Homological Properties of Differential Graded Algebras

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The candidate confirms that the work submitted is his own work and that appropriate credit has been given where reference has been made to the work of others.

Abstract

In this thesis we consider various homological properties of differential graded algebras, and more generally, properties of arbitrary triangulated categories which have set indexed coproducts. A major example of such a triangulated category is the derived category of a differential graded algebra. We present the background material to the theory of triangulated categories, derived categories and differential graded algebras as well as a brief resumé of classical homological algebra in Chapters 2 and 3.

In Chapter 4 we consider the following situation: let R and S be differential graded algebras and M be a differential graded R-S-bimodule satisfying some finiteness condition. In this chapter we give a characterisation of when a differential graded R-S-bimodule M induces a full embedding of derived categories

$$ {}_R M_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R). $$

In particular, this characterisation generalises the theory of Geigle and Lenzing's homological epimorphisms of rings, described in [24]. Furthermore, there is an example of the main result in relation to Dwyer and Greenlees's Morita theory and we are able to apply the main result to give a characterisation of when the co-induction functor yields a full embedding of derived categories.

In Chapter 5 we consider the new notion of a co-t-structure on a triangulated category. In the work of Hoshino, Kato and Miyachi, [31], the authors look at t-structures induced by a compact object, C, of a triangulated category, $\mathscr{T}$, which is rigid in the sense of Iyama and Yoshino, [32]. Hoshino, Kato and Miyachi show that such an object yields a non-degenerate t-structure on $\mathscr{T}$ whose heart is equivalent to $\mathsf{Mod}(\mathsf{End}(C)^{\mathsf{op}})$. Rigid objects in a triangulated category can the thought of as behaving like chain differential graded algebras (DGAs).

Analogously, looking at objects which behave like cochain DGAs naturally gives the

dual notion of a corigid object. Here, we see that a compact corigid object, S, of a triangulated category, $\mathscr{T}$, induces a structure similar to a t-structure which we shall call a co-t-structure. We also show that the coheart of this non-degenerate co-t-structure is equivalent to $\mathsf{Mod}(\mathsf{End}(S)^{\mathsf{op}})$, and hence an abelian subcategory of $\mathscr{T}$.

At the end of this chapter we present a generalisation of the main result to the case when we have more than one corigid object and make some conjectures about further generalisations. We also highlight the role which co-t-structures may play in the theory of torsion theories and their position complementing the theory of t-structures.

In Chapter 6 we consider generalised Moore spectra in a triangulated catgeory. This was first addressed at this level of generality by Jørgensen in [35]. Let R be a principal ideal domain and $\mathscr{T}$ be an R-linear triangulated category, then under certain nice assumptions on R, given an R-module A, Jørgensen looked for the “best approximation” of A in $\mathscr{T}$, $M(A)$, called the “Moore spectrum” of A.

In Chapter 6 we obtain the following generalised construction. Let $\mathscr{T}$ be a triangulated category and let C be a compact object of $\mathscr{T}$ satisfying certain nice “tilting” hypotheses. Let $S = \mathsf{End}(C)$. Then we obtain a full embedding $M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{M}$ which is left adjoint to the functor $\mathsf{Hom}_{\mathscr{T}}(C, -) : \mathscr{M} \to \mathsf{Mod}(S^{\mathsf{op}})$, where $\mathscr{M}$ is some auxilliary full subcategory of $\mathscr{T}$. We show that the functor M is well-behaved and relate the new construction with Jørgensen’s construction in [35]. As an example, we recover the canonical triangle embedding of the category of finitely presented right modules of a hereditary algebra into its u-cluster category of $u \geqslant 2$ from [40].

Contents

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The Logical Expressionist

by Beth Jervis

Every mathematician is
A poet truly at heart,
Who has chosen numerical reasoning
As the medium for his art.

Every mathematician has
A passion for rhyme and rhythm,
Flowing beautifully in algebra
And eccentric and inspiring in algorithm.

Every mathematician gets
Excited about a proof,
Erupts with pleasure at eureka time,
And cries out, exclaims, woo hoo!

Every mathematician does
His hard and tiresome trials,
He knows his soul from the deepest pits
Of painstaking unworkable hours.

But most of all,
The mathematician recognises art.
His beautiful maths

Is composed as such,
On the canvas of reason and logic.

He bends them a slight,
The rules of reality,
But only with faithful tools,
And arrives at singularity,
With contingence in mind,
That can infinitely and randomly disperse.

He takes his model,
His artist's gift,
And applies it to the world,
And sometimes it is applied mechanically,
And sometimes it turns out pure.

But statistically,
The mathematician knows this is art.
For every mathematician,
Is truly a poet at heart.

And every mathematician,
Has a story that springs from this heart.
He grapples with abstraction,
And invests his belief,
And reduces the world to number.
He counts on this number like
Sleepless counting sheep,
And pockets the world to keep.

Chapter 1

Introduction

The area of mathematics now known as homological algebra first arose in the early Twentieth Century from algebraic topology, in which the homology of chain complexes was used to provide invariants for topological spaces. In particular, topological spaces whose homologies are different cannot be homeomorphic to one another. In the introduction to [57], Weibel describes homological algebra as a “tool used to prove nonconstructive existence theorems in algebra and in algebraic topology”. Weibel also tells us that homological algebra can be used to provide us with obstructions to whether certain constructions can be carried out. Since the beginning of the second half of the Twentieth Century, homological algebra has blossomed, starting with Cartan and Eilenberg’s seminal text [17] which introduced the homological functors $\mathsf{Ext}(-,-)$ and $\mathsf{Tor}(-,-)$ which are the mainstay of classical homological algebra. It has become ubiquitous in many areas of mathematics, for example in the theory of representations of algebras, algebraic geometry, algebraic topology and in the theory of Banach algebras.

Since Grothendieck in the 1960s, the language of homological algebra has been highly categorical; the notions of abelian categories, projective and injective objects and resolutions by such objects, and derived functors have become central, providing powerful tools in diverse branches of mathematics. During the second half of the Twentieth Century the notions of triangulated categories and derived categories have become increasingly important tools in homological algebra. Derived categories of abelian categories (or, more generally, of additive categories) are mathematical objects which encode the homological information about the given mathematical object or property one is studying, and as such the properties of derived categories are important. They are useful invariants in the representation theory of associative algebras, algebraic geometry, algebraic topology and group theory. When two objects are derived equivalent it can often lead to a much better understanding of one or both objects. Indeed, the utility of derived categories has not escaped theoretical physicists, the application of derived categories and homological algebra being applied in such diverse areas as quantum theory and string theory in physics.

Derived categories are examples of triangulated categories. Triangulated categories were introduced in algebraic topology by Puppe in the 1960s, [51], and in algebra, in the thesis of Verdier, [56]. The definition which we present in this thesis (Definition 3.1.1) is that due to Verdier.

In an abelian category, the basic structure is provided by a sequence of objects and maps, $0 \to X \to Y \to Z \to 0$, which are called short exact sequences. A triangulated category is a weaker structure than that of an abelian category where the role played by short exact sequences in abelian categories is taken up by diagrams of the form $X \to Y \to Z \to \Sigma X$ which are called distinguished triangles. Triangulated categories provide just enough information for the use of homological algebra. For details regarding short exact sequences in abelian categories and distinguished triangles in triangulated categories see Chapters 2 and 3, respectively.

A central concept in the study of homological algebra is that of a complex. If $\mathscr{C}$ is a category (with a zero object), then a complex in $\mathscr{C}$ is a chain of objects and morphisms,

$$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to C_{-2} \to \cdots ,$$

such that the composition of any two consecutive morphisms is the zero morphism. Complexes are the tool by which homology and cohomology are computed. If, in addition, $\mathscr{C}$ is an abelian category, then one can consider the derived category of $\mathscr{C}$, denoted by $\mathscr{D}(\mathscr{C})$, which is the triangulated category whose objects are complexes of objects of $\mathscr{C}$ and whose morphisms are morphisms of complexes of objects of $\mathscr{C}$ modulo an equivalence relation called homotopy and with the formal inversion of quasi-isomorphisms (that is, morphisms which induce isomorphisms in homology). For details of the notions of complexes and homology see the expositions provided in section 2.2 of Chapter 2. The construction of the derived category is given in Chapter 3, section 3.2.

The main tools and objects of study in homological algebra are complexes. This highlights the importance of a class of algebraic objects known as differential graded algebras. In [1], Aldrich and García Rozas write that “differential graded algebras are a nexus between homological algebra and ring theory”. This is due to the fact that differential graded algebras are natural generalisations of rings and provide a natural environment in which to carry out homological algebra. Indeed, a ring can always be regarded as a differential graded algebra “concentrated in degree zero”, and the underlying structure of a differential graded algebra is that of a complex: the basic tool of homological algebra. Important examples of differential graded algebras occur as projective resolutions arising in commutative algebra, homology and cohomology of topological spaces, for example CW complexes. Differential graded algebras also occur in physics, where they are called super-algebras, see [1].

1.1 Outline of the thesis

In Chapter 2, we give a brief overview of some of the central aspects of classical homological algebra which trace their roots back to Cartan and Eilenberg in the 1950s. In particular, we give the definition of additive and abelian categories and short exact sequences in abelian categories. We sketch the construction of the classical derived functors $\mathsf{Ext}(-,-)$ and $\mathsf{Tor}(-,-)$ which have become essential tools in modern homological algebra and representation theory of associative algebras, providing a brief resumé of their construction by projective and injective resolutions. We also make a brief note of the projective dimension of modules over a ring and the global dimension of the ring. The main references for this chapter are [28] and [57].

In Chapter 3 we turn our attention to the background material required in modern homological algebra, sometimes known as “hyper-homological algebra”, which constitutes a generalisation and extension of the classical theory presented in Chapter 2. The concept of a triangulated category is introduced, including some results showing their elementary properties, such as a version of the Five Lemma for triangulated categories. We also sketch the construction of the derived category of an abelian category via the homotopy category of the abelian category and briefly discuss the existence of “hyper-homological” derived functors and relate them to the classical derived functors presented in Chapter 2. We see how the functors $\mathsf{RHom}(-,-)$ and $- \overset{\mathsf{L}}{\otimes} -$ are natural generalisations of the workhorses $\mathsf{Ext}(-,-)$ and $\mathsf{Tor}(-,-)$, respectively. These themselves become the workhorses of modern homological algebra. In the final part of Chapter 3 we introduce differential graded algebras and differential graded modules and give some elementary examples of them. We also describe the derived category of a differential graded algebra and consider the homological notions of K-projectivity, K-injectivity and K-flatness and their relationship with the computation of the "hyper-homological" derived functors. The main references for Chapter 3 are [10] and [27], and good introductions to the theory of differential graded algebras and differential graded modules can be found in [1] and [10].

Chapters 2 and 3 present only well known background material and do not contain any original work. Chapters 4, 5 and 6 contain the original research of the thesis.

In Chapter 4 we consider a generalisation of Geigle and Lenzing's homological epimorphisms of rings, [24]. In [24], Geigle and Lenzing ask the question: why does a homomorphism of rings $\varphi : R \to S$ induce a full embedding of bounded derived categories? They give a characterisation of when this occurs, obtaining five homological conditions which are equivalent to a ring homomorphism inducing a full embedding of derived categories. We summarise Geigle and Lenzing's theorem below, citing only one of the equivalent conditions, the one which we believe is most powerful.

Theorem 1.1.1 ([24], Theorem 4.4)**.** *Let R and S be rings. For a homomorphism of rings $\varphi : R \to S$ the following conditions are equivalent:*

(1) The multiplication map ${}_S S_R \otimes_R {}_R S_S \to {}_S S_S$ is an isomorphism and $\mathsf{Tor}_i^R(S,S) = 0$ for all $i \geqslant 1$.

(2) The induced functor $\mathscr{D}^b(\varphi_*) : \mathscr{D}^b(\mathsf{Mod}(S)) \to \mathscr{D}^b(\mathsf{Mod}(R))$ is a full embedding of derived categories.

In Theorem 1.1.1, the functor φ_* denotes the restriction of scalars functor induced by φ. A detailed explanation of the restriction of scalars functor can be found in the introduction to Chapter 4. The full statement of this theorem appears in this thesis as Theorem 4.4.10.

In light of Theorem 1.1.1, given that differential graded algebras are a natural generalisation of rings, it is a natural question to ask whether or not a similar characterisation also holds for differential graded algebras? It turns out that it does, and this is the main result of Chapter 4. Indeed, the generalisation of the result to differential graded algebras occurs as a special case of an analogous characterisation of when the functor

$${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$$

induces a full embedding of derived categories, where R and S are both differential graded algebras and M is a differential graded R-S-bimodule satisfying a finiteness condition. The version of Geigle and Lenzing's characterisation for differential graded algebras is stated below.

Theorem 1.1.2. *Let R and S be differential graded algebras and let $\varphi : R \to S$ be a morphism of differential graded algebras. The following conditions are equivalent:*

1. *The canonical map ${}_SS_R \overset{\mathsf{L}}{\otimes}_R {}_RS_S \to {}_SS_S$ is an isomorphism.*
2. *The induced functor $\mathscr{D}(\varphi_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories.*

In Theorem 1.1.2, $\mathscr{D}(\varphi_*)$ denotes the derived restriction of scalars functor analogous to $\mathscr{D}^b(\varphi_*)$ which occurs in Theorem 1.1.1. Theorem 1.1.2 is stated in full as Theorem 4.3.9. The more general version of this result for a differential graded R-S-bimodule satisfying a certain finiteness condition appears as Theorem 4.3.6.

Note, in the statement of Theorem 1.1.2, we refer to the canonical map ${}_SS_R \overset{\mathsf{L}}{\otimes}_R {}_RS_S \to {}_SS_S$. Throughout the generalised version for differential graded algebras we must employ this terminology as the setting of the derived category makes the consideration of the multiplication map and other natural maps difficult. A whole section of Chapter 4, section 4.2, is devoted to highlighting and resolving the difficulties that the "hyper-homological" setting of the derived category causes. Once these have been resolved, the power afforded by the "hyper-homological" tools at our disposal allows us to use the generalised characterisation of homological epimorphisms of differential graded algebras to extend Geigle and Lenzing's classical characterisation to the whole derived categories. The methods employed in [24] force the restriction to the bounded derived category which is relieved in the more general setting.

Chapter 4 also includes an example related to Dwyer and Greenlees' Morita theory in which one of the main results of [19] is seen to be an easy consequence of the generalised characterisation of homological epimorphisms of differential graded algebras. The more general setting in the realms of modern homological algebra also suggests an easy application to characterise when the co-induction functor yields a full embedding of derived categories which is not immediately apparent in the classical context.

