0 \begin{array}{c} \xleftarrow{M_0} \\ \xrightarrow{\mathbf{Hom}_{\mathcal{T}}(C, -)} \end{array} \mathbf{Proj}(S^{\text{op}}).$$

Hence the unit of this adjunction is an isomorphism (see Lemma 1.5), and for  $P \in \mathbf{Proj}(S^{\text{op}})$  and  $X \in \mathcal{M}_0 = \mathcal{T}$  we have a natural isomorphism

$$\mathbf{Hom}_{\mathbf{Mod}(S^{\text{op}})}(P, \mathbf{Hom}_{\mathcal{T}}(C, X)) \simeq \mathbf{Hom}_{\mathcal{T}}(M_0(P), X).$$

(Note that this isomorphism is stronger than just adjointness in this case because  $X \in \mathcal{T}$ .) We also have  $\mathcal{N}_0 = \mathbf{Ess.Im}(M_0) = \mathbf{Add}(C) \subseteq \mathcal{M}_0$ . This proves (1), (2) and (3) of Hypotheses 3.6 for  $k = 0$ .

Since  $M_0(P) \in \mathbf{Add}(C)$  for  $P \in \mathbf{Proj}(S^{\text{op}})$ , the assumption that  $\mathbf{Hom}_{\mathcal{T}}(C, \Sigma^i C) = 0$  for  $i = 1, \dots, n+1$  forces

$$\mathbf{Hom}_{\mathcal{T}}(C, \Sigma M_0(P)) = \cdots = \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{n+1} M_0(P)) = 0.$$

Hence hypothesis (4) is satisfied for  $k = 0$ .

For  $X \in \mathcal{M}_1$ , we have  $\text{Hom}_{\mathcal{S}}(M_0(P), \Sigma^{-1}X) = 0$  since  $M_0(P) \in \text{Add}(C)$ , so hypotheses (5) and (6) are also satisfied for  $k = 0$ . This completes the proof of the base step.  $\square$

**3.2. The construction of the map  $M_{k+1}$ .** We now turn our attention to the proof of the induction step, starting with the construction of a map  $M_{k+1} : \text{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{M}_{k+1}$ . Assume  $1 \leq k \leq n$  and assume that Hypotheses 3.6 are satisfied for  $k$ . Given any  $A \in \text{Mod}(S^{\text{op}})$  we shall choose a projective resolution

$$0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

where  $m = \text{projdim}_{S^{\text{op}}} A$ , that is, so that we have chosen a projective resolution of  $A$  of minimal length. For  $P \in \text{Proj}(S^{\text{op}})$  choose the obvious projective resolution

$$0 \rightarrow P \xrightarrow{\sim} P \rightarrow 0.$$

Note that the projective dimension of  $A$  is at most  $n$  because  $S^{\text{op}}$  has global dimension  $n$ .

By hypothesis (1) of Hypotheses 3.6, there exists a functor  $M_k : \text{Proj}^k(S^{\text{op}}) \rightarrow \mathcal{M}_k$ . Given the existence of this functor, we show how to construct a map  $M_{k+1} : \text{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{M}_{k+1}$ . In the subsequent section we shall explain how this becomes a functor.

Suppose  $A \in \text{Proj}^{k+1}(S^{\text{op}})$  and consider the projective resolution chosen for  $A$ :

$$0 \longrightarrow P_{k+1} \xrightarrow{\pi_{k+1}} P_k \xrightarrow{\pi_k} \cdots \longrightarrow P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A \longrightarrow 0$$

where  $P_i = 0$  whenever  $i > \text{projdim}_{S^{\text{op}}} A$ . This long exact sequence naturally splits into short exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & P_{k+1} & \rightarrow & P_k & \rightarrow & K_{k-1} \rightarrow 0 \\ & & & & & & \vdots \\ & & & & 0 & \rightarrow & K_i \rightarrow P_i \rightarrow K_{i-1} \rightarrow 0 \\ & & & & & & \vdots \\ (3.3) & & & & 0 & \rightarrow & K_0 \rightarrow P_0 \rightarrow A \rightarrow 0 \end{array}$$

where  $K_i = \ker \pi_i$ . Note  $K_0, P_0 \in \text{Proj}^k(S^{\text{op}})$ , so  $M_k(K_0), M_k(P_0) \in \mathcal{M}_k$ . We can apply the functor  $M_k$  to the homomorphism  $K_0 \rightarrow P_0$  in (3.3) and extend the resulting morphism in  $\mathcal{S}$  to a distinguished triangle:

$$(3.4) \quad M_k(K_0) \rightarrow M_k(P_0) \rightarrow M \rightarrow \Sigma M_k(K_0).$$

Let  $X, Y \in \mathcal{M}_{k+1}$  and apply the functors  $\mathbf{Hom}_{\mathbf{Mod}(S^{\text{op}})}(-, \mathbf{Hom}_{\mathcal{T}}(C, X))$  and  $\mathbf{Hom}_{\mathcal{T}}(-, X)$  to (3.3) and (3.4), respectively, to obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (A, (C, X)) & \longrightarrow & (P_0, (C, X)) & \longrightarrow & (K_0, (C, X)) \longrightarrow \\
& & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
(M_k(K_0), \Sigma^{-1}X) & \longrightarrow & (M, X) & \longrightarrow & (M_k(P_0), X) & \longrightarrow & (M_k(K_0), X) \longrightarrow
\end{array}$$

where, in the above diagram, we have used the notation  $(-, (C, X))$  and  $(-, X)$  as shorthand for the functors  $\mathbf{Hom}_{\mathbf{Mod}(S^{\text{op}})}(-, \mathbf{Hom}_{\mathcal{T}}(C, X))$  and  $\mathbf{Hom}_{\mathcal{T}}(-, X)$ , respectively. By hypothesis 3.6(5),  $\mathbf{Hom}_{\mathcal{T}}(M_k(K_0), \Sigma^{-1}X) = 0$  for  $X \in \mathcal{M}_{k+1}$ , hence, the broken arrow exists and is an isomorphism.

$\mathbf{Hom}_{\mathbf{Mod}(S^{\text{op}})}(-, \mathbf{Hom}_{\mathcal{T}}(C, X))$ ,  $\mathbf{Hom}_{\mathbf{Mod}(S^{\text{op}})}(-, \mathbf{Hom}_{\mathcal{T}}(C, Y))$ ,  $\mathbf{Hom}_{\mathcal{T}}(-, X)$  and  $\mathbf{Hom}_{\mathcal{T}}(-, Y)$  applied to (3.3) and (3.4) for  $X, Y \in \mathcal{M}_{k+1}$  yield the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (A, (C, Y)) & \longrightarrow & (P_0, (C, Y)) & \longrightarrow & (K_0, (C, Y)) \longrightarrow \dots \\
& & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
0 & \longrightarrow & (A, (C, X)) & \longrightarrow & (P_0, (C, X)) & \longrightarrow & (K_0, (C, X)) \longrightarrow \dots \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & 0 & \longrightarrow & (M, Y) & \longrightarrow & (M_k(P_0), Y) & \longrightarrow & (M_k(K_0), Y) \longrightarrow \dots \\
& & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
0 & \longrightarrow & (M, X) & \longrightarrow & (M_k(P_0), X) & \longrightarrow & (M_k(K_0), X) \longrightarrow \dots
\end{array}$$

and we obtain a natural isomorphism on  $\mathcal{M}_{k+1}$ :

$$\mathbf{Hom}_{\mathbf{Mod}(S^{\text{op}})}(A, \mathbf{Hom}_{\mathcal{T}}(C, -)) \simeq \mathbf{Hom}_{\mathcal{T}}(M, -).$$

This construction defines a map  $M_{k+1} : \mathbf{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{T}$  by setting  $M_{k+1}(A) = M$ . Now consider the full subcategory of  $\mathcal{T}$  defined by

$$\mathcal{N}_{k+1} := \{X \in \mathcal{T} \mid X \cong M_{k+1}(A) \text{ for some } A \in \mathbf{Proj}^{k+1}(S^{\text{op}})\}.$$

We want to show that  $\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1}$ , that is, that we have a map  $M_{k+1} : \mathbf{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{M}_{k+1}$ . Let  $X \in \mathcal{N}_{k+1}$ ; then there exists an  $S^{\text{op}}$ -module  $A$  of projective dimension at most  $k+1$  such that  $M_{k+1}(A) \cong X$ . As before, we have a short exact sequence  $0 \rightarrow K_0 \rightarrow P_0 \rightarrow A \rightarrow 0$  in which  $P_0$  is projective and  $K_0 \in \mathbf{Proj}^k(S^{\text{op}})$ . By construction, we have a distinguished triangle in  $\mathcal{T}$ :

$$M_k(K_0) \rightarrow M_k(P_0) \rightarrow M_{k+1}(A) \rightarrow \Sigma M_k(K_0).$$

Applying the functor  $\mathrm{Hom}_{\mathcal{F}}(C, -)$  to this distinguished triangle gives the following long exact sequence:

$$\mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-i}M_k(P_0)) \rightarrow \mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-i}M_{k+1}(A)) \rightarrow \mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-i+1}M_k(K_0)).$$

By Remark 3.7 (which follows from hypothesis 3.6(2)), we have

$$\mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-i}M_k(K_0)) = 0 \text{ for } i = 1, \dots, k,$$

and by assumption  $\mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-j}C) = 0$  for  $j = 1, \dots, n + 1$ . We also have  $M_k(P_0) \in \mathrm{Add}(C)$  by hypothesis (6) of 3.6. Hence, we obtain

$$\mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-i}M_{k+1}(A)) = 0 \text{ for } i = 2, \dots, k + 1.$$

By 3.6 hypothesis (2), we have an adjoint pair

$$\mathcal{N}_k \begin{array}{c} \xleftarrow{M_k} \\ \xrightarrow{\mathrm{Hom}_{\mathcal{F}}(C, -)} \end{array} \mathrm{Proj}^k(S^{\mathrm{op}})$$

whose unit  $\eta$  is an isomorphism (hypothesis (3)). So we have the following commutative square:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{F}}(C, M_k(K_0)) & \longrightarrow & \mathrm{Hom}_{\mathcal{F}}(C, M_k(P_0)) \\ \eta_{K_0} \uparrow \sim & & \eta_{P_0} \uparrow \sim \\ K_0 & \hookrightarrow & P_0 \end{array}$$

Hence the map  $\mathrm{Hom}_{\mathcal{F}}(C, M_k(K_0)) \rightarrow \mathrm{Hom}_{\mathcal{F}}(C, M_k(P_0))$  is an injection, which forces  $\mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-1}M_{k+1}(A)) = 0$ . Thus  $M_{k+1}(A) \in \mathcal{M}_{k+1}$  and we have  $\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1}$  and a map  $M_{k+1} : \mathrm{Proj}^{k+1}(S^{\mathrm{op}}) \rightarrow \mathcal{M}_{k+1}$ .

**Remark 3.9.** So far, all we have proved is that we have a map  $M_{k+1} : \mathrm{Proj}^{k+1}(S^{\mathrm{op}}) \rightarrow \mathcal{M}_{k+1}$  and that  $\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1}$ . In order to prove this we have used all the hypotheses of Hypotheses 3.6 with the exception of (4).

The next step is to make  $M_{k+1}$  into a functor.

**3.3. Functoriality of  $M_{k+1}$ .** Recall that by hypothesis (1) of Hypotheses 3.6 there exists a functor  $M_k : \mathrm{Proj}^k(S^{\mathrm{op}}) \rightarrow \mathcal{M}_k$ . By the work of section 3.2 we have constructed a map  $M_{k+1} : \mathrm{Proj}^{k+1}(S^{\mathrm{op}}) \rightarrow \mathcal{M}_{k+1}$  using the functor  $M_k$ . We now need to get a functor  $M_{k+1} : \mathrm{Proj}^{k+1}(S^{\mathrm{op}}) \rightarrow \mathcal{M}_{k+1}$  from the map  $M_{k+1}$ . This can be done indirectly by noticing that we have a natural isomorphism,

$$\mathrm{Hom}_{\mathrm{Mod}(S^{\mathrm{op}})}(A, \mathrm{Hom}_{\mathcal{F}}(C, -)) \simeq \mathrm{Hom}_{\mathcal{F}}(M_{k+1}(A), -),$$

which is natural on  $\mathcal{M}_{k+1}$ . If the functor  $\mathrm{Hom}_{\mathcal{F}}(C, -)$ , when it acts on  $\mathcal{N}_{k+1}$ , takes values in  $\mathrm{Proj}^{k+1}(S^{\mathrm{op}})$ , then we obtain an adjunction,

$$\mathcal{N}_{k+1} \begin{array}{c} \xleftarrow{M_{k+1}} \\ \xrightarrow{\mathrm{Hom}_{\mathcal{F}}(C, -)} \end{array} \mathrm{Proj}^{k+1}(S^{\mathrm{op}}),$$

via the following classical representability result.

**Lemma 3.10** ([13], Corollary IV.1.2). *Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  has a left adjoint if and only if for every object  $C$  of  $\mathcal{C}$  there is a natural isomorphism*

$$\varphi : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \simeq \mathrm{Hom}_{\mathcal{D}}(F_0(C), D)$$

which is natural in  $D \in \mathcal{D}$ . Then  $F_0$  is the object function of the left adjoint of  $G$ .

Composition of the functor  $M_{k+1} : \mathrm{Proj}^{k+1}(S^{\mathrm{op}}) \rightarrow \mathcal{N}_{k+1}$  with the inclusion functor  $\iota : \mathcal{N}_{k+1} \rightarrow \mathcal{M}_{k+1}$  gives the desired functor.

We now show that we do indeed have a functor  $\mathrm{Hom}_{\mathcal{F}}(C, -) : \mathcal{N}_{k+1} \rightarrow \mathrm{Proj}^{k+1}(S^{\mathrm{op}})$ . Suppose  $X \in \mathcal{N}_{k+1}$ , then there is an  $S^{\mathrm{op}}$ -module  $A$  of projective dimension at most  $k+1$  such that  $M_{k+1}(A) \cong X$ . We get the usual short exact sequence  $0 \rightarrow K_0 \rightarrow P_0 \rightarrow A \rightarrow 0$  and the usual distinguished triangle  $M_k(K_0) \rightarrow M_k(P_0) \rightarrow M_{k+1}(A) \rightarrow \Sigma M_k(K_0)$ . Applying the functor  $\mathrm{Hom}_{\mathcal{F}}(C, -)$  to the usual distinguished triangle gives the following long exact sequence:

$$0 \rightarrow (C, M_k(K_0)) \rightarrow (C, M_k(P_0)) \rightarrow (C, M_{k+1}(A)) \rightarrow (C, \Sigma M_k(K_0)),$$

where we have used the shorthand described earlier to denote the Hom-spaces. The zero on the left hand side comes by the fact that  $\mathrm{Hom}_{\mathcal{F}}(C, \Sigma^{-1} M_{k+1}(A)) = 0$  because  $M_{k+1}(A) \in \mathcal{M}_{k+1}$  (hypothesis (5) of 3.6). By condition (4) of Hypotheses 3.6, we have  $\mathrm{Hom}_{\mathcal{F}}(C, \Sigma M_k(K_0)) = 0$ . Hence, we have the commutative diagram below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & (C, M_k(K_0)) & \longrightarrow & (C, M_k(P_0)) & \longrightarrow & (C, M_{k+1}(A)) \longrightarrow 0 \\ & & \eta_{K_0} \uparrow \sim & & \eta_{P_0} \uparrow \sim & & \\ & & K_0 & \hookrightarrow & P_0 & & \end{array}$$

Therefore,  $\mathrm{Hom}_{\mathcal{F}}(C, M_{k+1}(A)) \in \mathrm{Proj}^{k+1}(S^{\mathrm{op}})$ . Moreover,  $\mathrm{Hom}_{\mathcal{F}}(C, M_{k+1}(A)) \cong A$  for all  $A \in \mathrm{Proj}^{k+1}(S^{\mathrm{op}})$ . It follows that,  $\mathrm{Hom}_{\mathcal{F}}(C, X) \in \mathrm{Proj}^{k+1}(S^{\mathrm{op}})$ .

Thus, we obtain a functor  $M_{k+1} : \mathrm{Proj}^{k+1}(S^{\mathrm{op}}) \rightarrow \mathcal{M}_{k+1}$ . Furthermore,  $\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1}$  and there is a natural isomorphism,

$$\mathrm{Hom}_{\mathrm{Mod}(S^{\mathrm{op}})}(A, \mathrm{Hom}_{\mathcal{F}}(C, X)) \simeq \mathrm{Hom}_{\mathcal{F}}(M_{k+1}(A), X),$$

which is natural on  $\mathrm{Proj}^{k+1}(S^{\mathrm{op}})$  and  $\mathcal{M}_{k+1}$ . This restricts to an adjoint pair,

$$\mathcal{N}_{k+1} \begin{array}{c} \xleftarrow{M_{k+1}} \\ \xrightarrow{\mathrm{Hom}_{\mathcal{F}}(C, -)} \end{array} \mathrm{Proj}^{k+1}(S^{\mathrm{op}}).$$

We now summarise the work of the preceding sections in the following proposition.

**Proposition 3.11.** *Under the assumptions of Setup 3.3 and the hypotheses of Hypotheses 3.6, we have:*

- (1) *There exists a functor  $M_{k+1} : \mathbf{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{M}_{k+1}$ .*
- (2) *Let  $\mathcal{N}_{k+1} = \{X \in \mathcal{T} \mid X \cong M_{k+1}(A) \text{ for some } A \in \mathbf{Proj}^{k+1}(S^{\text{op}})\}$ . We have  $\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1}$ . We also have a natural isomorphism*

$$\mathbf{Hom}_{\text{Mod}(S^{\text{op}})}(A, \mathbf{Hom}_{\mathcal{T}}(C, X)) \simeq \mathbf{Hom}_{\mathcal{T}}(M_{k+1}(A), X)$$

*which is natural for  $A \in \mathbf{Proj}^{k+1}(S^{\text{op}})$  and  $X \in \mathcal{M}_{k+1}$ . This restricts to an adjoint pair*

$$\mathcal{N}_{k+1} \begin{array}{c} \xleftarrow{M_{k+1}} \\ \xrightarrow{\mathbf{Hom}_{\mathcal{T}}(C, -)} \end{array} \mathbf{Proj}^{k+1}(S^{\text{op}}),$$

*where the left adjoint is written above the right adjoint, as in Hypotheses 3.6.*

**Remark 3.12.** In Proposition 3.11 we have accomplished the proof of the induction step for the first two hypotheses in Hypotheses 3.6. However, in obtaining this proof we have used all the hypotheses stated in Hypotheses 3.6. This demonstrates how the remaining hypotheses are analogous to scaffolding in the construction and the subsequent section is devoted to building the scaffolding.

**3.4. The scaffolding.** We start the construction of the scaffolding with the easier hypotheses, (4), (5) and (6) of Hypotheses 3.6.