We now turn our attention to Chapter 5. In this chapter we consider the behaviour of an object of an arbitrary triangulated category with set indexed coproducts which behaves in a way reminiscent of a cochain differential graded algebra. This is motivated by a theorem of Hoshino, Kato and Miyachi in which they consider an object behaving in a manner which reminds us of a chain differential graded algebra. Hoshino, Kato and Miyachi's theorem is stated in Theorem 5.2.7 in Chapter 5.The basic statement of Hoshino, Kato and Miyachi's theorem is that given a rigid object S of a triangulated category $\mathscr{T}$ satisfying a generating condition for $\mathscr{T}$, then we obtain a canonical t-structure on $\mathscr{T}$ given by

$$\begin{aligned} \mathscr{X} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}, \\ \mathscr{Y} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}. \end{aligned}$$

The notion of a t-structure on a triangulated category was first introduced by Beilinson, Bernstein and Deligne in [7], and it has the useful property of providing a means to obtain an abelian category from a triangulated category in the form of its heart. For explanations regarding notation and terminology, see Chapter 5, section 5.1. Hoshino, Kato and Miyachi's theorem goes further and obtains an equivalence of the heart of the induced t-structure with the category of right modules of the endomorphism algebra of the rigid object S.

A rigid object of a triangulated category is an object whose positive self-extensions all vanish. One can make a dual definition of a corigid object of a triangulated category as being an object whose negative self-extensions all vanish. Once this dualisation is made, it is then an obvious question to ask if there is a dual of Hoshino, Kato and Miyachi's theorem? If there is, one might also expect that the resulting structure which is canonically induced on the triangulated category is also a t-structure. However, this turns out not to be the case. There is a theorem dual to that of Hoshino, Kato and Miyachi, but the resulting structure is not that of a t-structure on $\mathscr{T}$. It is an entirely new structure which is almost dual to that of a t-structure which we have called a co-t-structure. While superficially very similar, the behaviours of t-structures and co-t-structures have significant differences, and included in the definition of a co-t-structure is an extra condition which does not occur in the definition of a t-structure. The definitions of a t-structure and co-t-structure appear as Definitions 5.1.4 and 5.1.5, respectively. This difference in behaviour seems to be due to the absence of a Wakamatsu-type lemma in the theory of co-t-structures which means that the existence of left and right adjoints cannot be obtained in as straighforward a manner as in the theory of t-structures. This accounts for some of the loss of functoriality which occurs in the theory of co-t-structures presented in Chapter 5 and the appearance of extra conditions in the analogue to Hoshino, Kato and Miyachi's theorem; we state this analogue below: this is the main result of Chapter 5.

Theorem 1.1.3. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S corigid object of $\mathscr{T}$ satisfying some suitably nice conditions. Then the following forms a co-t-structure on $\mathscr{T}$:*

$$\begin{aligned} \mathscr{A} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}. \end{aligned}$$

Moreover, its coheart $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ is an abelian subcategory of $\mathscr{T}$, and the functor

$$\mathsf{Hom}_{\mathscr{T}}(S, -) : \mathscr{C} \to \mathsf{Mod}(\mathsf{End}(S)^{\mathsf{op}})$$

is an equivalence of categories.

Some of the suitably nice conditions which are placed on S mirror exactly those which occur in Hoshino, Kato and Miyachi's dual theorem; the remaining conditions are required because of the absence of a Wakamatsu-type lemma.

At the end of this chapter we consider an extension of the main result to more than one corigid object of $\mathscr{T}$, and consider some conjectures regarding possible relaxations of the suitably nice conditions placed on S. One of the suitably nice conditions in Theorem 1.1.3 is that the endomorphism algebra of the object S must be a division ring. We conjecture in section 5.5 that the division ring hypothesis can be relaxed to that of the endomorphism algebra being a semisimple Artinian ring. In section 5.6 we discuss some of the implications of the absence of a Wakamatsu-type lemma for co-t-structures and the way in which the theory of co-t-structures may complement and provide insight into the theory of t-structures.

The final chapter of this thesis is Chapter 6. In this chapter we discuss the concept of generalised Moore spectra in a triangulated category. This notion was first addressed in a comparable level of generality by Jørgensen in [35]. Let R be a principal ideal domain and suppose $\mathscr{T}$ is an R-linear triangulated category. Let A be a left R-module; then Jørgensen's idea was to look for the "best approximation" of A in $\mathscr{T}$, which was denoted $M(A)$ and called the "Moore spectrum" of A. This approximation depends on the choice made of the object in $\mathscr{T}$ which approximates R itself. An outline of this construction is presented in section 6.1. In [35], this construction yields the following theorem.

Theorem 1.1.4 ([35], Proposition 4.7 and Theorem 4.9)**.** *Let $\mathscr{T}$ be an R-linear triangulated category with set indexed coproducts and C be an object of $\mathscr{T}$. Under certain nice hypotheses on C, the functor*

$$\mathsf{Hom}_{\mathscr{T}}(C,-):\mathscr{M}\to\mathsf{Mod}(R)$$

has a left adjoint

$$M:\mathsf{Mod}(R)\to\mathscr{M}.$$

If M is viewed as a functor $M:\mathsf{Mod}(R)\to\mathscr{T}$ by composition with the inclusion functor $i:\mathscr{M}\hookrightarrow\mathscr{T}$, then M is an R-linear functor, it has $M(R)\cong C$ and it preserves set indexed coproducts.

The nice hypotheses placed on C are outlined in Setup 6.1.1 and the auxilliary category $\mathscr{M}$ is defined in Definition 6.1.2. This theorem can be extended to a more general setting. The subsequent result is the following.

Theorem 1.1.5. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose C is an object of $\mathscr{T}$ satisfying certain nice "tilting" hyoptheses. Write $S=\mathsf{End}(C)$. The functor*

$$\mathsf{Hom}_{\mathscr{T}}(C,-):\mathscr{M}\to\mathsf{Mod}(S^{\mathsf{op}})$$

has a left adjoint

$$M:\mathsf{Mod}(S^{\mathsf{op}})\to\mathscr{M}.$$

Moreover, the functor M is a full embedding of the module category $\mathsf{Mod}(S^{\mathsf{op}})$ *into the full subcategory* $\mathscr{M}$ *of* $\mathscr{T}$.

The nice “tilting” hypotheses placed on C are explained in Setup 6.2.3 and the auxiliary category $\mathscr{M}$, in this case, is defined in Definition 6.2.4. The proof of this theorem is a long induction argument, but it explicitly constructs the functor M describing its action on objects and using a representability trick to obtain functoriality. The starting point for this construction is just a projective resolution of a right S-module. This construction is very different in character to the one which is constructed in [35].

In the remainder of this chapter we return to Jørgensen’s construction showing how Theorems 1.1.4 and 1.1.5 are related and show that the functor M is well behaved in a “natural” way. To finish the chapter, we present an example from u-cluster categories where we recover Keller’s canonical triangle embedding ([40, Section 4, Theorem]) of the category of finitely presented right modules of a hereditary algebra to its u-cluster category for $u \geqslant 2$ using Theorem 1.1.5.

Chapter 2

Classical Homological Algebra

2.1 Abelian categories

Abelian categories appear throughout algebra, representation theory and homological algebra. The prototype example of an abelian category is the category of abelian groups, Ab, whose objects are abelian groups and whose morphisms are just group homomorphisms. More generally, module categories, which occur throughout algebra, are abelian categories.

Before we introduce the definition of an abelian category, we first define a more general notion of an additive category. The following definitions can be found in [28] and [57].

For a general category $\mathscr{C}$ we shall employ the notation $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ to denote the set of morphisms from an object X to an object Y of $\mathscr{C}$.

Definition 2.1.1. A category $\mathscr{A}$ is called *additive* if the following hold:

(i) For any pair of objects X, Y in $\mathscr{A}$ the Hom-space $\mathsf{Hom}_{\mathscr{A}}(X, Y)$ has the structure of an abelian group;

(ii) For any objects X, Y, Z in $\mathscr{A}$ the composition map

$$\mathsf{Hom}_{\mathscr{A}}(X, Y) \times \mathsf{Hom}_{\mathscr{A}}(Y, Z) \to \mathsf{Hom}_{\mathscr{A}}(X, Z)$$

is bilinear;

(iii) For any pair of objects X, Y in $\mathscr{A}$, there exists a direct sum $X \coprod Y$.

Sometimes the notation $X \oplus Y$ is used for the direct sum $X \coprod Y$.

Abelian categories are defined in terms of kernels and cokernels; we first recall these categorical notions. For a category $\mathscr{A}$ a morphism $m : X \to Y$ (resp. $e : Y \to Z$) is called a *monomorphism* (resp. a *epimorphism*) if for all morphisms $f_1, f_2 : V \to X$ (resp. $g_1, g_2 : Z \to W$) with $m \circ f_1 = m \circ f_2$ (resp. $g_1 \circ e = g_2 \circ e$) we have $f_1 = f_2$ (resp. $g_1 = g_2$). For example, in the category of modules, a morphism is a monomorphism if and only if it is injective and an epimorphism if and only if it is surjective.

Connected to the concepts of monomorphisms and epimorphisms are the categorical notions of kernel and cokernel. Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism in $\mathscr{C}$. We say that f has a *kernel* if there is a morphism $i : X' \to X$ such that $f \circ i = 0$ and for any morphism $g : Z \to X$ such that $f \circ g = 0$ there exists a unique morphism $g' : Z \to X'$ such that $i \circ g' = g$:

$$\begin{tikzcd}
 & Z \arrow[dl, "g'"'] \arrow[d, "g"] \arrow[dr, dashed, "0"] & \\
X' \arrow[r, "i"'] & X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

We often write $\ker f$ for the object X' in $\mathscr{C}$. The dual construction is called the *cokernel.*

In addition to the categorical concepts of kernel and cokernel, there are also notions of image and coimage. Let $f : X \to Y$ be a morphism in $\mathscr{C}$ and suppose that f has kernel $i : X' \to X$ and cokernel $p : Y \to Y'$. Then a cokernel (if it exists) of i is called a *coimage* of f and a kernel (if it exists) of p is called a *image* of f. We write im f and coim f for the image and coimage, respectively.

See [28], [43], [50] and [57] for details on monomorphisms, epimorphisms, kernels and cokernels and images and coimages.

Definition 2.1.2. An additive category $\mathscr{A}$ is called *abelian* if the following hold:

(i) Any morphism f in $\mathscr{A}$ has a kernel, $\ker f$, and a cokernel, coker f;

(ii) Every monomorphism in $\mathscr{A}$ is the kernel of its cokernel and every epimorphism in $\mathscr{A}$ is the cokernel of its kernel;

(iii) Every morphism is expressible as the composite of an epimorphism and a monomorphism.

Definition 2.1.3. A *short exact sequence* in an abelian category $\mathscr{A}$ is a sequence

$$0 \to K \overset{m}{\hookrightarrow} X \overset{e}{\twoheadrightarrow} C \to 0$$

in which $m : K \hookrightarrow X$ is the kernel of $e : X \twoheadrightarrow C$, and e is the cokernel of m.

Definition 2.1.4. A sequence in an abelian category $\mathscr{A}$

$$\cdots \to X^{n-1} \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1} \to \cdots$$

is said to be *exact* at X^n if $\operatorname{im} f^{n-1} = \ker f^n$. It is said to be a *long exact sequence* if it is exact at X^n for all $n \in \mathbb{Z}$.

We now highlight two important examples of abelian categories.

Example 2.1.5. Let R be a ring. Let $\mathsf{Mod}(R)$ denote the category of left R-modules, that is the category whose objects are left R-modules and whose morphisms are R-homomorphisms. It is easy to check that this is an abelian category. Note that the categorical notions of kernels, cokernels, images and coimages coincide with their familiar algebraic counterparts.

Recall that a ring R is said to be a *$\mathbb{Z}$-graded ring* if there is a family of additive subgroups of R, $\{R_n \,|\, n \in \mathbb{Z}\}$, such that $R = \oplus_{n\in\mathbb{Z}} R_n$ and $R_n R_m \subseteq R_{n+m}$ for all $n, m \in \mathbb{Z}$. For a $\mathbb{Z}$-graded ring R, a module M is said to be a *$\mathbb{Z}$-graded left R-module* if there is a family of additive subgroups of M, $\{M_n \,|\, n \in \mathbb{Z}\}$, such that $M = \oplus_{n\in\mathbb{Z}} M_n$ and $R_n M_m \subseteq M_{n+m}$ for all $n, m \in \mathbb{Z}$.

Example 2.1.6. Let $\mathsf{GrMod}(R)$ be the category of $\mathbb{Z}$-graded left R-modules, that is the category whose objects are $\mathbb{Z}$-graded left R-modules and whose morphisms are just R-homomorphisms of degree 0. Again, it is now easy to check that $\mathsf{GrMod}(R)$ is an abelian category.

2.2 Homology

The notions of a complex and homology arose from algebraic topology in the form of simplicial and singular (co)chain complexes. Later these algebraic topological methods were applied throughout mathematics, ranging from their birthplace in algebraic topology to algebra and representation theory and algebraic geometry. Complexes are the basic tools of homological algebra which we use to define homology and cohomology. The following material can be found in textbooks such as [17], [28] and [57]. We use [28] as the main source of the material in this section.

2.2.1 Complexes

The following definitions are taken from [28].

Definition 2.2.1. A *cochain complex* $C = \{C, d\}$ of objects of an abelian category $\mathscr{A}$ is a family of objects $\{C^n \,|\, n \in \mathbb{Z}\}$ of $\mathscr{A}$ and a family of morphisms $\{d^n : C^n \to C^{n+1} \,|\, n \in \mathbb{Z}\}$ such that $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}$:

$$C : \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots .$$

The morphism d is called the *differential* (or sometimes the *(co)boundary operator*) of C. One defines a *chain complex* similarly.

Let $C = \{C, d_C\}$ and $D = \{D, d_D\}$ be cochain complexes of objects of an abelian category $\mathscr{A}$. A *morphism of degree* p,$\varphi : C \to D$, for some $p \in \mathbb{Z}$, consists of a family of morphisms, $\{\varphi^n : C^n \to D^{n+p}\}_{n\in\mathbb{Z}}$, of $\mathscr{A}$. Note that we do not require the components of f to commute with the differentials in C and D.

Definition 2.2.2. Let $C = \{C, d_C\}$ and $D = \{D, d_D\}$ be two cochain complexes of objects of an abelian category $\mathscr{A}$. A *morphism of complexes* or a *(co)chain map*, $\varphi : C \to D$, is a morphism of degree 0 such that $d_D^n \circ \varphi^n = \varphi^{n+1} \circ d_C^n$ for all $n \in \mathbb{Z}$, that is, such that the following diagram commutes:

$$\begin{tikzcd}
C : \arrow[d, "\varphi"'] & \cdots \arrow[r] & C^{n-1} \arrow[r, "d_C^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & C^n \arrow[r, "d_C^n"] \arrow[d, "\varphi^n"'] & C^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"'] & \cdots \\
D : & \cdots \arrow[r] & D^{n-1} \arrow[r, "d_D^{n-1}"'] & D^n \arrow[r, "d_D^n"'] & D^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Example 2.2.3. Let R be ring and $M = \oplus_{n\in\mathbb{Z}} M^n$ be a graded left R-module. Then M is a complex of R-modules with trivial differential. The morphisms of complexes between two graded left R-modules M and N are just the homomorphisms between M and N as a graded left R-module.

Note that the category of cochain complexes of objects of an abelian category is also an abelian category.

2.2.2 Homology and cohomology

Let $\mathscr{A}$ be an abelian category and let C be a chain complex of objects of $\mathscr{A}$, that is,

$$C : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots .$$

The condition that $d_n d_{n+1} = 0$ means that $\operatorname{im} d_{n+1} \subseteq \ker d_n$ for all $n \in \mathbb{Z}$. Hence, with the complex C we can associate a graded module $H(C)$ (sometimes written $H_*(C)$)

$$H(C) = \{H_n(C)\}, \text{ where } H_n(C) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

Then $H_n(C)$ is called the n^{th}*-homology module* of C, and $H(C)$ is called the *homology* of C. The elements of C_n are often called n*-chains*, those of $\ker d_n$ the n*-cycles* and those of $\operatorname{im} d_n$ the n*-boundaries*.

Cohomology and the n^{th}*-cohomology module* are defined analogously. That is, given a cochain complex of objects of $\mathscr{A}$, we can associate a graded module $H(C)$ (sometimes written $H^*(C)$ to avoid ambiguity)

$$H(C) = \{H^n(C)\}, \text{ where } H^n(C) = \frac{\ker d^n}{\operatorname{im} d^{n-1}}.$$

The notions of n-cochain, n-cocycle and n-coboundary are defined similarly.

Taking homology and cohomology also provides a means of obtaining a long exact sequence from a short exact sequence. Given a short exact sequence $0 \to A \to B \to C \to 0$ of complexes of objects of an abelian category $\mathscr{A}$, we obtain the following long exact sequence of homology

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\omega_n} H_{n-1}(A) \longrightarrow \cdots ,$$

and of cohomology

$$\cdots \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \xrightarrow{\omega^n} H^{n+1}(A) \longrightarrow \cdots .$$

The morphisms ω_n and ω^n for $n \in \mathbb{Z}$ are called connecting homomorphisms.

2.3 Classical derived functors

In this section we show how to compute the classical derived functors by taking projective and injective resolutions and then taking homology. Again, this treatment is based on [28]. Throughout this section $\mathscr{A}$ will be an abelian category.

2.3.1 Projective and injective objects

Projective and injective objects play a special role in abelian categories: they are objects which make the Hom functor exact.

Definitions 2.3.1. Let $\mathscr{A}$ be an abelian category.

(i) An object P of $\mathscr{A}$ is called *projective* if for any epimorphism $e : B \to C$ and any morphism $f : P \to C$ there exists a morphism $g : P \to B$ such that $e \circ g = f$, that is, such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\ B \arrow[r, two heads, "e"'] & C. \end{tikzcd}$$

(ii) An object I of $\mathscr{A}$ is called *injective* if for any monomorphism $m : A \to B$ and any morphism $f' : A \to I$ there exists a morphism $g' : B \to I$ such that $g' \circ m = f'$, that is, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "m"] \arrow[d, "f'"'] & B \arrow[dl, dashed, "g'"] \\ I. & \end{tikzcd}$$

A object P is projective if and only if given any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ the induced sequence is also exact:

$$0 \to \mathsf{Hom}_{\mathscr{A}}(P, X) \to \mathsf{Hom}_{\mathscr{A}}(P, Y) \to \mathsf{Hom}_{\mathscr{A}}(P, Z) \to 0.$$

Thus, P is projective if and only if the functor $\mathsf{Hom}_{\mathscr{A}}(P, -) : \mathscr{A} \to \mathsf{Ab}$ is *exact*, that is, the sequence induced by the application of the functor on a short exact sequence is also a short exact sequence. Analogously, I is in injective if and only if $\mathsf{Hom}_{\mathscr{A}}(-, I) : \mathscr{A} \to \mathsf{Ab}$ is exact.

2.3.2 Projective and injective presentations and resolutions

Suppose A is an object of $\mathscr{A}$. Then a *projective presentation* of A is a short exact sequence $0 \to R \to P \to A \to 0$, where P is a projective object of $\mathscr{A}$. Similarly, an *injective presentation* of A is a short exact sequence $0 \to A \to I \to E \to 0$, where I is an injective object of $\mathscr{A}$. An abelian category $\mathscr{A}$ is said to have *enough projectives* if every object of $\mathscr{A}$ has a projective presentation. The definition of *enough injectives* is analogous.

Recall, a chain complex of objects of $\mathscr{A}$

$$C: \cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

is called *acyclic* if $H_n(C) = 0$ for all $n \geq 1$. It is called *projective* if C_n is a projective object of $\mathscr{A}$ for all $n \geq 0$. Similarly, a cochain complex of objects of $\mathscr{A}$

$$C: 0 \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to \cdots$$

is called *acyclic* if $H^n(C) = 0$ for all $n \geq 1$ and *injective* if C^n is an injective object of $\mathscr{A}$ for all $n \geq 0$.

Definitions 2.3.2. Let A be an object of $\mathscr{A}$.

(i) A *projective resolution* of A consists of a projective and acyclic complex

$$P: \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0,$$

together with an isomorphism $H_0(P) \xrightarrow{\sim} A$.

(ii) A *injective resolution* of A consists of an injective and acyclic complex

$$I: 0 \to I^0 \to I^1 \to \cdots \to I^{n-1} \to I^n \to \cdots,$$

together with an isomorphism $A \xrightarrow{\sim} H^0(I)$.

Projective resolutions can be obtained from projective presentations: given an object A of $\mathscr{A}$, if $\mathscr{A}$ has enough projectives then there is a projective presentation

$$0 \to R_0 \hookrightarrow P_0 \twoheadrightarrow A \to 0.$$

We can then take a projective presentation of R_0:

$$0 \to R_1 \hookrightarrow P_1 \twoheadrightarrow R_0 \to 0,$$

obtaining a sequence,

$$0 \to R_1 \hookrightarrow P_1 \to P_0 \twoheadrightarrow A \to 0.$$

Continuing in this manner, by taking a projective presentation of R_1 and so on, we obtain a (possibly infinite) projective resolution of A. As such, if an abelian category has enough projectives then we can always obtain a projective resolution of any object. Similarly for injectives; see [28].

2.3.3 Projective and global dimensions

Projective and injective resolutions are used to define certain homological dimensions of objects in abelian categories, especially module categories. We give two of these definitions here: projective dimension and global dimension. Injective and flat dimensions are defined similarly. The following definitions are taken from [57].

Definition 2.3.3. Let R be a ring and $\mathsf{Mod}(R)$ be the category of left R-modules. Let $M \in \mathsf{Mod}(R)$; the *projective dimension* of M, written $\operatorname{projdim}_R M$, is the minimum integer n such that there is a projective resolution of M,

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow M \to 0.$$

If no finite resolution exists we write $\operatorname{projdim}_R M = \infty$.

Definition 2.3.4. Let R be a ring. The *left global dimension* of R is defined as

$$\operatorname{l.gldim} R := \sup\{\operatorname{projdim}_R M \mid M \in \mathsf{Mod}(R)\}.$$

The *right global dimension*, $\operatorname{r.gldim} R$, is defined similarly.

Note that the left and right global dimensions of a ring R are not necessary equal. When there is no ambiguity we will simply write $\operatorname{gldim} R$ and say *global dimension* of R when we mean left global dimension.

2.3.4 Derived functors

Let $\mathscr{A}$ be an abelian category with enough projectives and suppose $F : \mathscr{A} \to \mathsf{Ab}$ is an additive covariant functor. We aim to define a family of functors $L_nF : \mathscr{A} \to \mathsf{Ab}$, the so-called *left derived functors* of F. Let A be an object of $\mathscr{A}$, construct a projective resolution,

$$P: \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0,$$

of A. We define an abelian group $L_n^P F(A)$ as follows. Apply the functor F to the complex P to obtain the complex

$$FP: \cdots \to FP_n \to FP_{n-1} \to \cdots \to FP_1 \to FP_0 \to 0.$$

Define $L_n^P F(A) := H_n(FP)$.

It can be shown that $L_n^P(A)$ is independent of the projective resolution chosen and hence only depends on A. Moreover, given a morphism $\alpha : A \to A'$, one can define an induced map $\alpha_* : L_n^P F(A) \to L_n^{P'} F(A')$ which makes $L_n^P F(-)$ into a functor. We thus obtain the (classical) left derived functors $L_n F$ of F. Note that when F is a right exact functor, the functors $L_0 F$ and F are naturally equivalent.

The left derived functors occur in two important long exact sequences which are summarised in the following theorems.

Theorem 2.3.5 ([28], Theorem IV.6.1)**.** *Suppose $\mathscr{A}$ is an abelian category with enough projectives and $F : \mathscr{A} \to \mathsf{Ab}$ is an additive functor. Suppose $0 \longrightarrow A' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A'' \longrightarrow 0$ is a short exact sequence in $\mathscr{A}$, then there are connecting morphisms*

$$\omega_n : L_n F(A'') \to L_{n-1} F(A') \text{ for all } n \in \mathbb{N}$$

such that the following sequence is exact

$$\begin{aligned} \cdots \longrightarrow\ & L_n F(A') \xrightarrow{\alpha'_*} L_n F(A) \xrightarrow{\alpha_*} L_n F(A'') \xrightarrow{\omega_n} L_{n-1} F(A') \longrightarrow \cdots \\ \cdots \longrightarrow\ & L_1 F(A'') \xrightarrow{\omega_1} L_0 F(A') \xrightarrow{\alpha'_*} L_0 F(A) \xrightarrow{\alpha_*} L_0 F(A'') \longrightarrow 0. \end{aligned}$$

Recall from [28] that a sequence $F' \xrightarrow{\theta'} F \xrightarrow{\theta} F''$ of additive functors $\mathscr{A} \to \mathsf{Ab}$ and natural transformations θ', θ is called *exact on projectives* if, for any projective object P of $\mathscr{A}$, the sequence

$$0 \to F'P \xrightarrow{\theta'_P} FP \xrightarrow{\theta_P} F''P \to 0$$

is exact.

Theorem 2.3.6 ([28], Theorem IV.6.3)**.** *Suppose $\mathscr{A}$ is an abelian category with enough projectives and $F' \xrightarrow{\theta'} F \xrightarrow{\theta} F''$ is a sequence of additive functors $\mathscr{A} \to \mathsf{Ab}$ which is exact on projectives. Then, for any object A in $\mathscr{A}$, there are connecting morphisms*

$$\omega_n : L_nF''(A) \to L_{n-1}F'(A) \text{ for all } n \in \mathbb{N}$$

such that the following sequence is exact

$$\begin{array}{lllllllll} \cdots & \longrightarrow & L_nF'(A) \xrightarrow{\theta'} L_nF(A) \xrightarrow{\theta} L_nF''(A) \xrightarrow{\omega_n} L_{n-1}F'(A) \longrightarrow \cdots \\ \cdots & \longrightarrow & L_1F''(A) \xrightarrow{\omega_1} L_0F'(A) \xrightarrow{\theta'} L_0F(A) \xrightarrow{\theta} L_0F''(A) \longrightarrow 0. \end{array}$$

One can obtain the *right derived functors* $R^nG(-)$ of an additive contravariant functor $G : \mathscr{A} \to \mathsf{Ab}$ analogously. There are long exact sequences of right derived functors analogous to those obtained for left derived functors in Theorems 2.3.5 and 2.3.6; we shall see two important examples in the next section. We refer the reader to [28, Chapter IV] for details.

2.3.5 The functors $\mathsf{Ext}^i_R(-,-)$ and $\mathsf{Tor}^R_i(-,-)$

The main examples of classical derived functors are $\mathsf{Ext}(-,-)$ and $\mathsf{Tor}(-,-)$. Let R be a ring and $\mathsf{Mod}(R)$ be the category of left R-modules. Let B be a left R-module, the functor $\mathsf{Hom}_R(-,B)$ is an additive functor, and as such we can define its right derived functors.

Definition 2.3.7. Define $\mathsf{Ext}^n_R(-,B) = R^n\mathsf{Hom}_R(-,B)$, the right derived functor of $\mathsf{Hom}_R(-,B)$.

Similarly, one can obtain the right derived functors $\mathsf{Ext}^n_R(A,-)$ of the additive functor $\mathsf{Hom}_R(A,-)$ for a left R-module A by using an injective resolution. We thus obtain a bifunctor $\mathsf{Ext}^n_R(-,-)$ which can be computed by taking a projective resolution of the object in the first variable or an injective resolution of the object in the second variable; see [28, Proposition IV.7.3]. In general, the right derived functors of a covariant functor are computed using an injective resolution while the right derived functors of a contravariant functor are computed using projective resolutions.

From the right derived functor analogue of Theorem 2.3.5 we obtain the so called *long exact Ext-sequence in the first variable*: that is, given a short exact sequence of left R-modules $0 \to A' \to A \to A'' \to 0$ and a left R-module B, there is a long exact sequence:

$$\begin{aligned} &0 \to \mathsf{Hom}_R(A'', B) \to \mathsf{Hom}_R(A, B) \to \mathsf{Hom}_R(A', B) \to \mathsf{Ext}^1_R(A'', B) \to \cdots \\ &\cdots \to \mathsf{Ext}^n_R(A'', B) \to \mathsf{Ext}^n_R(A, B) \to \mathsf{Ext}^n_R(A', B) \to \mathsf{Ext}^{n+1}_R(A'', B) \to \cdots . \end{aligned} \tag{2.1}$$

Similarly, one can obtain the *long exact Ext-sequence in the second variable*: that is, given a left R-module A and a short exact sequence of left R-modules $0 \to B' \to B \to B'' \to 0$, there is a long exact sequence:

$$\begin{aligned} &0 \to \mathsf{Hom}_R(A, B') \to \mathsf{Hom}_R(A, B) \to \mathsf{Hom}_R(A, B'') \to \mathsf{Ext}^1_R(A, B') \to \cdots \\ &\cdots \to \mathsf{Ext}^n_R(A, B') \to \mathsf{Ext}^n_R(A, B) \to \mathsf{Ext}^n_R(A, B'') \to \mathsf{Ext}^{n+1}_R(A, B') \to \cdots . \end{aligned} \tag{2.2}$$

Let A be a right R-module and B be a left R-module. The tensor products $A \otimes_R -$ and $- \otimes_R B$ give additive functors $\mathsf{Mod}(R) \to \mathsf{Ab}$ and $\mathsf{Mod}(R^{\mathsf{op}}) \to \mathsf{Ab}$, respectively. As such we can define the left derived functors:

Definition 2.3.8. Define $\mathsf{Tor}^R_n(A, -) = L_n(A \otimes_R -)$, and $\mathsf{Tor}^R_n(-, B) = L_n(- \otimes_R B)$.

We thus get a bifunctor $\mathsf{Tor}^R_n(-, -)$ which can be computed by taking a projective (or even flat) resolution in either variable; see [28, Proposition IV.11.1].