**Lemma 3.13.** *Under the assumptions of Setup 3.3 and Hypotheses 3.6 we have:*

$$\mathbf{Hom}_{\mathcal{T}}(C, \Sigma M_{k+1}(A)) = \cdots = \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{n-k} M_{k+1}(A)) = 0$$

*for all  $A \in \mathbf{Proj}^{k+1}(S^{\text{op}})$ . (C.f. Hypotheses 3.6, condition (4).)*

**Proof:** Suppose  $A \in \mathbf{Proj}^{k+1}(S^{\text{op}})$ ; then we have the usual short exact sequence  $0 \rightarrow K_0 \rightarrow P_0 \rightarrow A \rightarrow 0$  from which we obtain the usual distinguished triangle  $M_k(K_0) \rightarrow M_k(P_0) \rightarrow M_{k+1}(A) \rightarrow \Sigma M_k(K_0)$  with  $K_0 \in \mathbf{Proj}^k(S^{\text{op}})$  and  $P_0$  a projective  $S^{\text{op}}$ -module. Now apply the functor  $\mathbf{Hom}_{\mathcal{T}}(C, -)$  to the usual distinguished triangle to get the long exact sequence

$$\mathbf{Hom}_{\mathcal{T}}(C, \Sigma^i M_k(P_0)) \rightarrow \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^i M_{k+1}(A)) \rightarrow \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{i+1} M_k(K_0)).$$

By Hypotheses 3.6,  $\mathbf{Hom}_{\mathcal{T}}(C, \Sigma^i M_k(K_0)) = 0$  for  $i = 1, \dots, n+1-k$ , and  $M_k(P_0) \in \text{Add}(C)$  so that  $\mathbf{Hom}_{\mathcal{T}}(C, \Sigma^i M_k(P_0)) = 0$  for  $i = 1, \dots, n+1$ . Condition (4) of Hypotheses 3.6 now follows for  $k+1$ .  $\square$

**Lemma 3.14.** *Under the assumptions of Setup 3.3 and Hypotheses 3.6 we have:*

$$\mathbf{Hom}_{\mathcal{T}}(M_{k+1}(A), \Sigma^{-1} X) = 0 \text{ for all } A \in \mathbf{Proj}^{k+1}(S^{\text{op}}) \text{ and } X \in \mathcal{M}_{k+2}.$$

*(C.f. Hypotheses 3.6, condition (5).)*

**Proof:** Let  $A \in \mathbf{Proj}^{k+1}(S^{\text{op}})$  and  $X \in \mathcal{M}_{k+2}$ . Consider the projective resolution we have chosen for  $A$ :

$$0 \longrightarrow P_{k+1} \xrightarrow{\pi_{k+1}} P_k \xrightarrow{\pi_k} \cdots \longrightarrow P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A \longrightarrow 0,$$

which breaks up into short exact sequences

$$\begin{aligned} 0 \rightarrow P_{k+1} \rightarrow P_k \rightarrow K_{k-1} \rightarrow 0 \\ \vdots \\ 0 \rightarrow K_i \rightarrow P_i \rightarrow K_{i-1} \rightarrow 0 \\ \vdots \\ 0 \rightarrow K_0 \rightarrow P_0 \rightarrow A \rightarrow 0 \end{aligned}$$

with  $K_i = \ker \pi_i$ . These short exact sequences in turn give rise to distinguished triangles:

$$\begin{aligned} M_k(P_{k+1}) \rightarrow M_k(P_k) \rightarrow M_k(K_{k-1}) \rightarrow \Sigma M_k(P_{k+1}) \\ \vdots \\ M_k(K_i) \rightarrow M_k(P_i) \rightarrow M_k(K_{i-1}) \rightarrow \Sigma M_k(K_i) \\ \vdots \\ (3.5) \quad M_k(K_0) \rightarrow M_k(P_0) \rightarrow M_{k+1}(A) \rightarrow \Sigma M_k(K_0) \end{aligned}$$

Consider the long exact sequence obtained by applying the functor  $\mathbf{Hom}_{\mathcal{F}}(-, X)$  to distinguished triangle (3.5) with  $X \in \mathcal{M}_{k+2}$ . We get:

$$\mathbf{Hom}_{\mathcal{F}}(\Sigma^2 M_k(K_0), X) \rightarrow \mathbf{Hom}_{\mathcal{F}}(\Sigma M_{k+1}(A), X) \rightarrow \mathbf{Hom}_{\mathcal{F}}(\Sigma M_k(P_0), X).$$

We know that  $\mathbf{Hom}_{\mathcal{F}}(\Sigma M_k(P_0), X) = 0$  because  $X \in \mathcal{M}_{k+2}$  and, by hypothesis (6) of 3.6, we have  $M_k(P_0) \in \mathbf{Add}(C)$ . So, we only need to see that  $\mathbf{Hom}_{\mathcal{F}}(\Sigma^2 M_k(K_0), X) = 0$ . By the same argument,  $\mathbf{Hom}_{\mathcal{F}}(\Sigma^2 M_k(K_0), X) = 0$  if  $\mathbf{Hom}_{\mathcal{F}}(\Sigma^3 M_k(K_1), X) = 0$ . Applying the same argument inductively yields the following implication:

$$\mathbf{Hom}_{\mathcal{F}}(\Sigma^{i+3} M_k(K_{i+1}), X) = 0 \implies \mathbf{Hom}_{\mathcal{F}}(\Sigma^{i+2} M_k(K_i), X) = 0.$$

Now,  $X \in \mathcal{M}_{k+2}$  and  $K_k = P_{k+1}$ , so

$$\mathbf{Hom}_{\mathcal{F}}(\Sigma^{k+2} M_k(K_k), X) = \mathbf{Hom}_{\mathcal{F}}(\Sigma^{k+2} M_k(P_{k+1}), X) = 0,$$

because  $M_k(P_{k+1}) \in \mathbf{Add}(C)$ , so it follows that  $\mathbf{Hom}_{\mathcal{F}}(\Sigma^{k+1} M_k(K_{k-1}), X) = 0$ , and eventually we see that

$$\mathbf{Hom}_{\mathcal{F}}(\Sigma M_{k+1}(A), X) = 0.$$

That is, for  $A \in \mathbf{Proj}^{k+1}(S^{\text{op}})$  and  $X \in \mathcal{M}_{k+2}$ , we have  $\mathbf{Hom}_{\mathcal{F}}(M_{k+1}(A), \Sigma^{-1} X) = 0$ , as required.  $\square$

**Lemma 3.15.** *Under the assumptions of Setup 3.3 and Hypotheses 3.6 then given any projective  $S^{\text{op}}$ -module  $P$  we have  $M_{k+1}(P) \in \mathbf{Add}(C)$ . (C.f. Hypotheses 3.6, condition (6).)*

**Proof:** Let  $P$  be a projective  $S^{\text{op}}$ -module and consider the short exact sequence coming from its projective resolution:

$$0 \longrightarrow 0 \longrightarrow P \xrightarrow{\sim} P \longrightarrow 0.$$

This gives a distinguished triangle:

$$0 \rightarrow M_k(P) \rightarrow M_{k+1}(P) \rightarrow \Sigma 0,$$

by construction. Hence  $M_{k+1}(P) \cong M_k(P)$  and  $M_k(P) \in \mathbf{Add}(C)$  by induction, so we have  $M_{k+1}(P) \in \mathbf{Add}(C)$ .  $\square$

We now turn our attention to the more difficult third Hypotheses 3.6(3).

**Proposition 3.16.** *Under the assumptions of Setup 3.3 and Hypotheses 3.6 then the unit,  $\eta$ , of the adjunction obtained in Proposition 3.11*

$$\mathcal{N}_{k+1} \begin{array}{c} \xleftarrow{M_{k+1}} \\ \xrightarrow{\text{Hom}_{\mathcal{S}}(C, -)} \end{array} \mathbf{Proj}^{k+1}(S^{\text{op}})$$

*is an isomorphism. (C.f. Hypotheses 3.6, condition (3).)*

**Proof:** The fact that the unit  $\eta$  is an isomorphism is equivalent, by Lemma 1.5, to the functor  $M_{k+1} : \mathbf{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{N}_{k+1}$  being full and faithful. Hence we shall show that  $M_{k+1}$  is full and faithful and then deduce that its unit must be an isomorphism. By hypothesis (3) of 3.6, the unit of the adjunction

$$\mathcal{N}_k \begin{array}{c} \xleftarrow{M_k} \\ \xrightarrow{\text{Hom}_{\mathcal{S}}(C, -)} \end{array} \mathbf{Proj}^k(S^{\text{op}})$$

is an isomorphism, so that the functor  $M_k : \mathbf{Proj}^k(S^{\text{op}}) \rightarrow \mathcal{N}_k$  is full and faithful. We start by showing the fidelity of  $M_{k+1}$ .