As in the case of $\mathsf{Ext}^R(-, -)$, Theorem 2.3.5 yields long exact sequences of the left derived functors of $- \otimes_R -$. Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of right R-modules and B be a left R-module, then the long exact sequence

$$\begin{aligned} &\cdots \to \mathsf{Tor}^R_n(A', B) \to \mathsf{Tor}^R_n(A, B) \to \mathsf{Tor}^R_n(A'', B) \to \mathsf{Tor}^R_{n-1}(A', B) \to \cdots \\ &\qquad \cdots \to \mathsf{Tor}^R_1(A'', B) \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to 0. \end{aligned} \tag{2.3}$$

is called the *long exact Tor-sequence in the first variable.*

Similarly, given a right R-module A and a short exact sequence $0 \to B' \to B \to B'' \to 0$ of left R-modules, then one obtains the *long exact Tor-sequence in the second variable*:

$$\begin{aligned} &\cdots \to \mathsf{Tor}^R_n(A, B') \to \mathsf{Tor}^R_n(A, B) \to \mathsf{Tor}^R_n(A, B'') \to \mathsf{Tor}^R_{n-1}(A, B') \to \cdots \\ &\qquad \cdots \to \mathsf{Tor}^R_1(A, B'') \to A \otimes_R B' \to A \otimes_R B \to A \otimes_R B'' \to 0. \end{aligned} \tag{2.4}$$

Detailed treatments of the classical derived functors of $\mathsf{Hom}_R(-,-)$ and $- \otimes_R -$ and their long exact sequences can be found in [28, Chapter IV.7] and [28, Chapter IV.11], respectively.

Chapter 3

Triangulated and Derived Categories and Differential Graded Algebras

3.1 Triangulated categories

Introductions to the theory of triangulated categories can be found in Gelfand and Manin's book [25], Hartshorne's book [27, Chapter I], Krause's notes [41] and Neeman's book [46]. We shall mainly follow the expositions given in [27] and [46].

Let $\mathscr{T}$ be an additive category which is equipped with an automorphism $\Sigma : \mathscr{T} \to \mathscr{T}$. Let X, Y, Z be objects in $\mathscr{T}$, a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \tag{3.1}$$

such that the composites $g \circ f$, $h \circ g$ and $\Sigma f \circ h$ are zero is called a *triangle* in $\mathscr{T}$. A *morphism of triangles* is a commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "h"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X',
\end{tikzcd}$$

in which each row is a triangle. A morphism of triangles is called an *isomorphism of triangles* if each of the morphisms α, β, γ are isomorphisms in the category $\mathscr{T}$. If we have an isomorphism of triangles, the two triangles are said to be *isomorphic*.

Definition 3.1.1. A *triangulated category* is an additive category $\mathscr{T}$ which is equipped with an automorphism $\Sigma : \mathscr{T} \to \mathscr{T}$, which is called the *suspension functor*, and a class of triangles, Δ, called *distinguished triangles*, which satisfy the following axioms.

(TR0) Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Given any object X of $\mathscr{T}$ the triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

is a distinguished triangle.

(TR1) For any morphism $f : X \to Y$ in $\mathscr{T}$ there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

(TR2) The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a distinguished triangle if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is a distinguished triangle.

(TR3) Given two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ in $\mathscr{T}$, and a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "h"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

then there exists a morphism $\gamma : Z \to Z'$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "h"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dashed, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'.
\end{tikzcd}$$

Note that $\gamma : Z \to Z'$ may not be unique.

(TR4) (The Octahedral Axiom) Consider three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{i} Z' \xrightarrow{i'} \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{j} X' \xrightarrow{j'} \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{k} Y' \xrightarrow{k'} \Sigma X$$

Then there exist morphisms $u : Z' \to Y'$ and $v : Y' \to X'$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "g\circ f"'] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"'] \\
Y \arrow[r, "g"'] \arrow[d, "i"'] & Z \arrow[r, "j"'] \arrow[d, "k"'] & X' \arrow[r, "j'"'] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma i"'] \\
Z' \arrow[r, "u"'] \arrow[d, "i'"'] & Y' \arrow[r, "v"'] \arrow[d, "k'"'] & X' \arrow[r, "\Sigma i\circ j'"'] \arrow[d] & \Sigma Z' \arrow[d, "\Sigma i'"'] \\
\Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0 \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

and each of its rows and columns are distinguished triangles.

The suspension functor is sometimes referred to as the *shift functor* or *translation functor*. Often in the literature, in the first case, the n^{th}-power of the suspension functor, Σ^n for $n \in \mathbb{Z}$, is denoted by $[n]$, and in the second case by T^n. In this thesis we shall only use the Σ notation for the suspension functor.

Definition 3.1.2. Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories with suspension functors Σ and Σ', respectively. Suppose $F : \mathscr{T} \to \mathscr{T}'$ is an additive functor. Then F is called a *triangulated functor* if there is a natural equivalence $\varphi : F\Sigma \to \Sigma' F$. As a result if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle in $\mathscr{T}$ then the following diagram is commutative

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, equal] & FY \arrow[r, "Fg"] \arrow[d, equal] & FZ \arrow[r, "Fh"] \arrow[d, equal] & F\Sigma X \arrow[d, "\varphi_X"] \\
FX \arrow[r, "Ff"'] & FY \arrow[r, "Fg"'] & FZ \arrow[r, "\varphi_X \circ Fh"'] & \Sigma' FX
\end{tikzcd}$$

where both rows are distinguished triangles in $\mathscr{T}'$ and are isomorphic.

A triangulated functor is sometimes called a *(covariant) ∂-functor* or a *triangle functor*.

Definition 3.1.3. Let $\mathscr{T}$ be a triangulated category and $\mathscr{A}$ be an abelian category. An additive functor $H : \mathscr{T} \to \mathscr{A}$ is called a *(covariant) cohomological functor* if given an distinguished triangle,

$$X \to Y \to Z \to \Sigma X,$$

H induces a long exact sequence

$$\cdots \to H(\Sigma^i X) \to H(\Sigma^i Y) \to H(\Sigma^i Z) \to H(\Sigma^{i+1} X) \to \cdots .$$

If H is a cohomological functor, we write $H^i(X) = H(\Sigma^i X)$ for $i \in \mathbb{Z}$.

Similarly, H is called a *contravariant cohomological functor* if it induces a long exact sequence

$$\cdots \to H(\Sigma^i Z) \to H(\Sigma^i Y) \to H(\Sigma^i X) \to H(\Sigma^{i-1} Z) \to \cdots .$$

The following theorem is well known.

Theorem 3.1.4. *Let $\mathscr{T}$ be a triangulated category, and suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished triangle and W is an arbitrary object of $\mathscr{T}$. Then*

(i) There is a long exact sequence

$$\mathsf{Hom}_{\mathscr{T}}(W, X) \xrightarrow{f_*} \mathsf{Hom}_{\mathscr{T}}(W, Y) \xrightarrow{g_*} \mathsf{Hom}_{\mathscr{T}}(W, Z) \xrightarrow{h_*} \mathsf{Hom}_{\mathscr{T}}(W, \Sigma X),$$

where f_* denotes the induced morphism $\mathsf{Hom}_{\mathscr{T}}(W, f)$.

(ii) There is a long exact sequence

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma X, W) \xrightarrow{h^*} \mathsf{Hom}_{\mathscr{T}}(Z, W) \xrightarrow{g^*} \mathsf{Hom}_{\mathscr{T}}(Y, W) \xrightarrow{f^*} \mathsf{Hom}_{\mathscr{T}}(X, W),$$

where f^* denotes the induced morphism $\mathsf{Hom}_{\mathscr{T}}(f, W)$.

Part (i) of Theorem 3.1.4 just says that the functor $\mathsf{Hom}_{\mathscr{T}}(W, -)$ is a covariant cohomological functor, and part (ii) says that the functor $\mathsf{Hom}_{\mathscr{T}}(-, W)$ is a contravariant cohomological functor.

Corollary 3.1.5 (The Five Lemma for triangulated categories)**.** *Let $\mathscr{T}$ be a triangulated category. Given a commutative diagram of distinguished triangles:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "h"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X',
\end{tikzcd}$$

where $\alpha : X \to X'$ and $\beta : Y \to Y'$ are isomorphisms, then $\gamma : Z \to Z'$ is an isomorphism.

Proof: Apply Theorem 3.1.4 and the usual Five Lemma. □

Note that the isomorphism $\gamma : Z \to Z'$ obtained from (TR3) is not necessarily unique. Hence the distinguished triangle starting with $f : X \to Y$ is unique up to a non-canonical isomorphism. A corollary of Corollary 3.1.5 is that distinguished triangles recognise isomorphisms:

Corollary 3.1.6. *Let $\mathscr{T}$ be a triangulated category and suppose that $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$ is a distinguished triangle in $\mathscr{T}$. The morphism $f : X \to Y$ is an isomorphism if and only if $Z \cong 0$.*

3.2 Derived categories

For any abelian category $\mathscr{A}$ there is a category $\mathscr{D}(\mathscr{A})$, called the *derived category* of $\mathscr{A}$, whose objects are complexes of objects. We shall outline its construction following the treatments in [10], [25] and [27].

3.2.1 The homotopy category

The first step on the road to the derived category of $\mathscr{A}$ is the construction of the homotopy category of $\mathscr{A}$; for this we need to define the notion of homotopy of complexes.

Definition 3.2.1. Let $f, g : X \to Y$ be morphisms of complexes of objects of $\mathscr{A}$. Then f and g are said to be *homotopic* if there is a collection of maps $h = (h^n)$, $h^n : X^n \to Y^{n-1}$ (which do not necessarily commute with the differentials in X and Y) such that

$$f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$. The collection h is called a *homotopy*. A morphism of complexes f is called *null homotopic* if it is homotopic to the zero map.

It is easy to see that homotopy is an equivalence relation on morphisms of complexes and that homotopy respects composition, that is: if f and f' are homotopic, g and g' are homotopic and the composites $f \circ g$ and $f' \circ g'$ exist, then $f \circ g$ is homotopic to $f' \circ g'$. In light of this, we define a new category $\mathscr{K}(\mathscr{A})$, called the *homotopy category of $\mathscr{A}$*, whose objects are just complexes of objects of $\mathscr{A}$ and whose morphisms are morphisms of complexes modulo homotopy.

A useful property of the homotopy category is that it is triangulated, in order to show this we must endow $\mathscr{K}(\mathscr{A})$ with a suspension functor $\Sigma : \mathscr{K}(\mathscr{A}) \to \mathscr{K}(\mathscr{A})$ and a class of distinguished triangles, Δ.

We define the suspension functor on $\mathscr{K}(\mathscr{A})$ as follows: let X be an object of $\mathscr{K}(\mathscr{A})$, then ΣX will be the complex X with $(\Sigma X)^n = X^{n+1}$ for all $n \in \mathbb{Z}$ with differential $d_{\Sigma X}$ given by $d^i_{\Sigma X} = -d^{i+1}_X$. The action of Σ on a morphism will be just to shift it one degree to the left. It is easy to verify that this defines an automorphism of $\mathscr{K}(\mathscr{A})$.

To define the class of distinguished triangles we first define a class of standard triangles which are obtained from a construction known as the mapping cone, or sometimes, simply cone.

Definition 3.2.2. Let $f : X \to Y$ be a morphism of complexes of objects of $\mathscr{A}$. The *mapping cone* of f, denoted $\mathsf{Cone}(f)$, is the complex which has $(\mathsf{Cone}(f))^n = X^{n+1} \oplus Y^n$ sitting in degree n with differential $d_{\mathsf{Cone}(f)}$ given by $d^n_{\mathsf{Cone}(f)} = (-d^{n+1}_X, d^n_Y + f^n)$.

There is a natural inclusion $Y \overset{i}{\hookrightarrow} \Sigma X \oplus Y$ and a natural projection $\Sigma X \oplus Y \xrightarrow{p} \Sigma X$ leading to the following definition.

Definition 3.2.3. Let $f : X \to Y$ be a morphism of complexes. Then a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{i} \mathsf{Cone}(f) \xrightarrow{p} \Sigma X$$

is called a *standard triangle*.

The composites $i \circ f$ and $p \circ i$ are not necessarily zero, they are, however, null homotopic, and hence equal to zero in the homotopy category $\mathscr{K}(\mathscr{A})$, hence a standard triangle is a triangle in the sense of diagram (3.1).

Definition 3.2.4. A triangle in $\mathscr{K}(\mathscr{A})$ is a *distinguished triangle* in $\mathscr{K}(\mathscr{A})$ if it is a standard triangle or isomorphic, in $\mathscr{K}(\mathscr{A})$, to a standard triangle.

Axioms (TR0)-(TR4) are verified routinely, once one has made the observation that the mapping cone of the identity map is null homotopic; see [25, Chapter IV], for instance. We state the theorem without proof.

Theorem 3.2.5. *For an abelian category $\mathscr{A}$, the homotopy category $\mathscr{K}(\mathscr{A})$ is triangulated.*

3.2.2 Quasi-isomorphisms, localisation and the derived category

The derived category of $\mathscr{A}$ is constructed by taking the homotopy category $\mathscr{K}(\mathscr{A})$ as outlined above and then formally inverting a special class of morphisms called quasi-isomorphisms. This is a process of localisation; localisation in categories is just a generalisation of the familiar concept of localisation in rings.

Definition 3.2.6. A morphism $f : X \to Y$ in $\mathscr{K}(\mathscr{A})$ is called a *quasi-isomorphism* if it induces an isomorphism in homology, that is if $H(f) : H(X) \to H(Y)$ is an isomorphism.

Note that for $f : X \to Y$ in $\mathscr{K}(\mathscr{A})$, $H(f)$ is well-defined since if f and f' are homotopic then $H(f) = H(f')$.

Definition 3.2.7. Let $\mathscr{C}$ be a category. A collection S of morphisms is called a *multiplicative system* if it satisfies the following axioms:

(MS1) If $f, g \in S$ and $f \circ g$ exists, then $f, g \in S$. For any object X of $\mathscr{C}$ the identity map $\mathrm{id}_X \in S$;

(MS2) Any diagram

$$\begin{tikzcd}
& Z \arrow[d, "s"] & \text{resp.} & & Z \\
X \arrow[r, "u"'] & Y & & X & Y \arrow[l, "u"] \arrow[u, "s"']
\end{tikzcd}$$

with $s \in S$ can be completed to a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "v"] \arrow[d, "t"'] & Z \arrow[d, "s"] & \text{resp.} & W & Z \arrow[l, "v"'] \\
X \arrow[r, "u"'] & Y & & X \arrow[u, "t"] & Y \arrow[l, "u"] \arrow[u, "s"']
\end{tikzcd}$$

with $t \in S$.

(MS3) If $f, g : X \to Y$ are morphisms in $\mathscr{C}$, the following conditions are equivalent:

(i) There exists an $s : Y \to Y'$ in S such that $s \circ f = s \circ g$;

(ii) There exists a $t : X' \to X$ in S such that $f \circ t = g \circ t$.

The hypotheses of (MS2) are, essentially, the right and left Øre conditions. For instance, the left hand diagram can be interpreted as representing the left fraction $s^{-1}u$, and the completion to a commutative square says that this is equal to a right fraction vt^{-1}. In the case of a triangulated category, we are required to ensure that a multiplicative system is compatible with the triangulated structure, hence the following defintion.

Definition 3.2.8. Let $\mathscr{T}$ be a triangulated category and S a multiplicative system of morphisms in $\mathscr{T}$. Then S is said to be *compatible with the triangulation* if the following two assumptions are satisifed:

(MS4) $s \in S$ if and only if $\Sigma s \in S$, where Σ is the suspension functor in $\mathscr{T}$.

(MS5) Given two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ in $\mathscr{T}$, and a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "h"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

with $\alpha, \beta \in S$ then there exists a morphism $\gamma : Z \to Z'$, also in S, making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "h"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dashed, "\gamma"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'.
\end{tikzcd}$$

Proposition 3.2.9. *Let $\mathscr{A}$ be an abelian category and $\mathscr{K}(\mathscr{A})$ be its homotopy category. Then the set of quasi-isomorphisms of $\mathscr{K}(\mathscr{A})$ is a multiplicative system which is compatible with the triangulation.*

Proof: See [27, Chapter I], Propositions 4.1 and 4.2. □

Definition 3.2.10. If $\mathscr{C}$ is a category and S is a multiplicative system of morphisms in $\mathscr{C}$, then the *localisation of $\mathscr{C}$ with respect to S* is a category $S^{-1}\mathscr{C}$, together with a functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ such that

(L1) $Q(s)$ is an isomorphism for every $s \in S$;

(L2) Any functor $F : \mathscr{C} \to \mathscr{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through Q.

Let $\mathscr{C}$ be a category and S a multiplicative system in $\mathscr{C}$. Consider the category $S^{-1}\mathscr{C}$ whose objects are just the objects of $\mathscr{C}$ but whose morphisms are defined as follows: for objects $X, Y \in S^{-1}\mathscr{C}$, $\mathsf{Hom}_{S^{-1}\mathscr{C}}(X, Y)$ is the set of equivalence classes of diagrams of the form

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd} \tag{3.2}$$

with $s \in S$; where the equivalence relation is given by

$$\begin{tikzcd}
& X' \arrow[dl, "s'"'] \arrow[dr, "f'"] & & \sim & & X'' \arrow[dl, "s''"'] \arrow[dr, "f''"] & \\
X & & Y & & X & & Y,
\end{tikzcd}$$

where $s', s'' \in S$, if and only if there exists a commutative diagram

$$\begin{tikzcd}
& & X''' \arrow[dl, "g'"'] \arrow[dr, "g''"] \arrow[ddll, bend right, "t"'] & & \\
& X' \arrow[dl, "s'"'] \arrow[drrr, "f'"] & & X'' \arrow[dlll, "s''"] \arrow[dr, "f''"] & \\
X & & & & Y
\end{tikzcd}$$

with $t \in S$, that is, such that

$$s' \circ g' = t = s'' \circ g''.$$

One can check that this is well-defined. Using this construction of the morphisms in $S^{-1}\mathscr{C}$, one can define a well-defined composition of morphisms on $S^{-1}\mathscr{C}$; see [25], [27] or [41] for details.

Remark 3.2.11. Note that there are some set-theoretic considerations when determining whether the class, $\mathsf{Hom}_{S^{-1}\mathscr{C}}(X, Y)$, of equivalence classes of diagrams of the form (3.2) is a set, and therefore whether the localised category $S^{-1}\mathscr{C}$ is indeed a category. In this case, the fact that $\mathsf{Hom}_{S^{-1}\mathscr{C}}(X, Y)$ is a set is a consequence of work by Bousfield, first appearing in [13] and [14], from which the term *Bousfield localisation* is derived. A formal exposition can be found in [46, Chapter 9]. Set-theoretic considerations of categorical localisation in connection with the derived category can also be found in [57, Chapter 10].

Definition 3.2.12. Let $\mathscr{C}$ be a category and S be a multiplicative system. Define a functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ as follows. For any object X of $\mathscr{C}$, $Q(X) = X$. For any morphism $f : X \to Y$, $Q(f)$ is given by the equivalence class of

$$\begin{tikzcd}
& X \arrow[dl, equal, "\mathrm{id}_X"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

in $S^{-1}\mathscr{C}$.

One can show that $S^{-1}\mathscr{C}$ together with $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ as defined above is the localisation of $\mathscr{C}$ with respect to S. We are now ready for the definition of the derived category of an additive or abelian category.

Definition 3.2.13. Let $\mathscr{A}$ be an abelian category and let $\mathscr{C} = \mathscr{K}(\mathscr{A})$ be its homotopy category and suppose S is the multiplicative system consisting of quasi-isomorphisms of $\mathscr{K}(\mathscr{A})$. Then the *derived category* of $\mathscr{A}$, denoted by $\mathscr{D}(\mathscr{A})$, is the category $S^{-1}\mathscr{C}$.

It can be shown that the derived category inherits the triangulated structure of the homotopy category and is thus a triangulated category itself.

3.3 Derived functors

Given abelian categories $\mathscr{A}$ and $\mathscr{B}$ and an additive functor $F : \mathscr{A} \to \mathscr{B}$. This functor extends to a triangulated functor $F : \mathscr{K}(\mathscr{A}) \to \mathscr{K}(\mathscr{B})$. In general, such an extension will not take quasi-isomorphisms in $\mathscr{K}(\mathscr{A})$ to quasi-isomorphisms in $\mathscr{K}(\mathscr{B})$, and thus would not necessarily take isomorphisms in $\mathscr{D}(\mathscr{A})$ to isomorphisms in $\mathscr{D}(\mathscr{B})$, and hence would not induce a functor $F : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$. The idea of a derived functor is to find a functor $\mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ which comes from, or is close to, F in some natural sense. In this section we present a sketch based on the exposition in [27, Chapter I]. A detailed treatment of derived functors can also be found in [25, Chapter III].

3.3.1 Right derived functors

We shall present the definition of derived functors in the case of right derived functors, following [27]. Left derived functors can be defined analogously.

Definition 3.3.1. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{K}(\mathscr{A}) \to \mathscr{K}(\mathscr{B})$ be a triangulated functor. Let Q denote the localisation functors $\mathscr{K}(\mathscr{A}) \to \mathscr{D}(\mathscr{A})$ and $\mathscr{K}(\mathscr{B}) \to \mathscr{D}(\mathscr{B})$. The *right derived functor* of F is a triangulated functor $RF : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(B)$ together with a natural transformation $\xi : Q \circ F \to RF \circ Q$ with the following universal property: given any triangulated functor $G : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ and natural transformation $\zeta : Q \circ F \to G \circ Q$, then there exists a unique natural transformation $\eta : RF \to G$ such that $\zeta = (\eta \circ Q) \circ \xi$.

It is clear that should RF exist then it is unique up to a (unique) natural isomorphism. The existence of right derived functors is proved in [27, Theorem I.5.1].

We write R^iF for $H^i(RF)$ for $i \in \mathbb{Z}$. It is well known that if F comes from a left exact functor $F : \mathscr{A} \to \mathscr{B}$ and $\mathscr{A}$ has enough injectives, then the R^iF coincide with the usual right derived functors sketched in Section 2.3.

Similarly, one can define left derived functors and prove their existence and uniqueness up to a unique natural isomorphism. Moreover, we write L_iF for $H_i(LF)$ for $i \in \mathbb{Z}$ and can show that when F comes from a right exact functor $F : \mathscr{A} \to \mathscr{B}$ and $\mathscr{A}$ has enough projectives, then the L_iF are the usual left derived functors, defined in Section 2.3.

3.3.2 The functors $\mathsf{RHom}_R(-,-)$ and $\mathsf{Ext}^i_R(-,-)$

Let $\mathscr{A}$ be an abelian category with enough projectives and $F : \mathscr{A} \to \mathsf{Ab}$ be an additive functor. Recall from section 2.3 that one can compute the classical right derived functors R^nF for $n \in \mathbb{N} \cup \{0\}$. Moreover, when F is left exact there is a natural equivalence between R^0F and F. In particular, for a ring R, the functors $\mathsf{Ext}^n_R(A,-)$ are the classical right derived functors of $\mathsf{Hom}_R(A,-)$, and $\mathsf{Ext}^0_R(A,-)$ is naturally equivalent to $\mathsf{Hom}_R(A,-)$ since $\mathsf{Hom}_R(A,-)$ is left exact. Here, we show how the classical right derived functors Ext are related to the “hyper-homological” derived functor RHom.

Definition 3.3.2. Let $\mathscr{A}$ be an abelian category and $\mathscr{D}(\mathscr{A})$ its derived category. Let X, Y

be objects of $\mathscr{D}(\mathscr{A})$, the i^{th}-*hyperext* of X and Y is defined by

$$\mathsf{Ext}^i(X,Y) = \mathsf{Hom}_{\mathscr{D}(\mathscr{A})}(X, \Sigma^i Y),$$

where Σ denotes the suspension functor in $\mathscr{D}(\mathscr{A})$.

Remark 3.3.3. Given a short exact sequence $0 \to X \to Y \to Z \to 0$ of complexes of objects of $\mathscr{A}$, there is a morphism $Z \to \Sigma X$ in $\mathscr{D}(\mathscr{A})$, where Σ is the suspension functor of $\mathscr{D}(\mathscr{A})$, which makes $X \to Y \to Z \to \Sigma X$ into a distinguished triangle; see the proof of [27, Proposition I.6.1].

Remark 3.3.4. In this case, since $\mathscr{D}(\mathscr{A})$ is a triangulated category, Theorem 3.1.4 yields the following long exact sequences

$$\cdots \to \mathsf{Ext}^i(W,X) \to \mathsf{Ext}^i(W,Y) \to \mathsf{Ext}^i(W,Z) \to \mathsf{Ext}^{i+1}(W,X) \to \cdots$$

and

$$\cdots \to \mathsf{Ext}^i(Z,W) \to \mathsf{Ext}^i(Y,W) \to \mathsf{Ext}^i(X,W) \to \mathsf{Ext}^{i+1}(Z,W) \to \cdots$$

in the first and second variables, analogous to those obtained for the classical Ext in section 2.3.5.

Definition 3.3.5. Let X and Y be complexes of objects of $\mathscr{A}$ and define a complex $\mathsf{Hom}(X,Y)$ by

$$(\mathsf{Hom}(X,Y))_p = \prod_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathscr{A}}(X_n, Y_{n+p}),$$

where $\mathsf{Hom}_{\mathscr{A}}(X_n, Y_{n+p})$ is the group of morphisms $X_n \to Y_{n+p}$ in $\mathscr{A}$, and where, for $f \in (\mathsf{Hom}(X,Y))_p$, the differential $d_p^{\mathsf{Hom}(X,Y)} f$ is given by the family

$$(d_p^{\mathsf{Hom}(X,Y)} f)_n = f_{n-1} d_n^X + (-1)^{p+1} d_{n+p}^Y f_n,$$

for $n \in \mathbb{Z}$, where f_n denotes the n^{th}-component of f.

This definition will provide examples of differential graded algebras and differential graded modules, see Examples 3.4.3 and 3.4.9.

Observation 3.3.6. Let $\mathscr{A} = \mathsf{Mod}(\mathbb{K})$ for some commutative ground ring $\mathbb{K}$. If X, Y and Z are complexes of $\mathbb{K}$-modules, then the map

$$\begin{aligned} \mathsf{Hom}(Y,Z) \times \mathsf{Hom}(X,Y) &\to \mathsf{Hom}(X,Z) \\ (g,f) &\mapsto g \circ f \end{aligned}$$

is a morphism of complexes of $\mathbb{K}$-modules. That is, we have the following Leibniz rule:

$$d^{\mathsf{Hom}(X,Z)}(g \circ f) = d^{\mathsf{Hom}(Y,Z)}(g) \circ f + (-1)^{|g|} g \circ d^{\mathsf{Hom}(X,Y)}(f),$$

where g is a morphism of degree $|g|$.

Observation 3.3.7. For the complex $\mathsf{Hom}(X, Y)$ we make two observations:

(i) The n-cycles of $\mathsf{Hom}(X, Y)$ are in one-to-one correspondence with morphisms of complexes $X \to \Sigma^n Y$.

(ii) The n-boundaries correspond to the null homotopic morphisms;

that is,

$$H^n(\mathsf{Hom}(X, Y)) = \mathsf{Hom}_{\mathscr{K}(\mathscr{A})}(X, Y),$$

and $\mathsf{Hom}(-, -)$ defines a bi-triangulated functor $\mathscr{K}(\mathscr{A})^{\mathsf{op}} \times \mathscr{K}(\mathscr{A}) \to \mathscr{K}(\mathsf{Ab})$.

If $\mathscr{A}$ has enough injectives then one can calculate its derived functors in the classical sense. It can be shown that if $\mathscr{A}$ has enough injectives and enough projectives, the right derived functor of $\mathsf{Hom}(-, -)$ exists:

$$\mathsf{RHom}(-, -) : \mathscr{D}(\mathscr{A})^{\mathsf{op}} \times \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathsf{Ab}).$$

One can then obtain the following useful theorem.

Theorem 3.3.8 (Yoneda). *Let $\mathscr{A}$ be an abelian category which has enough injectives. Then for any X, Y in $\mathscr{D}(\mathscr{A})$ we have*

$$H^i(\mathsf{RHom}(X, Y)) = \mathsf{Ext}^i(X, Y).$$

Moreover, for any X, Y in $\mathscr{A}$, the $\mathsf{Ext}^i(X, Y)$ defined above is the usual $\mathsf{Ext}^i_{\mathscr{A}}(X, Y)$.

Proof: [27, Chapter I], Theorem 6.4 and Corollary 6.5. □

3.4 Differential graded algebras and modules

Differential graded algebras (DGAs) arise throughout mathematics, particularly in algebra and representation theory and algebraic topology. Examples include Koszul complexes, endomorphism algebras, fibres of ring homomorphisms, singular chain and cochain algebras of topological spaces, and bar resolutions. Introductions to the theory of DGAs can be found in [1], [6], [10], [22] and [23].

3.4.1 Differential graded algebras

Definition 3.4.1. A *differential graded algebra*, R, over the commutative ground ring $\mathbb{K}$ is a graded algebra, $R = \bigoplus_{i\in\mathbb{Z}} R_i$, over $\mathbb{K}$ together with a differential, that is a $\mathbb{K}$-linear map $d^R : R \to R$ of degree -1 with $d^2 = 0$, satisfying the Leibniz rule

$$d^R(rs) = d^R(r)s + (-1)^{|r|} r d^R(s)$$

where $r, s \in R$ and r is a graded element of degree $|r|$.

Note that a DGA R can be naturally thought of as a complex

$$\cdots \longrightarrow R_{n+1} \xrightarrow{d^R_{n+1}} R_n \xrightarrow{d^R_n} R_{n-1} \longrightarrow \cdots .$$

This fact makes the setting of DGAs the ideal place for the study of homological algebra, because the objects are inherently complexes.

As with rings there is a notion of homomorphism of DGAs. Let R and S be DGAs. A *morphism of DGAs* is a chain map $\varphi : R \to S$ satisfying the conditions $\varphi(r_1r_2) = \varphi(r_1)\varphi(r_2)$ and $\varphi(1_R) = \varphi(1_S)$.

We denote by R^{op} the *opposite differential graded algebra* of R, that is, R^{op} consists of the same underlying complex, but the multiplication is given by $r \cdot s = (-1)^{|r||s|} sr$ for $r, s \in R$, where concatenation denotes the original multiplication in R. The notion of the opposite DGA of R will be useful in the next section in characterising differential graded right modules as differential graded left modules.