**Claim:** The functor  $M_{k+1}$  is faithful.

**Proof of claim:** Suppose  $A_1, A_2 \in \mathbf{Proj}^{k+1}(S^{\text{op}})$  and we have a map  $\alpha : A_1 \rightarrow A_2$ . We want to show that if  $M_{k+1}(\alpha) = 0$  then  $\alpha = 0$ . So assume that  $M_{k+1}(\alpha) = 0$ . By elementary homological algebra, see for instance [7],  $\alpha$  lifts to a map of projective resolutions of  $A_1$  and  $A_2$ , so we obtain a commutative

diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & P_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \downarrow \kappa & & \downarrow \pi & & \downarrow \alpha & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & P_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

with  $K_1, K_2 \in \text{Proj}^k(S^{\text{op}})$  and  $P_1, P_2 \in \text{Proj}(S^{\text{op}})$ . By construction we obtain the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & (M_{k+1}(A_1), -) & \longrightarrow & (M_k(P_1), -) & \longrightarrow & (M_k(K_1), -) & \longrightarrow & \dots & & \\ & & \nearrow \sim & \uparrow & \nearrow \sim & \uparrow & \nearrow \sim & \uparrow & & & \\ 0 & \longrightarrow & (A_1, (C, -)) & \longrightarrow & (P_1, (C, -)) & \longrightarrow & (K_1, (C, -)) & \longrightarrow & \dots & & \\ & & \uparrow & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & (M_{k+1}(A_2), -) & \longrightarrow & (M_k(P_2), -) & \longrightarrow & (M_k(K_2), -) & \longrightarrow & \dots & & \\ & & \nearrow \sim & \uparrow & \nearrow \sim & \uparrow & \nearrow \sim & \uparrow & & & \\ 0 & \longrightarrow & (A_2, (C, -)) & \longrightarrow & (P_2, (C, -)) & \longrightarrow & (K_2, (C, -)) & \longrightarrow & \dots & & \end{array}$$

where the arrows on the back face of the cuboid above are given by the obvious compositions and where the blanks are assumed to take values in  $\mathcal{M}_{k+1}$ . The back face then restricts to the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & (M_{k+1}(A_1), -) & \longrightarrow & (M_k(P_1), -) & \longrightarrow & (M_k(K_1), -) & \longrightarrow \\ & & \uparrow & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & (M_{k+1}(A_2), -) & \longrightarrow & (M_k(P_2), -) & \longrightarrow & (M_k(K_2), -) & \longrightarrow \end{array}$$

Note that this links the functoriality of  $M_{k+1}$  with the functoriality of  $M_k$ .

Using this link in the functoriality of the functors  $M_{k+1}$  and  $M_k$ , we then get the commutative diagram below.

$$\begin{array}{ccccccc} M_k(K_1) & \longrightarrow & M_k(P_1) & \longrightarrow & M_{k+1}(A_1) & \longrightarrow & \\ M_k(\kappa) \downarrow & \nearrow M_k(\rho) & \downarrow M_k(\pi) & \searrow 0 & \downarrow M_{k+1}(\alpha) & & \\ M_k(K_2) & \longrightarrow & M_k(P_2) & \longrightarrow & M_{k+1}(A_2) & \longrightarrow & \end{array}$$

Since  $M_{k+1}(\alpha) : M_{k+1}(A_1) \rightarrow M_{k+1}(A_2)$  is zero, it follows that  $M_k(\pi) : M_k(P_1) \rightarrow M_k(P_2)$  factors through  $M_k(K_2)$  as shown. Since, by hypothesis (3) of 3.6,  $M_k$  is full and faithful, the factoring map has the form  $M_k(\rho)$  for some  $\rho : P_1 \rightarrow K_2$ .

Hence we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & P_1 & \longrightarrow & A_1 \longrightarrow 0 \\
& & \downarrow \kappa & & \swarrow \rho & \downarrow \pi & \downarrow \alpha \\
0 & \longrightarrow & K_2 & \longrightarrow & P_2 & \longrightarrow & A_2 \longrightarrow 0
\end{array}$$

whence, we have  $\alpha = 0$  and  $M_{k+1}$  is a faithful functor, proving the claim.

**Claim:** The functor  $M_{k+1}$  is full.

**Proof of claim:** Suppose  $A_1, A_2 \in \text{Proj}^{k+1}(S^{\text{op}})$  and suppose we have a map  $a : M_{k+1}(B_1) \rightarrow M_{k+1}(B_2)$ . By construction we have two distinguished triangles, as shown below.

$$(3.6) \quad \begin{array}{ccccccc}
M_k(K_1) & \longrightarrow & M_k(P_1) & \longrightarrow & M_{k+1}(A_1) & \longrightarrow & \Sigma M_k(K_1) \\
M_k(\kappa) \downarrow & & M_k(\pi) \downarrow & & a \downarrow & & \Sigma M_k(\kappa) \downarrow \\
M_k(K_2) & \longrightarrow & M_k(P_2) & \longrightarrow & M_{k+1}(A_2) & \longrightarrow & \Sigma M_k(K_2)
\end{array}$$

Since the composite  $M_k(P_1) \rightarrow M_{k+1}(A_1) \rightarrow M_{k+1}(A_2) \rightarrow \Sigma M_k(K_2)$  is zero, the broken arrow  $M_k(\pi) : M_k(P_1) \rightarrow M_k(P_2)$  exists making the middle square commutative. It follows that a map  $M_k(\kappa) : M_k(K_1) \rightarrow M_k(K_2)$  exists, making the whole diagram commutative. Note that the maps  $M_k(\pi)$  and  $M_k(\kappa)$  have this form because  $M_k$  is full by hypothesis (3) of 3.6. This then gives a commutative (because  $M_k$  is faithful) diagram,

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & P_1 & \longrightarrow & A_1 \longrightarrow 0 \\
& & \downarrow \kappa & & \downarrow \pi & & \downarrow \alpha \\
0 & \longrightarrow & K_2 & \longrightarrow & P_2 & \longrightarrow & A_2 \longrightarrow 0
\end{array}$$

in which the rows are exact. Applying the functors  $M_k$  and  $M_{k+1}$  we get, by construction, the following commutative diagram of distinguished triangles.

$$(3.7) \quad \begin{array}{ccccccc}
M_k(K_1) & \longrightarrow & M_k(P_1) & \longrightarrow & M_{k+1}(A_1) & \longrightarrow & \Sigma M_k(K_1) \\
M_k(\kappa) \downarrow & & M_k(\pi) \downarrow & & M_{k+1}(\alpha) \downarrow & & \Sigma M_k(\kappa) \downarrow \\
M_k(K_2) & \longrightarrow & M_k(P_2) & \longrightarrow & M_{k+1}(A_2) & \longrightarrow & \Sigma M_k(K_2)
\end{array}$$

We need to show that  $M_{k+1}(\alpha) = a$ . By the commutativity of diagrams (3.6) and (3.7), we can consider the following commutative diagram:

$$\begin{array}{ccccccc}
 M_k(K_1) & \longrightarrow & M_k(P_1) & \longrightarrow & M_{k+1}(A_1) & \longrightarrow & \Sigma M_k(K_1) \\
 & & \downarrow & \searrow^{0} & \downarrow^{M_{k+1}(\alpha)-a} & & \swarrow^{\exists} \\
 & & M_k(P_2) & \longrightarrow & M_{k+1}(A_2) & & 
 \end{array}$$

The composite straight broken arrow is zero by the commutativity of diagrams (3.6) and (3.7), so the curved broken arrow  $\Sigma M_k(K_1) \rightarrow M_{k+1}(A_2)$  exists making the right hand triangle commute. However, we already know any map  $\Sigma M_k(K_1) \rightarrow M_{k+1}(A_2)$  has to be zero. Hence  $M_{k+1}(\alpha) - a = 0$ , that is  $M_{k+1}(\alpha) = a$ . Thus, the functor  $M_{k+1} : \mathbf{Proj}^{k+1}(S^{\text{op}}) \rightarrow \mathcal{N}_{k+1}$  is full, proving the second claim.

Since the functor  $M_{k+1}$  is full and faithful and it appears in an adjunction

$$\mathcal{N}_{k+1} \begin{array}{c} \xleftarrow{M_{k+1}} \\ \xrightarrow{\text{Hom}_{\mathcal{F}}(C, -)} \end{array} \mathbf{Proj}^{k+1}(S^{\text{op}})$$

it follows that the unit  $\eta$  of this adjunction is an isomorphism, thus proving hypothesis (3) for  $k + 1$ .  $\square$

Hence we have shown that if Hypotheses 3.6 are true for  $k$  then they are true for  $k + 1$ . We know, by Lemma 3.8 that they are true for  $k = 0$ . It is clear that the induction terminates at  $k = n + 1$ , thus setting  $M = M_{n+1}$  and  $\mathcal{M} = \mathcal{M}_{n+1}$  gives the existence of a functor

$$M : \mathbf{Mod}(S^{\text{op}}) \rightarrow \mathcal{M}$$

which is left adjoint to the functor  $\text{Hom}_{\mathcal{F}}(C, -) : \mathcal{M} \rightarrow \mathbf{Mod}(S^{\text{op}})$ , completing the proof of Theorem 3.5.

It is useful for the next sections to highlight two important aspects of the proof of Theorem 3.5. The first is a general remark about a specific distinguished triangle which is obtained in the proof of Theorem 3.5; the second relates to a useful construction which occurs in the proof of Proposition 3.16.

**Remark 3.17.** Given a short exact sequence,  $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$ , in  $\mathbf{Mod}(S^{\text{op}})$  with  $K \in \mathbf{Proj}^k(S^{\text{op}})$ ,  $P \in \mathbf{Proj}(S^{\text{op}})$  and  $A \in \mathbf{Proj}^{k+1}(S^{\text{op}})$ , the proof of Theorem 3.5 gives a distinguished triangle  $MK \rightarrow MP \rightarrow MA \rightarrow \Sigma MK$ .

**Remark 3.18.** In the proof of Proposition 3.16 we observed the following construction: if we are given a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{i_1} & P_1 & \xrightarrow{p_1} & A_1 & \longrightarrow & 0 \\ & & \downarrow \kappa & & \downarrow \pi & & \downarrow \alpha & & \\ 0 & \longrightarrow & K_2 & \xrightarrow{i_2} & P_2 & \xrightarrow{p_2} & A_2 & \longrightarrow & 0, \end{array}$$

in which the rows are short exact sequences,  $A_1, A_2 \in \mathbf{Proj}^{k+1}(S^{\text{op}})$ ,  $K_1, K_2 \in \mathbf{Proj}^k(S^{\text{op}})$  and  $P_1, P_2 \in \mathbf{Proj}(S^{\text{op}})$ , we get a commutative diagram,

$$\begin{array}{ccccccc} MK_1 & \xrightarrow{Mi_1} & MP_1 & \xrightarrow{Mp_1} & MA_1 & \longrightarrow & \Sigma MK_1 \\ M\kappa \downarrow & & M\pi \downarrow & & M\alpha \downarrow & & \Sigma M\kappa \downarrow \\ MK_2 & \xrightarrow{Mi_2} & MP_2 & \xrightarrow{Mp_2} & MA_2 & \longrightarrow & \Sigma MK_2, \end{array}$$

in which the rows are distinguished triangles in  $\mathcal{T}$ .

We shall now regard the functor  $M : \mathbf{Mod}(S^{\text{op}}) \rightarrow \mathcal{M}$  obtained in Theorem 3.5 as taking values in  $\mathcal{T}$  via composition with the inclusion functor  $i : \mathcal{M} \hookrightarrow \mathcal{T}$  (c.f. Theorem 2.3).

**Corollary 3.19.** *Viewing the functor obtained in Theorem 3.5 as taking values in  $\mathcal{T}$ , the functor  $M : \mathbf{Mod}(S^{\text{op}}) \rightarrow \mathcal{T}$  is a full embedding.*

#### 4. JØRGENSEN'S CONSTRUCTION REVISITED

In this brief section we return to Jørgensen's theorem (stated in this paper as Theorem 2.3). We first note the following specialisation of Theorem 3.5.

**Proposition 4.1.** *Let  $\mathcal{T}$  be an  $R$ -linear triangulated category with set indexed coproducts. Suppose  $C$  is a compact object of  $\mathcal{T}$  satisfying the following assumptions:*

- (1) *Its endomorphism algebra  $S^{\text{op}} = \mathbf{End}_{\mathcal{T}}(C)^{\text{op}}$  has global dimension 1; and,*
- (2) *We have  $\mathbf{Hom}_{\mathcal{T}}(C, \Sigma C) = \mathbf{Hom}_{\mathcal{T}}(C, \Sigma^{-1}C) = 0$ .*

Let  $\mathcal{M} = \mathcal{M}_1$  (see Definition 3.4). Then, the functor

$$\mathbf{Hom}_{\mathcal{T}}(C, -) : \mathcal{M} \rightarrow \mathbf{Mod}(S^{\text{op}})$$

has a left adjoint

$$M : \mathbf{Mod}(S^{\text{op}}) \rightarrow \mathcal{M}.$$

Moreover, the functor  $M$  is a full embedding of the module category  $\mathbf{Mod}(S^{\text{op}})$  into the full subcategory  $\mathcal{M}$  of  $\mathcal{T}$ .

Proposition 4.1 can be viewed as a generalisation of Theorem 2.3. Indeed, the hypotheses of Setup 2.1 imply the hypotheses of Proposition 4.1. In addition, if the unit of the adjunction obtained in Theorem 2.3 is an isomorphism then we have

$$R \cong \mathrm{Hom}_{\mathcal{T}}(C, M'(R)) \cong \mathrm{Hom}_{\mathcal{T}}(C, C) = S,$$

where  $M'$  denotes the functor obtained in Theorem 2.3. Hence, in this case,  $\mathrm{Mod}(R^{\mathrm{op}}) \simeq \mathrm{Mod}(S^{\mathrm{op}})$  and the functors  $M$ , obtained in Proposition 4.1, and  $M'$ , obtained in Theorem 2.3, coincide.

**Remark 4.2.** Note that the full subcategory  $\mathcal{A}_1$ , defined in Hypotheses 3.6, of the auxiliary category  $\mathcal{M}_1$  defined in Definition 3.4 coincides with the auxiliary category  $\mathcal{M}$  defined in Definition 2.2.

## 5. THE FUNCTOR $M$ IS WELL BEHAVED

We now show that the functor  $M : \mathrm{Mod}(S^{\mathrm{op}}) \rightarrow \mathcal{T}$  constructed in Theorem 3.5 is well behaved with respect to short exact sequences in  $\mathrm{Mod}(S^{\mathrm{op}})$  and distinguished triangles in  $\mathcal{T}$ . The hard work carried out in section 3 provides the setting for a functorial proof of Theorem 5.1 which differs in character entirely with the corresponding result which it generalises ([10, Theorem 4.11]).

**Theorem 5.1.** *Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts. Let  $C$  be an object of  $\mathcal{T}$  satisfying the assumptions of Setup 3.3. Let  $M : \mathrm{Mod}(S^{\mathrm{op}}) \rightarrow \mathcal{T}$  be the functor obtained in Theorem 3.5. If  $0 \rightarrow A' \xrightarrow{a'} A \xrightarrow{a} A'' \rightarrow 0$  is a short exact sequence in  $\mathrm{Mod}(S^{\mathrm{op}})$ , then there is a distinguished triangle:*

$$MA' \xrightarrow{Ma'} MA \xrightarrow{Ma} MA'' \rightarrow \Sigma MA'$$

in  $\mathcal{T}$ .

**Proof:** By Theorem 3.5 we have an adjoint pair

$$\mathcal{M} \begin{array}{c} \xleftarrow{M} \\ \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(C, -)} \end{array} \mathrm{Mod}(S^{\mathrm{op}}),$$

where  $\mathcal{M} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(C, \Sigma^{-i}X) = 0 \text{ for } i = 1, \dots, n+1\}$ . Suppose we have a short exact sequence

$$(5.1) \quad 0 \rightarrow B' \xrightarrow{b'} B \xrightarrow{b} B'' \rightarrow 0.$$

We can extend the morphism  $Mb' : MB' \rightarrow Mb$  to get a distinguished triangle:

$$(5.2) \quad MB' \xrightarrow{Mb'} MB \xrightarrow{f} Z \xrightarrow{g} \Sigma MB'.$$

We first claim that  $Z \in \mathcal{M}$ . Applying the functor  $\mathrm{Hom}_{\mathcal{T}}(C, -)$  to distinguished triangle (5.2) gives the following long exact sequence.

$$(5.3) \quad \mathrm{Hom}_{\mathcal{T}}(C, \Sigma^{-i}MB) \rightarrow \mathrm{Hom}_{\mathcal{T}}(C, \Sigma^{-i}Z) \rightarrow \mathrm{Hom}_{\mathcal{T}}(C, \Sigma^{-i+1}MB').$$

Since  $MB$  and  $MB'$  are objects of  $\mathcal{M}$ , we have  $\mathbf{Hom}_{\mathcal{F}}(C, \Sigma^{-i}Z) = 0$  for  $i = 2, \dots, n+1$ . We need to check that  $\mathbf{Hom}_{\mathcal{F}}(C, \Sigma^{-1}Z) = 0$ . Looking at the right hand end of (5.3), we have the exact sequence

$$0 \rightarrow \mathbf{Hom}_{\mathcal{F}}(C, \Sigma^{-1}Z) \rightarrow \mathbf{Hom}_{\mathcal{F}}(C, MB') \hookrightarrow \mathbf{Hom}_{\mathcal{F}}(C, MB),$$

where  $\mathbf{Hom}_{\mathcal{F}}(C, MB') \hookrightarrow \mathbf{Hom}_{\mathcal{F}}(C, MB)$  is injective by the fact that the unit of the adjunction in Theorem 3.5 is an isomorphism. It follows that  $\mathbf{Hom}_{\mathcal{F}}(C, \Sigma^{-1}Z) = 0$  and, thus, that  $Z \in \mathcal{M}$ , as claimed.

We shall now show that not only is  $Z$  an object of  $\mathcal{M}$ , but that  $Z$  is isomorphic to  $MB''$  in  $\mathcal{M}$ . Let  $X$  be an object of  $\mathcal{M}$  and apply the functor  $\mathbf{Hom}_{\mathcal{S}^{\text{op}}}(-, \mathbf{Hom}_{\mathcal{F}}(C, X))$  to the given short exact sequence, (5.1). This gives a long exact sequence, which forms the top row of a commutative diagram which is given by the adjunction obtained from Theorem 3.5:

$$(5.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (B'', (C, X)) & \longrightarrow & (B, (C, X)) & \longrightarrow & (B', (C, X)) \longrightarrow \\ & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ 0 & \longrightarrow & (MB'', X) & \longrightarrow & (MB, X) & \longrightarrow & (MB', X) \longrightarrow \end{array}$$

where we use  $(B, (C, X))$  as a shorthand for  $\mathbf{Hom}_{\mathcal{S}^{\text{op}}}(B, \mathbf{Hom}_{\mathcal{F}}(C, X))$  and  $(MB, X)$  as a shorthand for  $\mathbf{Hom}_{\mathcal{F}}(MB, X)$ , and so on. Applying the functor  $\mathbf{Hom}_{\mathcal{F}}(-, X)$  to distinguished triangle (5.2) also gives an exact sequence

$$0 \longrightarrow (Z, X) \longrightarrow (MB, X) \longrightarrow (MB', X) \longrightarrow$$

which combined with the exact sequence obtained from the bottom row of diagram (5.4) yields the following commutative diagram.

$$(5.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (MB'', X) & \xrightarrow{(Mb)^*} & (MB, X) & \xrightarrow{(Mb')^*} & (MB', X) \longrightarrow \\ & & \downarrow \xi^* \downarrow \sim & & \parallel & & \parallel \\ 0 & \longrightarrow & (Z, X) & \xrightarrow{f^*} & (MB, X) & \xrightarrow{(Mb)^*} & (MB', X) \longrightarrow \end{array}$$

Hence there is an isomorphism  $\xi^* : \mathbf{Hom}_{\mathcal{F}}(MB'', X) \rightarrow \mathbf{Hom}_{\mathcal{F}}(Z, X)$  for all  $X \in \mathcal{M}$ . It follows that  $\xi : MB'' \rightarrow Z$  is an isomorphism in  $\mathcal{M}$ , as desired.