In the remainder of this section we shall consider some examples of DGAs.

Examples 3.4.2. The following are trivial examples of DGAs.

(1) Let R be an algebra over a commutative ground ring $\mathbb{K}$, then R can be considered to be a DGA concentrated in degree zero:

$$R : \cdots \to 0 \to 0 \to R \to 0 \to 0 \to \cdots .$$

(2) Similarly, a graded $\mathbb{K}$-algebra $R = \bigoplus_{i\in\mathbb{Z}} R_i$ can be considered as a DGA with trivial differential:

$$R : \cdots \longrightarrow R_{n+1} \xrightarrow{0} R_n \xrightarrow{0} R_{n-1} \longrightarrow \cdots .$$

(3) Let k be a field. Then consider the complex

$$R: \cdots \to 0 \to k \to 0 \to \cdots \to 0 \to k \to 0 \to \cdots$$

with copies of k in degree 0 and in degree $d \geq 1$. Then R is a DGA with trivial differential.

(4) Given a DGA R, the homology of R is a graded algebra, and thus, like above, it can be considered to be a DGA with trivial differential.

Example 3.4.3. Let R be a $\mathbb{K}$-algebra for a commutative ground ring $\mathbb{K}$ and let M be a complex of left R-modules:

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}^M} M_n \xrightarrow{d_n^M} M_{n-1} \longrightarrow \cdots .$$

Let $\mathscr{E}$ be the endomorphism algebra $\mathsf{Hom}_R(M,M)$ which is given by

$$\mathscr{E}_p := (\mathsf{Hom}_R(M,M))_p = \prod_{n\in\mathbb{Z}} \mathsf{Hom}_R(M_n, M_{n+p}),$$

where $\mathsf{Hom}_R(M_n, M_{n+p})$ is the group of R-module homomorphisms $M_n \to M_{n+p}$. Equip $\mathscr{E}$ with the following differential to make it into a DGA,

$$d_p^{\mathscr{E}} = \prod_{n\in\mathbb{Z}} (d_n^M + (-1)^{p+1} d_{n+p}^M).$$

Compare with Definition 3.3.5. The Leibniz rule follows from Observation 3.3.6.

3.4.2 Differential graded modules

Definition 3.4.4. A *differential graded left R-module (DG left R-module)*, M, is a graded left module, $M = \bigoplus_{i\in\mathbb{Z}} M_i$, over R (viewed as a graded algebra) together with a differential, that is a $\mathbb{K}$-linear map $d^M : M \to M$ of degree -1 with $d^2 = 0$, satisfying the Leibniz rule

$$d^M(rm) = d^R(r)m + (-1)^{|r|} r d^M(m)$$

where $m \in M$ and $r \in R$ is a graded element of degree $|r|$. *DG right R-modules* are defined similarly.

Like a DGA R, a DG left R-module is also inherently a complex. There is a notion of a morphism of DG modules analogous to that of a chain map or a homomorphism of modules.

Definition 3.4.5. Let R be a DGA and M and N be DG R-modules. A *morphism of DG R-modules* is a chain map $\varphi : M \to N$ such that $\varphi(rm) = r\varphi(m)$.

Note that DG right R-modules can be canonically identified with DG left R^{op}-modules via the following action: let M be a DG right R-module, then M becomes a DG left R^{op}-module via $r \cdot m = (-1)^{|r||m|}mr$ for $r \in R$ and $m \in M$. We denote by $\mathsf{DGMod}(R)$ the *category of DG left R-modules*, that is the category whose objects are DG left R-modules and whose morphisms are just morphisms of DG R-modules. Similarly, $\mathsf{DGMod}(R^{\mathsf{op}})$ denotes the *category of DG right R-modules*. From now on, when we say "M is a DG R-module" we will mean that M is a DG left R-module. Similarly, "M is a DG R^{op}-module" will mean that M is a DG right R-module.

Definition 3.4.6. Let R and S be DGAs, M is said to be a *DG R-S-bimodule* if it is a DG R-module and a DG S^{op}-module, with the R and S^{op} structures compatible, that is, $r(ms) = (rm)s$ for all $r \in R$, $s \in S$ and $m \in M$.

We will often denote a DG R-module M by ${}_RM$ and a DG R^{op}-module N by N_R to emphasise, when necessary, the left and right R-structures. Similarly, a DG R-S-bimodule M will be denoted by ${}_RM_S$.

We next consider some examples of DG modules.

Examples 3.4.7. The following examples of DG modules correspond to the trivial examples of DGAs given in Examples 3.4.2.

(1) Let R be the DGA given by a $\mathbb{K}$-algebra concentrated in degree zero. Then a DG R-module is just a complex of R-modules.

(2) Let R be the DGA given by the graded $\mathbb{K}$-algebra $R = \bigoplus_{i\in\mathbb{Z}} R_i$ with trivial differential. Then a DG R-module M consists of a graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ with a differential d^M satisfying the property $d^M(rm) = (-1)^{|r|}rd^M(m)$.

(3) Let k be a field and consider the DGA R consisting of copies of k in degrees zero and d, for $d \geq 1$, with trivial differential. Let kx denote the copy of k sitting in degree d, then $R \cong k \oplus kx$. Then a DG R-module M is a complex of k-vector spaces such that $xM_i \subseteq M_{i+d}$ and $d^M(xm) = (-1)^d xd^M(m)$.

The next example is a useful observation.

Example 3.4.8. Let R be a DGA and M and N be DG R-modules. Suppose $f : M \to N$ is a morphism of DG R-modules. Then $\ker f$, coker f, im f and coim f are also DG R-modules.

We now consider an example generalising Example 3.4.3.

Example 3.4.9. Let M and N be complexes of $\mathbb{K}$-modules for the commutative ground ring $\mathbb{K}$. As in Example 3.4.3, $\mathsf{Hom}(M, N)$ is a complex of $\mathbb{K}$-modules with

$$(\mathsf{Hom}(M, N))_p = \prod_{n \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(M_n, N_{n+p}),$$

and differential d given by

$$d_p = \prod_{n \in \mathbb{Z}} (d_n^M + (-1)^{p+1} d_{n+p}^N).$$

Now $\mathsf{Hom}(M, N)$ is a DG $\mathscr{E}$-module for the endomorphism DGA $\mathscr{E} = \mathsf{Hom}_{\mathbb{K}}(N, N)$, where the action is given by $\alpha\mu = \alpha \circ \mu$, where $\alpha \in \mathscr{E}$ and $\mu \in \mathsf{Hom}(M, N)$. The Leibniz rule follows from Observation 3.3.6.

3.5 The categories $\mathscr{K}(R)$ and $\mathscr{D}(R)$

Let R be a DGA. Since the objects of $\mathsf{DGMod}(R)$ are inherently complexes, there are two possible constructions of the derived category of R. One is to follow the construction outlined in section 3.2 and obtain the derived category $\mathscr{D}(\mathsf{DGMod}(R))$.

The other is to use the inherent complex structure of the objects of $\mathsf{DGMod}(R)$. Define the homotopy category of R, $\mathscr{K}(R)$, as the category which consists of the same objects as $\mathsf{DGMod}(R)$ but whose morphisms are morphisms of DG R-modules up to homotopy. Then the derived category of R, $\mathscr{D}(R)$, is defined as the category whose objects are just those of $\mathsf{DGMod}(R)$ and whose morphisms are obtained from morphisms in $\mathscr{K}(R)$ by formally inverting quasi-isomorphisms following the construction outlined in section 3.2.

Definition 3.5.1. A DG R-module M is said to be *(homologically) bounded below* (or *(homologically) bounded to the left*) if there exists $n \in \mathbb{Z}$ such that $H^i(M) = 0$ whenever $i < n$. By $\mathscr{D}^+(R)$ we denote the full subcategory of $\mathscr{D}(R)$ consisting of DG R-modules which are homologically bounded below.

Definition 3.5.2. A DG R-module M is said to be *(homologically) bounded above* (or *(homologically) bounded to the right*) if there exists $n \in \mathbb{Z}$ such that $H^i(M) = 0$ whenever $i > n$. By $\mathscr{D}^-(R)$ we denote the full subcategory of $\mathscr{D}(R)$ consisting of DG R-modules which are homologically bounded above.

Definition 3.5.3. A DG R-module M is said to be *(homologically) bounded* if M is homologically bounded below and homologically bounded above. The bounded derived category of R, denoted $\mathscr{D}^b(R)$, is the full subcategory of $\mathscr{D}(R)$ consisting of homologically bounded DG R-modules.

Definition 3.5.4. A DG R-module M is said to be *(homologically) finitely presented* if each $H_i(M)$ is finitely presented as a $H_0(R)$-module. By $\mathscr{D}^f(R)$ we denote the full subcategory of the derived category $\mathscr{D}(R)$ which consists of DG R-modules M such that M is (homologically) bounded and (homologically) finitely presented. The category $\mathscr{D}^f(R)$ shall be referred to as the *finite derived category of* R.

Versions of Definitions 3.5.1, 3.5.2, 3.5.3 and 3.5.4 exist for the homotopy category $\mathscr{K}(R)$, giving the corresponding full subcategories: $\mathscr{K}^+(R)$, $\mathscr{K}^-(R)$, $\mathscr{K}^b(R)$ and $\mathscr{K}^f(R)$.

3.5.1 The functor $\mathsf{Hom}_R(-,-)$

Let R be a DGA. The *forgetful functor*,

$$(-)^\natural : \mathsf{DGMod}(R) \to \mathsf{GrMod}(R),$$

is given by the assignment $M \mapsto M^\natural$, where $M^\natural$ is obtained from M by forgetting the differential.

We define the functor $\mathsf{Hom}_R(-,-)$ in a manner analogous to Definition 3.3.5 and Examples 3.4.3 and 3.4.9. For DG R-modules M and N, we define a complex $\mathsf{Hom}_R(M,N)$ by

$$(\mathsf{Hom}_R(M,N))^\natural := \mathsf{Hom}_{R^\natural}(M^\natural, N^\natural) = \prod_{n\in\mathbb{Z}} \mathsf{Hom}_{R^\natural}(M_n, N_{n+p})$$

together with the differential

$$d_n^{\mathsf{Hom}_R(M,N)} = \prod_{n\in\mathbb{Z}} (d_{p-1}^M + (-1)^{p+1} d_{n+p}^N).$$

It can be shown that $\mathsf{Hom}_R(-,-) : \mathsf{DGMod}(R)^{\mathsf{op}} \times \mathsf{DGMod}(R) \to \mathsf{DGMod}(\mathbb{Z})$ is a bifunctor. Moreover, it preserves homotopies and hence we obtain a bifunctor $\mathsf{Hom}_R(-,-) : \mathscr{K}(R)^{\mathsf{op}} \times \mathscr{K}(R) \to \mathscr{K}(\mathbb{Z})$; see [6] or [10].

3.5.2 The functors $- \otimes_R -$, $- \overset{\mathsf{L}}{\otimes}_R -$ and $\mathsf{Tor}_i^R(-,-)$

As with modules over rings and graded modules over graded algebras there is a notion of tensor product of DG modules; see [6] or [10].

Throughout this section R and S will be a DGAs.

Let M be a DG R^{op}-module and N a DG R-module and consider the graded algebra $R^\natural$, graded $(R^\natural)^{\mathsf{op}}$-module $M^\natural$ and graded $R^\natural$-module $N^\natural$ obtained by applying the forgetful functor defined in section 3.5.1. Then there is a graded $\mathbb{Z}$-module $M^\natural \otimes_{R^\natural} N^\natural$ with

$$(M^\natural \otimes_{R^\natural} N^\natural)_i = \coprod_{p+q=i} (M_p \otimes_{R^\natural} N_q).$$

The *tensor product*, $M \otimes_R N$, of a DG R^{op}-module M and a DG R-module N is defined by

$$(M \otimes_R N)^\natural = M^\natural \otimes_{R^\natural} N^\natural,$$

together with the differential

$$d_i^{M \otimes_R N}(m \otimes n) = d_p^M(m) \otimes n + (-1)^p m \otimes d_q^N(n),$$

where $|m| = p$, $|n| = q$ and $p + q = i$.

It can be seen that $- \otimes_R - : \mathsf{DGMod}(R^{\mathsf{op}}) \times \mathsf{DGMod}(R) \to \mathsf{DGMod}(\mathbb{Z})$ is a bifunctor. Moreover, it preserves homotopies and hence we obtain a bifunctor $- \otimes_R - : \mathscr{K}(R^{\mathsf{op}}) \times \mathscr{K}(R) \to \mathscr{K}(\mathbb{Z})$; see [6] or [10].

In the case that M is a DG S-R-bimodule, then tensor product $M \otimes_R N$ above obtains the structure of a DG S-module via the S-action on M. Similarly, in the case that N is a DG R-S-bimodule, $M \otimes_R N$ has the structure of a DG S^{op}-module.

The tensor product of DG modules has the following natural properties.

Properties 3.5.5. Let R and S be DGAs and suppose that L is a DG R^{op}-module, M is a DG R-S-bimodule and N is a DG S-module. Then we have the natural isomorphism:

$$L \otimes_R (M \otimes_S N) \;\cong\; (L \otimes_R M) \otimes_S N \text{ (associativity of tensor product).}$$

If L is a DG S-module, M a DG R-S-bimodule and N a DG R-module, then we have the following natural adjunction isomorphisms:

$$\begin{aligned}\mathsf{Hom}_S(L,\mathsf{Hom}_R(M,N)) &\cong \mathsf{Hom}_R(M\otimes_S L,N)\\ \mathsf{Hom}_{\mathscr{K}(S)}(L,\mathsf{Hom}_R(M,N)) &\cong \mathsf{Hom}_{\mathscr{K}(R)}(M\otimes_S L,N).\end{aligned}$$

That is, the bifunctor $-\otimes_S-$ is the left adjoint of the bifunctor $\mathsf{Hom}_S(-,-)$.

It is also useful to note how the bifunctors $-\otimes_R-$ and $\mathsf{Hom}_R(-,-)$ interact with the suspension functor.

Properties 3.5.6. Let R be a DGA. Let M and N be DG R-modules and L be a DG R^{op}-module. For $n\in\mathbb{Z}$ we have the following canonical isomorphisms:

$$\begin{aligned}\mathsf{Hom}_R(M,N) &\cong \mathsf{Hom}_R(\Sigma^n M,\Sigma^n N)\\ \mathsf{Hom}_R(M,\Sigma^n N) &\cong \Sigma^n\mathsf{Hom}_R(M,N)\\ \mathsf{Hom}_R(\Sigma^n M,N) &\cong \Sigma^{-n}\mathsf{Hom}_R(M,N)\\ (\Sigma^n L)\otimes_R M &\cong \Sigma^n(L\otimes_R M)\\ L\otimes_R(\Sigma^n M) &\cong \Sigma^n(L\otimes_R M),\end{aligned}$$

where, on the left hand side, Σ denotes the suspension functor in $\mathscr{K}(R)$ or $\mathscr{K}(R^{\mathsf{op}})$, depending on context, and on the right hand side Σ denotes the suspension functor in $\mathscr{K}(\mathbb{Z})$, except for the first isomorphism where it is the suspension functor in $\mathscr{K}(R)$.

The definition of left derived functors is analogous to that of right derived functors; see section 3.3.1. As such, we can compute the left derived functor $-\overset{\mathsf{L}}{\otimes}_R-:\mathscr{D}(R^{\mathsf{op}})\times\mathscr{D}(R)\to\mathscr{D}(\mathbb{Z})$ of the exact bifunctor $-\otimes_R-:\mathscr{K}(R^{\mathsf{op}})\times\mathscr{K}(R)\to\mathscr{K}(\mathbb{Z})$. The classical left derived functors $\mathsf{Tor}_i^R(-,-)$ are related to the “hyper-homological” derived functor $-\overset{\mathsf{L}}{\otimes}_R-$.

Definition 3.5.7. Let $M\in\mathscr{D}(R^{\mathsf{op}})$ and $N\in\mathscr{D}(R)$ be DG left and right R-modules, respectively. The *i^{th}-hypertor* of M and N is defined by

$$H_i(M\overset{\mathsf{L}}{\otimes}_R N)=\mathsf{Tor}_i^R(M,N).$$

It can be seen that when R is a ring and M and N consist of complexes concentrated in degree zero that this definition coincides with the usual definition of Tor. We also obtain long exact sequences analogous to those obtained for the i^{th}-hyperext in Remarks 3.3.4.

The functors $- \otimes_R -$ and $\mathsf{Hom}_R(-,-)$ can be replaced by their derived functors in Properties 3.5.5 and 3.5.6 to obtain similar isomorphisms.

3.6 K-projectivity, K-injectivity and K-flatness

In section 3.2 we introduced the derived category as the localisation of the homotopy category with respect to quasi-isomorphisms. As such, the Hom spaces in the derived category are equivalence classes of diagrams of morphisms in the homotopy category. This means that the computation of the Hom spaces of the derived category is more difficult than those of the homotopy category.

In section 3.3 we introduced "hyper-homological" derived functors, and looked at the specific example of $\mathsf{RHom}(-,-)$ in relation to the classical Ext functors. In the classical case, derived functors are computed by means of projective and injective resolutions; see section 2.3.

This gives rise to two natural questions. How does one compute the Hom spaces for the derived category? And, how are the "hyper-homological" derived functors computed? The notions of *K-projective*, *K-injective* and *K-flat* objects were introduced by Spaltenstein in [54] in order to address these questions. We note that in some sources these notions are referred to as *homotopically projective*, *homotopically injective* and *homotopically flat*; see, for instance, [6].

3.6.1 K-projective, K-injective and K-flat objects

We start with the definitions of K-projective and K-injective objects.

Definitions 3.6.1. Let $\mathscr{C}$ be a category with a multiplicative system S.

(i) An object P of $\mathscr{C}$ is called *K-projective* if for any morphism $s : X \to Y$ in S the induced map

$$\mathsf{Hom}_{\mathscr{C}}(\mathsf{id}_P, s) : \mathsf{Hom}_{\mathscr{C}}(P, X) \to \mathsf{Hom}_{\mathscr{C}}(P, Y)$$

is a bijection.

(ii) An object I of $\mathscr{C}$ is called *K-injective* if for any morphism $s : X \to Y$ in S the induced map

$$\mathsf{Hom}_{\mathscr{C}}(s, \mathrm{id}_I) : \mathsf{Hom}_{\mathscr{C}}(Y, I) \to \mathsf{Hom}_{\mathscr{C}}(X, I)$$

is a bijection.

Examples 3.6.2. Let R be a ring and let $\mathscr{C} = \mathscr{K}(R)$ be the homotopy category of left R-modules. Let S be the set of quasi-isomorphisms in $\mathscr{C}$ so that $S^{-1}\mathscr{C}$ is the derived category $\mathscr{D}(R)$ of the module category $\mathsf{Mod}(R)$.

(1) The complex

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0 \to \cdots$$

with P_i projective for all $i \geq 0$ is a K-projective object in $\mathscr{K}(R)$. Thus any projective resolution of a left R-module is a K-projective object in the homotopy category of left R-modules, and indeed, in the corresponding derived category.

(2) The complex

$$\cdots \to 0 \to 0 \to I \to 0 \to 0 \to \cdots ,$$

where I is injective, is a K-injective complex in $\mathscr{C}$.

The following useful propositions allow us to view the Hom spaces in $\mathscr{D}(\mathscr{A})$ via the Hom spaces in $\mathscr{K}(\mathscr{A})$, see [10, Lemma 10.12.2.2] or [54, section 1], for instance.

Proposition 3.6.3. *Let $\mathscr{A}$ be an abelian category, $\mathscr{K}(\mathscr{A})$ its homotopy category and $\mathscr{D}(\mathscr{A})$ is derived category and suppose $Q : \mathscr{K}(\mathscr{A}) \to \mathscr{D}(\mathscr{A})$ is the localisation functor.*

(1) The following conditions are equivalent:

(a) P is a K-projective object of $\mathscr{K}(A)$;

(b) For any object A in $\mathscr{K}(\mathscr{A})$ we have that

$$Q : \mathsf{Hom}_{\mathscr{K}(\mathscr{A})}(P, A) \to \mathsf{Hom}_{\mathscr{D}(\mathscr{A})}(P, A)$$

is a bijection.

(c) The functor $\mathsf{Hom}_{\mathscr{K}(\mathscr{A})}(P, -)$ sends quasi-isomorphisms to isomorphisms.

(2) The following conditions are equivalent:

(a) I is a K-injective object of $\mathscr{K}(A)$;

(b) For any object A in $\mathscr{K}(\mathscr{A})$ we have that

$$Q : \mathsf{Hom}_{\mathscr{K}(\mathscr{A})}(A, I) \to \mathsf{Hom}_{\mathscr{D}(\mathscr{A})}(A, I)$$

is a bijection.

(c) The functor $\mathsf{Hom}_{\mathscr{K}(\mathscr{A})}(-, I)$ sends quasi-isomorphisms to isomorphisms.

Definition 3.6.4. Let R be a DGA and $\mathscr{K}(R)$ be its homotopy category. A DG R^{op}-module F in $\mathscr{K}(R)$ is called *K-flat* if the functor $F \otimes_R -$ preserves quasi-isomorphisms. Similarly a DG R-module F' is called K-flat if the functor $- \otimes_R F'$ preserves quasi-isomorphisms.

Example 3.6.2 leads nicely into the following definition of resolutions by K-projective, K-injective and K-flat objects.

Definitions 3.6.5. Let $\mathscr{A}$ be an abelian category and $\mathscr{K}(\mathscr{A})$ its homotopy category. Let A be an object of $\mathscr{K}(\mathscr{A})$. Then

(i) A *K-projective resolution* of A consists of a K-projective object P together with a quasi-isomorphism $\pi : P \to A$.

(ii) A *K-injective resolution* of A consists of a K-injective object I together with a quasi-isomorphism $\iota : A \to I$.

Moreover, if R is a DGA, then

(iii) A *K-flat resolution* of a DG R-module M consists of a K-flat DG R-module F together with a quasi-isomorphism $\varphi : F \to M$.

Remark 3.6.6. Suppose A, B are objects of $\mathscr{D}(\mathscr{A})$ for some abelian category $\mathscr{A}$ and consider their Hom space $\mathsf{Hom}_{\mathscr{D}(\mathscr{A})}(A, B)$. If A has a K-projective resolution $\pi : P \to A$ then we have the following canonical isomorphisms:

$$\mathsf{Hom}_{\mathscr{K}(\mathscr{A})}(P, B) \xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(\mathscr{A})}(P, B) \xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(\mathscr{A})}(A, B),$$

where the first isomorphism is by Proposition 3.6.3 and the second isomorphism by the fact that π is an isomorphism in $\mathscr{D}(\mathscr{A})$. A similar sequence of canonical isomorphisms is obtained if B has a K-injective resolution. Hence, K-projective and K-injective resolutions provide a means of realising the Hom spaces of the derived category.

Definition 3.6.7. For an abelian category $\mathscr{A}$, the homotopy category $\mathscr{K}(\mathscr{A})$ has *enough K-projectives* if any object A of $\mathscr{K}(\mathscr{A})$ has a K-projective resolution. Similarly for *enough K-injectives* and *enough K-flats*.

Thus, if $\mathscr{K}(\mathscr{A})$ has enough K-projectives or enough K-injectives then Remark 3.6.6 answers the first question regarding the computation of Hom spaces.

3.6.2 K-projective resolutions for DG modules

The following sketch is based on [10, Section 10.12]. Let M be a DG R-module for a DGA R. Recall that $\mathscr{K}(\mathbb{Z})$ denotes the homotopy category of complexes of abelian groups. The following theorem is due to Spaltenstein and can be found in [54].

Theorem 3.6.8 (Spaltenstein)**.** *For any complex of abelian groups $A \in \mathscr{K}(\mathbb{Z})$, there exists a K-projective resolution $\pi : P \to A$ in $\mathscr{K}(\mathbb{Z})$. That is, the category $\mathscr{K}(\mathbb{Z})$ has enough K-projectives.*

Let $M \in \mathscr{K}(R)$ and consider M as a complex of abelian groups, that is, as an object in $\mathscr{K}(\mathbb{Z})$. By Theorem 3.6.8, there is a K-projective resolution $S_0 = S(M) \xrightarrow{\varepsilon} M$ in $\mathscr{K}(\mathbb{Z})$; one may assume that ε is surjective. Let $P_0 = R \otimes_{\mathbb{Z}} S_0$ be a DG R-module (corresponding to the natural homomorphism $\mathbb{Z} \to R$). Then there exists a natural map $\delta_0 : P_0 \to M$ given by $\delta_0(r \otimes s) = r\varepsilon(s)$. It is clear that δ_0 is surjective because ε is a quasi-isomorphism.

Now let $K = \ker \delta_0$. The exact sequence $0 \longrightarrow K \longrightarrow P_0 \xrightarrow{\delta_0} M \longrightarrow 0$ induces an exact sequence in homology $0 \to H(K) \to H(P_0) \to H(M) \to 0$. Repeating this procedure with K replacing M we obtain a complex of DG R-modules

$$\cdots \longrightarrow P_{-2} \xrightarrow{\delta_{-2}} P_{-1} \xrightarrow{\delta_{-1}} P_0 \longrightarrow 0.$$

Define a DG R-module $B(M)$ as follows. Let $B(M) = \bigoplus_{i=0}^{\infty} \Sigma^i P_{-i}$, where the differential $d : \Sigma^i P_{-i} \to \Sigma^i P_{-i} \oplus \Sigma^{i-1} P_{-i+1}$ is given by

$$d(p) = (d_{P_{-i}}(p), (-1)^{|p|}\delta_{-i}(p)).$$

There is a natural quasi-isomorphism of DG R-modules $\delta : B(M) \to M$ with $\delta|_{p_0} = \delta_0$ and $\delta|_{P_{-i}}$ for $i > 0$. $B(M)$ is called the *bar resolution* of M. We have the following theorem.

Theorem 3.6.9. *For a DG R-module M the bar resolution $B(M)$ of M is a K-projective object of $\mathscr{K}(R)$.*

Proof: See [10, Proposition 10.12.2.6]. □

Corollary 3.6.10. *$\mathscr{K}(R)$ has enough K-projectives.*

A similar construction can be obtained to show that $\mathscr{K}(R)$ has enough K-injectives.

3.6.3 Computing $\mathsf{RHom}_R(-,-)$ and $- \overset{\mathsf{L}}{\otimes}_R -$

Let R be a DGA. Recall from section 3.3.2 that we obtain the right derived functor

$$\mathsf{RHom}_R(-,-) : \mathscr{D}(R)^{\mathsf{op}} \times \mathscr{D}(R) \to \mathscr{D}(\mathbb{Z})$$

of $\mathsf{Hom}_R(-,-)$. Similarly, one obtains the left derived functor

$$- \overset{\mathsf{L}}{\otimes}_R - : \mathscr{D}(R^{\mathsf{op}}) \times \mathscr{D}(R) \to \mathscr{D}(\mathbb{Z})$$

of $- \otimes_R -$. We explain how these derived functors are computed in terms of the more concrete functors.

First note that by Corollary 3.6.10 that $\mathscr{D}(R)$ has enough K-projectives. Similar results ensure that $\mathscr{D}(R)$ has enough K-injectives and enough K-flats.

Let M and N be a DG R-modules (that is, objects of $\mathscr{D}(R)$), and consider the functors $\mathsf{RHom}_R(M,-) : \mathscr{D}(R) \to \mathscr{D}(\mathbb{Z})$ and $\mathsf{RHom}_R(-,N) : \mathscr{D}(R)^{\mathsf{op}} \to \mathscr{D}(\mathbb{Z})$. Since $\mathscr{D}(R)$ has enough K-projectives and enough K-injectives, there exists a K-projective resolution $\pi : P \to M$ of M and a K-injective resolution $\iota : N \to I$ of N in $\mathscr{D}(R)$. We then have the following canonical isomorphisms which can be used to compute $\mathsf{RHom}_R(M,-)$ and $\mathsf{RHom}_R(-,N)$:

$$\begin{aligned} \mathsf{RHom}_R(M,-) &\simeq \mathsf{RHom}_R(P,-) \simeq \mathsf{Hom}_R(P,-), \text{ and} \\ \mathsf{RHom}_R(-,N) &\simeq \mathsf{RHom}_R(-,I) \simeq \mathsf{Hom}_R(-,I). \end{aligned}$$

Similarly, let M be a DG R^{op}-module and N a DG R-module, and consider the functors $M \overset{\mathsf{L}}{\otimes}_R - : \mathscr{D}(R) \to \mathscr{D}(\mathbb{Z})$ and $- \overset{\mathsf{L}}{\otimes}_R N : \mathscr{D}(R^{\mathsf{op}}) \to \mathscr{D}(\mathbb{Z})$. Since $\mathscr{D}(R^{\mathsf{op}})$ and $\mathscr{D}(R)$ both have enough K-flats, there exist K-flat resolutions $\varphi : F \to M$ of M in $\mathscr{D}(R^{\mathsf{op}})$ and $\varphi' : F' \to N$ of N in $\mathscr{D}(R)$. As above, we have the following canonical isomorphisms:

$$\begin{aligned} M \overset{\mathsf{L}}{\otimes}_R - &\simeq F \overset{\mathsf{L}}{\otimes}_R - \simeq F \otimes_R -, \text{ and} \\ - \overset{\mathsf{L}}{\otimes}_R N &\simeq - \overset{\mathsf{L}}{\otimes}_R F' \simeq - \otimes_R F'. \end{aligned}$$

Note that since a K-projective object is also K-flat that we could have used K-projective resolutions in place of K-flat resolutions above.

Chapter 4

Homological Epimorphisms of Differential Graded Algebras

In [24], Geigle and Lenzing characterise, using the classical derived functors $\mathsf{Ext}(-,-)$ and $\mathsf{Tor}(-,-)$, when a homomorphism of rings $\varphi : R \to S$ induces a full embedding of bounded derived categories, $\mathscr{D}^b(S) \hookrightarrow \mathscr{D}^b(R)$. Geigle and Lenzing refer to such ring homomorphisms as homological epimorphisms of rings.