From this we obtain the following diagram:

$$\begin{array}{ccccccc} MB' & \xrightarrow{Mb'} & MB & \xrightarrow{Mb} & MB'' & \xrightarrow{g \circ \xi} & \Sigma MB' \\ \parallel & & \parallel & & \downarrow \xi \downarrow \sim & & \parallel \\ MB' & \xrightarrow{Mb'} & MB & \xrightarrow{f} & Z & \xrightarrow{g} & \Sigma MB' \end{array}$$

The squares on the left and right of the diagram above are commutative. If we can show the middle square commutes, then we shall have obtained an

isomorphism of our candidate triangle with a distinguished triangle in  $\mathcal{T}$ , thus giving us the desired distinguished triangle. In order to do this we have to remember more information from the construction of the isomorphism obtained in (5.5). Diagram (5.5) gives the equality  $(Mb)^* = f^* \circ \xi^* = (\xi \circ f)^*$ , and hence  $Mb = \xi \circ f$ , making the central square commute, thus proving the theorem.  $\square$

We now aim to prove that for  $A, B \in \text{Mod}(S^{\text{op}})$  there are natural maps

$$\text{Ext}_{S^{\text{op}}}^n(A, B) \xrightarrow{\Delta_{A,B}^n} \text{Hom}_{\mathcal{T}}(MA, \Sigma^n MB).$$

We need to appeal to the definitions of a  $\delta$ -functor, a universal  $\delta$ -functor and the fact that the functor  $\text{Ext}(-, -)$  is a universal  $\delta$ -functor. The following definitions are taken from [5].

**Definition 5.2.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be abelian categories. A (covariant)  $\delta$ -functor from  $\mathcal{A}$  to  $\mathcal{B}$  is a collection of functors  $T = (T^i)_{i \geq 0}$  together with a morphism

$$\delta^i : T^i(A'') \rightarrow T^{i+1}(A')$$

for each short exact sequence  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  and each  $i \geq 0$ , such that

- (1) For each short exact sequence, as above, there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow T^0(A') \longrightarrow T^0(A) \longrightarrow T^0(A'') \xrightarrow{\delta^0} T^1(A') \longrightarrow \dots \\ \dots \longrightarrow T^i(A') \longrightarrow T^i(A) \longrightarrow T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \longrightarrow \dots \end{aligned}$$

- (2) For each morphism of one short exact sequence, as above, into another  $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ , the  $\delta$ s give a commutative diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B'). \end{array}$$

A *contravariant*  $\delta$ -functor is defined similarly.

**Definition 5.3.** A  $\delta$ -functor  $T = (T^i)_{i \geq 0} : \mathcal{A} \rightarrow \mathcal{B}$  is said to be a *universal*  $\delta$ -functor if, given any other  $\delta$ -functor  $U = (U^i)_{i \geq 0} : \mathcal{A} \rightarrow \mathcal{B}$  and any given morphism of functors  $f^0 : T^0 \rightarrow U^0$ , there exists a unique sequence of morphisms  $f^i : T^i \rightarrow U^i$  for each  $i \geq 0$ , starting with the given  $f$ , which commute with the  $\delta^i$ s for each short exact sequence.

It is a well known fact that  $\text{Ext}^n(A, -)$  is a covariant universal  $\delta$ -functor and  $\text{Ext}^n(-, B)$  is a contravariant universal  $\delta$ -functor. We will need the following lemma.

**Lemma 5.4.** *Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts. Let  $C$  be an object of  $\mathcal{T}$  satisfying the assumptions of Setup 3.3. Recall that  $S^{\text{op}} = \text{End}(C)^{\text{op}}$ . Then, we have the following:*

- (i) *For  $A \in \text{Mod}(S^{\text{op}})$  the functor  $(\text{Hom}_{\mathcal{T}}(MA, \Sigma^n M(-)))_{n \geq 0}$  is a covariant  $\delta$ -functor.*
- (ii) *For  $B \in \text{Mod}(S^{\text{op}})$  the functor  $(\text{Hom}_{\mathcal{T}}(M(-), \Sigma^n MB))_{n \geq 0}$  is a contravariant  $\delta$ -functor.*

**Proof:** Let  $A \in \text{Mod}(S^{\text{op}})$  and consider  $U^n(-) = \text{Hom}_{\mathcal{T}}(MA, \Sigma^n M(-))$ . We claim that  $(U^n)_{n \geq 0}$  is a covariant  $\delta$ -functor. Suppose we have a short exact sequence  $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$  in  $\text{Mod}(S^{\text{op}})$ . By Theorem 5.1, there is a distinguished triangle  $MB' \rightarrow MB \rightarrow MB'' \rightarrow \Sigma MB'$ . This distinguished triangle induces a long exact sequence

$$\dots \rightarrow U^i(B') \rightarrow U^i(B) \rightarrow U^i(B'') \rightarrow U^{i+1}(B') \rightarrow \dots$$

and gives a morphism  $\delta^i : U^i(B'') \rightarrow U^{i+1}(B')$ .

Now suppose we have another short exact sequence  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  and a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0, \end{array}$$

which gives a commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc} MB' & \longrightarrow & MB & \longrightarrow & MB'' & \longrightarrow & \Sigma MB' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ MC' & \longrightarrow & MC & \longrightarrow & MC'' & \longrightarrow & \Sigma MC' \end{array}$$

This, in turn, yields a commutative diagram:

$$\begin{array}{ccc} U^i(B'') & \xrightarrow{\delta^i} & U^{i+1}(B') \\ \downarrow & & \downarrow \\ U^i(C'') & \xrightarrow{\delta^i} & U^{i+1}(C') \end{array}$$

Hence,  $(U^n)_{n \geq 0}$  is a covariant  $\delta$ -functor. This proves assertion (i); assertion (ii) is proved similarly.  $\square$

**Theorem 5.5.** *Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts. Let  $C$  be an object of  $\mathcal{T}$  satisfying the assumptions of Setup 3.3. Recall that  $S^{\text{op}} = \text{End}(C)^{\text{op}}$ . Then, we have the following:*

(i) For  $A, B \in \text{Mod}(S^{\text{op}})$  there exist maps,

$$\text{Ext}_{S^{\text{op}}}^n(A, B) \xrightarrow{\Delta_{A,B}^n} \text{Hom}_{\mathcal{T}}(MA, \Sigma^n MB),$$

which are natural in  $A$  and  $B$ .

(ii) Given a short exact sequence  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  in  $\text{Mod}(S^{\text{op}})$ , we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{S^{\text{op}}}(A'', B) & \longrightarrow & \text{Hom}_{S^{\text{op}}}(A, B) & \longrightarrow & \text{Hom}_{S^{\text{op}}}(A', B) \longrightarrow \\ & & \downarrow \Delta_{A'',B}^0 & & \downarrow \Delta_{A,B}^0 & & \downarrow \Delta_{A',B}^0 \\ 0 & \longrightarrow & (MA'', MB) & \longrightarrow & (MA, MB) & \longrightarrow & (MA', MB) \longrightarrow \\ \dots & \longrightarrow & \text{Ext}_{S^{\text{op}}}^n(A'', B) & \longrightarrow & \text{Ext}_{S^{\text{op}}}^n(A, B) & \longrightarrow & \text{Ext}_{S^{\text{op}}}^n(A', B) \longrightarrow \dots \\ & & \downarrow \Delta_{A'',B}^n & & \downarrow \Delta_{A,B}^n & & \downarrow \Delta_{A',B}^n \\ \dots & \longrightarrow & (MA'', \Sigma^n MB) & \longrightarrow & (MA, \Sigma^n MB) & \longrightarrow & (MA', \Sigma^n MB) \longrightarrow \dots \end{array}$$

(iii) Given a short exact sequence  $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$  in  $\text{Mod}(S^{\text{op}})$ , we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{S^{\text{op}}}(A, B') & \longrightarrow & \text{Hom}_{S^{\text{op}}}(A, B) & \longrightarrow & \text{Hom}_{S^{\text{op}}}(A, B'') \longrightarrow \\ & & \downarrow \Delta_{A,B'}^0 & & \downarrow \Delta_{A,B}^0 & & \downarrow \Delta_{A,B''}^0 \\ 0 & \longrightarrow & (MA, MB') & \longrightarrow & (MA, MB) & \longrightarrow & (MA, MB'') \longrightarrow \\ \dots & \longrightarrow & \text{Ext}_{S^{\text{op}}}^n(A, B') & \longrightarrow & \text{Ext}_{S^{\text{op}}}^n(A, B) & \longrightarrow & \text{Ext}_{S^{\text{op}}}^n(A, B'') \longrightarrow \dots \\ & & \downarrow \Delta_{A,B}^n & & \downarrow \Delta_{A,B}^n & & \downarrow \Delta_{A,B''}^n \\ \dots & \longrightarrow & (MA, \Sigma^n MB') & \longrightarrow & (MA, \Sigma^n MB) & \longrightarrow & (MA, \Sigma^n MB'') \longrightarrow \dots \end{array}$$

Note that the use of  $(X, \Sigma^n Y)$  on the bottom row in statements (ii) and (iii) is shorthand for  $\text{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ .

**Proof:** The functors  $(\text{Ext}_{S^{\text{op}}}^n(A, -))_{n \geq 0}$  and  $(\text{Ext}_{S^{\text{op}}}^n(-, B))_{n \geq 0}$  are universal  $\delta$ -functors. We also know that the functor  $(\text{Hom}_{\mathcal{T}}(MA, \Sigma^n M(-)))_{n \geq 0}$  is a covariant  $\delta$ -functor and the functor  $(\text{Hom}_{\mathcal{T}}(-, \Sigma^n MB))_{n \geq 0}$  is a contravariant  $\delta$ -functor by Lemma 5.4; the theorem now follows.  $\square$

## 6. AN EXAMPLE FROM $u$ -CLUSTER CATEGORIES

In this section we shall consider a special case of Theorem 3.5 when the endomorphism ring of the compact object  $C$  of the triangulated category  $\mathcal{T}$  is right coherent. In particular, this allows us to specialise Theorem 3.5 to the category  $\text{mod}(S^{\text{op}})$ , the full subcategory of  $\text{Mod}(S^{\text{op}})$  consisting of finitely presented  $S^{\text{op}}$ -modules. We shall then apply this specialisation to the case of a

path algebra of a quiver which has no oriented cycles. Such a path algebra is well known to be hereditary, and as such coherent, see [12] for example. When applying this specialisation of the main theorem to this case, the full embedding of the theorem recovers the canonical embedding of the module category into its  $u$ -cluster category, where  $u \geq 2$  is an integer.

**6.1. A version of the main result for finitely presented modules.** We first specialise the main result to the case for finitely presented  $S^{\text{op}}$ -modules. For the basic facts on coherent rings and modules we refer to [12].

**Definition 6.1.** A ring  $R$  is said to be *right coherent* if every finitely generated right ideal of  $R$  is also finitely presented as a right  $R$ -module.

A finitely generated right  $R$ -module  $A$  is said to be *coherent* if every finitely generated submodule of  $A$  is finitely presented.

Recall that a ring  $R$  is right coherent if and only if any finitely presented right  $R$ -module is coherent. It follows that, if  $R$  is right coherent then every finitely generated projective right  $R$ -module is coherent. It is well known that the kernel of a homomorphism of finitely generated projective  $R$ -modules is also finitely generated, see [8, Lemma 2.11]. Hence, the kernel of a homomorphism of finitely generated right  $R$ -modules is finitely presented. The following is now an easy lemma.

**Lemma 6.2.** *Suppose  $R$  is a right coherent ring. A finitely presented right  $R$ -module  $A$  with finite projective dimension  $k$  has a projective resolution of length  $k$  consisting of finitely generated projective right  $R$ -modules.*

In light of Lemma 6.2 and the usual finite version of Proposition 3.1, we now obtain the following version of Theorem 3.5.

**Theorem 6.3.** *Let  $\mathcal{T}$  be a triangulated category with set indexed coproducts. Suppose  $C$  is an object of  $\mathcal{T}$  satisfying the following assumptions:*

- (1) *Its endomorphism algebra  $S^{\text{op}} = \text{End}_{\mathcal{T}}(C)^{\text{op}}$  has finite global dimension  $n$ ;*
- (2) *We have  $\text{Hom}_{\mathcal{T}}(C, \Sigma^i C) = \text{Hom}_{\mathcal{T}}(C, \Sigma^{-i} C) = 0$  for  $i = 1, \dots, n + 1$ ;*  
*and,*
- (3) *The endomorphism algebra  $S$  is right coherent.*

*Let  $\mathcal{M} = \mathcal{M}_{n+1}$ , then there exists a full embedding  $M : \text{mod}(S^{\text{op}}) \rightarrow \mathcal{M}$ .*

Recall that the category  $\text{mod}(S^{\text{op}})$  of finitely presented right  $S$ -modules is an abelian category if and only if  $S$  is right coherent. Thus, given Lemma 6.2, it follows that the proofs of Theorems 5.1 and 5.5 can be used to prove versions of these theorems for finitely presented right  $S$ -modules.

**Remark 6.4.** Note that in Theorem 6.3 we do not obtain that  $M$  is left adjoint to  $\text{Hom}_{\mathcal{T}}(C, -)$  because it is not clear that  $\text{Hom}_{\mathcal{T}}(C, -)$  applied to  $\mathcal{M}$

necessarily takes values in  $\mathbf{mod}(S^{\text{op}})$ . The construction of Theorem 3.5 applies to Theorem 6.3 by virtue of Lemma 6.2 and the fact that one can still show at each stage of the construction that there is an adjunction

$$\mathcal{N}_k \begin{array}{c} \xleftarrow{M_k} \\ \xrightarrow{\text{Hom}_{\mathcal{T}}(C, -)} \end{array} \mathbf{proj}^k(S^{\text{op}}),$$

where  $\mathcal{N}_k = \{X \in \mathcal{T} \mid X \cong M_k(A) \text{ for some } A \in \mathbf{proj}^k(S^{\text{op}})\}$  and where  $\mathbf{proj}^k(S^{\text{op}})$  denotes the full subcategory of  $\mathbf{mod}(S^{\text{op}})$  consisting of finitely presented  $S^{\text{op}}$ -modules of projective dimension at most  $k$ .

**6.2.  $u$ -cluster categories.** Cluster categories were introduced by Buan, Marsh, Reineke, Reiten and Todorov in [3]. They were also introduced independently for the type  $A$  case in [4]. We give a brief sketch of the definition of the  $u$ -cluster category following the exposition given in [9]. The  $u$ -cluster category was first introduced by Bernhard Keller in [11, Section 8.4]. Let  $k$  be an algebraically closed field and  $H$  be a finite dimensional hereditary  $k$ -algebra. For an integer  $u \geq 1$ , the  $u$ -cluster category  $\mathcal{C}$  is defined by  $\mathcal{D}^f(H^{\text{op}})/\tau^{-1}\Sigma^u$ , where  $\tau$  is the AR translation of  $D^f(H^{\text{op}})$  (see [1] or [2], for example) and  $\Sigma$  is its suspension. Here  $\mathcal{D}^f(H^{\text{op}})$  is shorthand for  $\mathcal{D}^f(\mathbf{mod}(H^{\text{op}}))$ .

By Keller, [11, Section 4, Theorem], the canonical projection functor  $\pi : \mathcal{D}^f(H^{\text{op}}) \rightarrow \mathcal{C}$  is triangulated. Hence by composition with the inclusion functor we obtain a full embedding

$$(6.1) \quad \begin{array}{ccc} \mathbf{mod}(H^{\text{op}}) & \xhookrightarrow{\iota} & \mathcal{D}^f(H^{\text{op}}) \\ & \searrow & \downarrow \pi \\ & & \mathcal{C}, \end{array}$$

for  $u \geq 2$ .

Now let  $H = kQ$  be the path algebra of a quiver  $Q$  with no loops or oriented cycles. Then  $H$  is an hereditary algebra, hence coherent, and  $H \in \mathcal{D}^f(H^{\text{op}})/\tau^{-1}\Sigma^u = \mathcal{C}$  is maximal  $u$ -orthogonal (see [9]). In particular, we have:

- $\text{Hom}_{\mathcal{C}}(H, \Sigma^i H) = 0$  for  $i = 1, \dots, u$ ;
- $\text{Hom}_{\mathcal{C}}(H, \Sigma^{-i} H) = 0$  for  $i = 1, \dots, u$ .

In addition, we have that  $\text{Hom}_{\mathcal{C}}(H, H) \cong H$ , therefore the endomorphism algebra has global dimension 1, and for  $u \geq 2$ ,  $H$  satisfies the hypotheses of Theorem 6.3. Therefore there exists a full embedding  $M : \mathbf{mod}(H^{\text{op}}) \rightarrow \mathcal{M}$

where  $\mathcal{M} = \mathcal{M}_2$  in Definition 3.4. Hence, composing with the inclusion functor, we have a full embedding

$$(6.2) \quad \begin{array}{ccc} \text{mod}(H^{\text{op}}) & \xrightarrow{M} & \mathcal{M} \\ & \searrow & \downarrow \\ & & \mathcal{C}. \end{array}$$

We claim that this embedding coincides with that of (6.1) from [11].

It is clear that embeddings (6.1) and (6.2) are canonically equivalent on  $\text{add}(H)$ . Thus they are equivalent on  $\text{proj}(H^{\text{op}})$ , so we only need to extend the equivalence to projective dimension 1 since  $H$  is hereditary. Let  $A$  and  $B$  be  $H^{\text{op}}$ -modules of projective dimension 1 and suppose we have a module homomorphism  $a : A \rightarrow B$ . Take projective resolutions of  $A$  and  $B$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & A \longrightarrow 0, \\ 0 & \longrightarrow & Q_1 & \xrightarrow{g_1} & Q_0 & \xrightarrow{g_0} & B \longrightarrow 0. \end{array}$$

By elementary homological algebra, see [7], we can lift the homomorphism  $a : A \rightarrow B$  to a commutative diagram:

$$(6.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & A \longrightarrow 0 \\ & & \downarrow p_1 & & \downarrow p_0 & & \downarrow a \\ 0 & \longrightarrow & Q_1 & \xrightarrow{g_1} & Q_0 & \xrightarrow{g_0} & B \longrightarrow 0. \end{array}$$

Since there is a natural isomorphism  $\tau : M|_{\text{proj}(H^{\text{op}})} \rightarrow \pi \circ \iota|_{\text{proj}(H^{\text{op}})}$  there are commutative diagrams:

$$(6.4) \quad \begin{array}{ccc} MP_1 \xrightarrow{Mf_1} MP_0 & \text{and} & MQ_1 \xrightarrow{Mg_1} MQ_0 \\ \tau_{P_1} \downarrow \sim & & \tau_{Q_1} \downarrow \sim \\ \tau_{P_0} \downarrow \sim & & \tau_{Q_0} \downarrow \sim \\ \pi \circ \iota(P_1) \xrightarrow{\pi \circ \iota(f_1)} \pi \circ \iota(P_0) & & \pi \circ \iota(Q_1) \xrightarrow{\pi \circ \iota(g_1)} \pi \circ \iota(Q_0). \end{array}$$

In addition, applying the functors  $M$  and  $\pi \circ \iota$  to diagram (6.3) yields the following commutative diagrams, respectively:

$$(6.5) \quad \begin{array}{ccccccc} MP_1 & \xrightarrow{Mf_1} & MP_0 & \xrightarrow{Mf_0} & MA & \xrightarrow{h} & \Sigma MP_1 \\ Mp_1 \downarrow & & Mp_0 \downarrow & & Ma \downarrow & & \Sigma Mp_1 \downarrow \\ MQ_1 & \xrightarrow{Mg_1} & MQ_0 & \xrightarrow{Mg_0} & MB & \xrightarrow{j} & \Sigma MQ_1, \end{array}$$

and

$$(6.6) \quad \begin{array}{ccccccc} \pi \circ \iota(P_1) & \xrightarrow{\pi \circ \iota(f_1)} & \pi \circ \iota(P_0) & \xrightarrow{\pi \circ \iota(f_0)} & \pi \circ \iota(A) & \xrightarrow{\theta} & \Sigma(\pi \circ \iota(P_1)) \\ \pi \circ \iota(p_1) \downarrow & & \pi \circ \iota(p_0) \downarrow & & \pi \circ \iota(a) \downarrow & & \Sigma(\pi \circ \iota(p_1)) \downarrow \\ \pi \circ \iota(Q_1) & \xrightarrow{\pi \circ \iota(g_1)} & \pi \circ \iota(Q_0) & \xrightarrow{\pi \circ \iota(g_0)} & \pi \circ \iota(B) & \xrightarrow{\varphi} & \Sigma(\pi \circ \iota(Q_1)). \end{array}$$

Note that diagram (6.6) comes by virtue of Keller's theorem that the canonical projection functor is triangulated [11, Section 4, Theorem].

Applying the functor  $M$  and  $\pi \circ \iota$  to the top row of diagram (6.3) and using the natural isomorphism highlighted in diagrams (6.4) gives the following commutative diagram:

$$(6.7) \quad \begin{array}{ccccccc} MP_1 & \xrightarrow{Mf_1} & MP_0 & \xrightarrow{Mf_0} & MA & \xrightarrow{h} & \Sigma MP_1 \\ \tau_{P_1} \downarrow \sim & & \tau_{P_0} \downarrow \sim & & \sigma_A \downarrow \sim & & \Sigma \tau_{P_1} \downarrow \sim \\ \pi \circ \iota(P_1) & \xrightarrow{\pi \circ \iota(f_1)} & \pi \circ \iota(P_0) & \xrightarrow{\pi \circ \iota(f_0)} & \pi \circ \iota(A) & \xrightarrow{\theta} & \Sigma(\pi \circ \iota(P_1)), \end{array}$$

where  $\sigma_A$  exists by an axiom of triangulated categories, see, for instance [16, Definition 1.1.1 (TR3)] and is an isomorphism by the Five Lemma for triangulated categories, see [16, Proposition 1.1.20].

Likewise, one obtains the following diagram:

$$(6.8) \quad \begin{array}{ccccccc} MQ_1 & \xrightarrow{Mg_1} & MQ_0 & \xrightarrow{Mg_0} & MB & \xrightarrow{j} & \Sigma MQ_1 \\ \tau_{Q_1} \downarrow \sim & & \tau_{Q_0} \downarrow \sim & & \sigma_B \downarrow \sim & & \Sigma \tau_{Q_1} \downarrow \sim \\ \pi \circ \iota(Q_1) & \xrightarrow{\pi \circ \iota(g_1)} & \pi \circ \iota(Q_0) & \xrightarrow{\pi \circ \iota(g_0)} & \pi \circ \iota(B) & \xrightarrow{\varphi} & \Sigma(\pi \circ \iota(Q_1)). \end{array}$$

Combining diagrams (6.5), (6.6), (6.7), and (6.8) gives the following three-dimensional diagram:

$$(6.9) \quad \begin{array}{ccccccc} & \pi \circ \iota(P_1) & \longrightarrow & \pi \circ \iota(P_0) & \longrightarrow & \pi \circ \iota(A) & \longrightarrow & \Sigma(\pi \circ \iota(P_1)) \\ & \nearrow \sim & & \nearrow \sim & & \nearrow \sim & & \nearrow \sim \\ MP_1 & \longrightarrow & MP_0 & \longrightarrow & MA & \longrightarrow & \Sigma MP_1 & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & \pi \circ \iota(Q_1) & \longrightarrow & \pi \circ \iota(Q_0) & \longrightarrow & \pi \circ \iota(B) & \longrightarrow & \Sigma(\pi \circ \iota(Q_1)) \\ & \nearrow \sim & & \nearrow \sim & & \nearrow \sim & & \nearrow \sim \\ MQ_1 & \longrightarrow & MQ_0 & \longrightarrow & MB & \longrightarrow & \Sigma MQ_1, & \end{array}$$

where each square and cube is known to commutative except for those involving the broken arrows. We claim that the commutativity of the rest of the diagram forces the whole diagram to commute.

By the commutativity of the rest of the diagram, we have another diagram of distinguished triangles:

$$\begin{array}{ccccccc}
 MP_1 & \xrightarrow{Mf_1} & MP_0 & \xrightarrow{Mf_0} & MA & \xrightarrow{h} & \Sigma MP_1 \\
 \gamma_1 \downarrow & & \gamma_0 \downarrow & & \gamma \downarrow & & \Sigma \gamma_1 \downarrow \\
 \pi \circ \iota(Q_1) & \xrightarrow{\pi \circ \iota(g_1)} & \pi \circ \iota(Q_0) & \xrightarrow{\pi \circ \iota(g_0)} & \pi \circ \iota(B) & \xrightarrow{\varphi} & \Sigma(\pi \circ \iota(Q_1)),
 \end{array}$$

where we have

$$\begin{aligned}
 \gamma_1 &= \pi \circ \iota(p_1) \circ \tau_{P_1} - \tau_{Q_1} \circ Mp_1 \\
 \gamma_0 &= \pi \circ \iota(p_0) \circ \tau_{P_0} - \tau_{Q_0} \circ Mp_0 \\
 \gamma &= \pi \circ \iota(a) \circ \sigma_A - \sigma_B \circ Ma.
 \end{aligned}$$

By the commutativity of diagrams (6.4), we have  $\gamma_1 = \gamma_0 = \Sigma\gamma_1 = 0$ . Thus we obtain the diagram:

$$\begin{array}{ccccccc}
 MP_1 & \xrightarrow{Mf_1} & MP_0 & \xrightarrow{Mf_0} & MA & \xrightarrow{h} & \Sigma MP_1 \\
 & & & \searrow & \downarrow \gamma & & \swarrow \exists \\
 & & & & \pi \circ \iota(B) & & 
 \end{array}$$

From diagram (6.8) we have that  $\pi \circ \iota(B) \cong MB$ , thus  $\pi \circ \iota(B) \in \mathcal{M}$ . The broken arrow,  $\Sigma MP_1 \rightarrow \pi \circ \iota(B)$  must be zero by the proof of Theorem 3.5. Hence, it follows that  $\gamma = 0$ , so that

$$\pi \circ \iota(a) \circ \sigma_A = \sigma_B \circ Ma.$$

This then forces diagram (6.9) to commute, as claimed. In particular, we obtain the following commutative diagram for any module homomorphism  $a : A \rightarrow B$  in  $\text{mod}(H^{\text{op}})$ :

$$\begin{array}{ccc}
 MA & \xrightarrow{Ma} & MB \\
 \sigma_A \downarrow \sim & & \sigma_B \downarrow \sim \\
 \pi \circ \iota(A) & \xrightarrow{\pi \circ \iota(a)} & \pi \circ \iota(B).
 \end{array}$$

Hence there exists a natural isomorphism  $\sigma : M \rightarrow \pi \circ \iota$  on  $\text{mod}(H^{\text{op}})$ . We have, therefore, proved the following theorem.

**Theorem 6.5.** *Let  $H = kQ$  be the path algebra of a quiver  $Q$  with no loops or oriented cycles and let  $\mathcal{C} = \mathcal{D}^f(H)/\tau^{-1}\Sigma^u$  be the  $u$ -cluster category as defined in [9] and [11] for  $u \geq 2$ . Then the canonical embedding,  $\pi \circ \iota$ , obtained in*

[11, Section 4, Theorem] (see diagram (6.1)) and the full embedding obtained in Theorem 6.3 (see diagram (6.2)) are naturally equivalent.

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