As we stated in Chapters 3 and 1, DGAs may be regarded as a generalisation of rings. It is, therefore, natural to ask whether the characterisation of Geigle and Lenzing also works for DGAs? It turns out that it does. Moreover, the characterisation for DGAs is a special case of a more general result regarding DG bimodules. Given two DGAs R and S and a DG R-S-bimodule M we can look at the functor

$$ {}_R M_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R) \tag{4.1}$$

and ask when this is a full embedding of derived categories. The case of homological epimorphisms of DGAs then becomes the situation when $M = S$, with S acquiring the left R-structure via a morphism of DGAs $\varphi : R \to S$. In the work of Keller, [38, Remarks 3.2], DG bimodules are regarded as generalised morphisms of DGAs. Thus, asking when ${}_R M_S \overset{\mathsf{L}}{\otimes}_S -$ is a full embedding of derived categories is analogous to asking when M is a generalised homological epimorphism of DGAs. This characterisation appears in this chapter as Theorem 4.3.6.

We now define the restriction of scalars functor, this treatment is broadly based on

[10]. Let R and S be DGAs and $\varphi : R \to S$ be a morphism of DGAs. We can equip S with either a left R-structure or a right R-structure via the morphism φ as follows. The left R-structure on S is defined by the action $r \cdot s = \varphi(r)s$. Similarly, the right R-structure on S is defined by the action $s \cdot r = s\varphi(r)$.

Let N be a DG S-module considered as an object in the homotopy category $\mathscr{K}(S)$. Then N can be made into a DG R-module, that is an object of $\mathscr{K}(R)$, via the *restriction of scalars* functor ${}_RS_S \otimes_S - : \mathscr{K}(S) \to \mathscr{K}(R)$, where the DG R-module structure on S is acquired through φ as above. The restriction of scalars functor is sometimes denoted by φ_* to highlight the fact that the R-structure on S is induced by φ. The restriction of scalars functor also induces a derived functor,

$${}_RS_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R),$$

which we shall also call the *restriction of scalars* functor. This functor is sometimes denoted by $\mathscr{D}(\varphi_*)$ (see [24], for example) to emphasise that it is induced by φ.

The definition of the restriction of scalars functor can be generalised in the following natural way. Let M be a DG R-S-bimodule, then the functor appearing in (4.1),

$${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R),$$

can be regarded as a generalisation of the restriction of scalars functor. In this chapter we shall be asking when this functor induces a full embedding of derived categories, this is the subject of section 4.3, culminating in Theorem 4.3.6. We shall also give an example of this characterisation in section 4.4 in relation to Dwyer and Greenlees' Morita theory from [19]. We specialise this characterisation to the DG R-S-bimodule ${}_RS_S$ (Theorem 4.3.9), where the R structure on S is obtained via the restriction of scalars functor, and, following [24], we define the notion of homological epimorphisms of DGAs. We specialise further to the bounded and finite derived categories, and using the characterisation for the bounded derived categories $\mathscr{D}^b(R)$ and $\mathscr{D}^b(S)$ we obtain Geigle and Lenzing's classical characterisation of homological epimorphisms of rings (Theorem 4.4.10).

In addition to asking when the restriction of scalars functor yields a full embedding of derived categories, we can also ask when the *co-induction* functor yields a full embedding of derived categories. Recall from [28] that given a morphism $\varphi : R \to S$ of DGAs, the co-induction functor is given by

$$\mathsf{Hom}_R({}_RS_S, -) : \mathscr{K}(R) \to \mathscr{K}(S),$$

where the DG R-structure on S is induced as above. As with the restriction and extensions of scalars functors, there is an induced derived functor and a generalised co-induction functor for any DG R-S-bimodule M:

$$\mathsf{RHom}_R({}_RS_S, -) : \mathscr{D}(R) \to \mathscr{D}(S),$$
$$\mathsf{RHom}_R({}_RM_S, -) : \mathscr{D}(R) \to \mathscr{D}(S).$$

As a corollary of Theorem 4.3.6 in section 4.4.3, we give a characterisation of when this generalised coinduction functor is a full embedding of derived categories.

This more general setting makes the structural reasons behind the characterisation of homological epimorphisms of rings in [24] more transparent. Indeed, the characterisation of a homological epimorphism of DGAs in terms of the restriction of scalars functor, (4.1), extends the result of Geigle and Lenzing, and the modern categorical language shows that this result is no accident. Furthermore, the "hyper-homological" framework suggests an analogous characterisation of when the coinduction functor induces a full embedding of derived categories. The results appearing in this chapter are an expansion of those presented in the paper [49].

Before proceeding with these characterisations, we present a brief review of some facts about compact objects in triangulated categories which will also be of use in subsequent chapters. Section 4.2 is a technical section which gives examples on how to define "canonical maps" and then how to perform computations with them using appropriate K-projective and K-injective resolutions; see Chapter 3, section 3.6.

4.1 Compact objects

Let R be a DGA. In this section we shall introduce two notions: firstly, that of a compact object C in an arbitrary triangulated category $\mathscr{T}$; and secondly, that of an object M in $\mathscr{D}(R)$ which is finitely built from R. It turns out that in the triangulated category $\mathscr{D}(R)$ these notions coincide.

Definition 4.1.1 ([44], Definition 1.6)**.** Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. An object C of $\mathscr{T}$ is called *compact* if the functor $\mathsf{Hom}_{\mathscr{T}}(C,-)$ commutes with set indexed coproducts; that is given a coproduct, $\coprod_{i\in I} X_i$, of objects in $\mathscr{T}$, where I is an indexing set, we have a canonical isomorphism

$$\coprod_{i\in I} \mathsf{Hom}_{\mathscr{T}}(C, X_i) \xrightarrow{\sim} \mathsf{Hom}_{\mathscr{T}}(C, \coprod_{i\in I} X_i).$$

Example 4.1.2. Let R be a DGA and $\mathscr{D}(R)$ be its derived category. Then the left DG R-module ${}_RR$ is a compact object of $\mathscr{D}(R)$.

Definition 4.1.3. Let R be a DGA. A DG R-module M is *finitely built from* ${}_RR$ in $\mathscr{D}(R)$ if M can be obtained from ${}_RR$ using finitely many distinguished triangles, suspensions, direct summands and finite coproducts (c.f. [34]).

We recall the following definition from [44].

Definition 4.1.4. Let $\mathscr{T}$ be a triangulated category with set-indexed coproducts. Then $\mathscr{T}$ is *compactly generated* if there exists a set of compact objects $\mathscr{S}$ of $\mathscr{T}$ such that $\mathsf{Hom}_{\mathscr{T}}(S, X) = 0$ for all objects S of $\mathscr{S}$ implies $X = 0$.

If R is a DGA, then its derived category $\mathscr{D}(R)$ is compactly generated, just take the left DG R-module R itself and the set of compact objects $S = \{\Sigma^i R \mid i \in \mathbb{Z}\}$. Now let $\mathscr{S}$ be the smallest full subcategory of $\mathscr{D}(R)$ containing S which is closed under finite coproducts, distinguished triangles, taking direct summands and suspension, that is $\mathscr{S}$ consists of the objects of $\mathscr{D}(R)$ which are finitely built from R. Then by Thomason's Localisation Theorem, [44, Theorem 2.1], we obtain the following proposition relating objects which are finitely built from R in $\mathscr{D}(R)$ and compact objects in $\mathscr{D}(R)$.

Proposition 4.1.5. *Let R be a DGA. A DG R-module M is finitely built from R in $\mathscr{D}(R)$ if and only if it is a compact object of $\mathscr{D}(R)$.*

The following useful theorem is well-known.

Theorem 4.1.6. *Let R and S be a DGAs and suppose that ${}_RP$ is finitely built from ${}_RR$ in $\mathscr{D}(R)$. Suppose that M and K are DG R-S-bimodules, N is a DG S-module and L is a*

DG S^{op}-module. Then we have the following canonical isomorphisms:

$$\mathsf{RHom}_R({}_RP, {}_RM_S) \overset{\mathsf{L}}{\otimes}_S {}_SN \xrightarrow{\sim} \mathsf{RHom}_R({}_RP, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN), \tag{4.2}$$

$$\mathsf{RHom}_{S^{\mathsf{op}}}({}_RK_S, L_S) \overset{\mathsf{L}}{\otimes}_R {}_RP \xrightarrow{\sim} \mathsf{RHom}_{S^{\mathsf{op}}}(\mathsf{RHom}_R({}_RP, {}_RK_S), L_S). \tag{4.3}$$

4.2 Canonical maps

In many instances involving classical derived functors one is able to talk about "natural maps". For example, if R and S are rings and $\varphi : R \to S$ is a ring homomorphism then the restriction of scalars functor $\varphi_* : \mathsf{Mod}(S) \to \mathsf{Mod}(R)$ gives rise to natural maps $\mathsf{Hom}_S(M,N) \to \mathsf{Hom}_R(\varphi_*(M), \varphi_*(N))$ and $\mathsf{Ext}^n_S(M,N) \to \mathsf{Ext}^n_R(\varphi_*(M), \varphi_*(N))$ for all $n \in \mathbb{N}$ and for all left S-modules M and N. However, due to the way in which "hyper-homological" functors are computed, see section 3.6.3, it is not clear what constitutes a natural map in the generalised setting. The answer to this problem is the notion of a "canonical map".

4.2.1 Defining canonical maps

Unfortunately, there is no general technique with which to obtain a canonical map, only ad hoc methods. We give examples of such methods below, in each case giving the classical example followed by its "hyper-homological" counterpart.

Example 4.2.1. Let R and S be rings and $\varphi : R \to S$ be a ring homomorphism. Recall that the restriction of scalars functor allows us to view S as an R-S-bimodule. Let M be a left S-module, then there is a natural map

$$\begin{aligned} {}_SM &\longrightarrow \mathsf{Hom}_R({}_RS_S, {}_RS_S \otimes_S {}_SM), \\ m &\longmapsto (s \mapsto s \otimes m). \end{aligned}$$

Consider the following way in which this natural map can be obtained. There is a natural isomorphism given by adjunction:

$$\mathsf{Hom}_R({}_RS_S \otimes_S {}_SM, {}_RS_S \otimes_S {}_SM) \xrightarrow{\sim} \mathsf{Hom}_S({}_SM, \mathsf{Hom}_R({}_RS_S, {}_RS_S \otimes_S {}_SM)).$$

The natural map $m \mapsto (s \mapsto s \otimes m)$ corresponds to the identity map under the adjunction.

Example 4.2.2. Let R and S be DGAs and M a DG S-module. In this case, it is not clear what the natural map ${}_SN \to \mathsf{RHom}_R({}_RS_S, {}_RS_S \overset{\mathsf{L}}{\otimes}_S {}_SN)$ is. However, as above, we have a natural isomorphism given by adjunction:

$$\mathsf{Hom}_{\mathscr{D}(R)}({}_RS_S \overset{\mathsf{L}}{\otimes}_S {}_SN, {}_RS_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(S)}({}_SN, \mathsf{RHom}_R({}_RS_S, {}_RS_S \overset{\mathsf{L}}{\otimes}_S {}_SN)).$$

Following Example 4.2.1, we define the canonical map ${}_SN \to \mathsf{RHom}_R({}_RS_S, {}_RS_S \overset{\mathsf{L}}{\otimes}_S {}_SN)$ to be the image of the identity map on ${}_RS_S \overset{\mathsf{L}}{\otimes}_S {}_SN$ under the adjunction.

We next consider two examples involving the tensor product and the derived tensor product.

Example 4.2.3. Let $\varphi : R \to S$ be a ring homomorphism as in Example 4.2.1. There is a natural map ${}_SS_R \otimes_R {}_RS_S \to {}_SS_S$ given by the multiplication map $s \otimes t \mapsto st$. As before, there is a natural sequence of isomorphisms:

$$\begin{aligned}\mathsf{Hom}_R({}_RS_S, {}_RS_S) &\xrightarrow{\sim} \mathsf{Hom}_R({}_RS_S, \mathsf{Hom}_S({}_SS_R, {}_SS_S))\\ &\xrightarrow{\sim} \mathsf{Hom}_S({}_SS_R \otimes_R {}_RS_S, {}_SS_S),\end{aligned}$$

where the first isomorphism is induced by ${}_RS_S \xrightarrow{\sim} \mathsf{Hom}_S({}_SS_R, {}_SS_S)$ and the second isomorphism is the adjunction isomorphism. Applying the sequence of natural isomorphisms to the identity map on ${}_RS_S$ one obtains the multiplication map ${}_SS_R \otimes_R {}_RS_S \to {}_SS_S$.

Instead of considering the immediate generalisation of Example 4.2.3, we consider a further generalisation by replacing the ring homomorphism $\varphi : R \to S$ with a generalised morphism of DGAs, rather than just a morphism of DGAs, that is, we consider a DG R-S-bimodule M in place of the morphism φ.

Let R and S be DGAs and suppose M is a DG R-S-bimodule. Moreover, assume that M is finitely built from S_S in $\mathscr{D}(S^{\mathsf{op}})$. Let Z be the DG S-R-bimodule defined by ${}_SZ_R = \mathsf{RHom}_{S^{\mathsf{op}}}({}_RM_S, {}_SS_S)$. We make the following observation; c.f. [34, Setup 2.1].

Remark 4.2.4. Since M is finitely built from S_S in $\mathscr{D}(S^{\mathsf{op}})$, Theorem 4.1.6 yields the following canonical isomorphism:

$$\begin{aligned}{}_RM_S \overset{\mathsf{L}}{\otimes}_S - &\cong {}_RM_S \overset{\mathsf{L}}{\otimes}_S \mathsf{RHom}_S({}_SS_S, -)\\ &\cong \mathsf{RHom}_S(\mathsf{RHom}_{S^{\mathsf{op}}}({}_RM_S, {}_SS_S), -)\\ &= \mathsf{RHom}_S({}_SZ_R, -).\end{aligned}$$

The first isomorphism is the canonical evaluation isomorphism, the second is the canonical isomorphism yielded by Theorem 4.1.6 and the final equality is by definition of Z as an S-dual of M.

Example 4.2.5. In Example 4.2.3 we looked at how to obtain the canonical map ${}_SS_R \otimes_R {}_RS_S \to {}_SS_S$ when we are given a ring homomorphism $\varphi : R \to S$. Here we shall examine how to define the canonical map ${}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$.

Consider the following natural sequence of isomorphisms:

$$\begin{aligned}\mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S, {}_RM_S) &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SS_S)\\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S, \mathsf{RHom}_S({}_SZ_R, {}_SS_S))\\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(S)}({}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S, {}_SS_S),\end{aligned}$$

where the first isomorphism is induced by the canonical isomorphism $M \to M \overset{\mathsf{L}}{\otimes}_S S$, the second isomorphism is induced by the canonical isomorphism observed in Remark 4.2.4, and the final isomorphism is just adjunction. The canonical map ${}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is now defined to be the image of the identity map on ${}_RM_S$ under these isomorphisms, as in Example 4.2.3.

We give one further example which when combined with Example 4.2.5 will be useful in giving an example of how computations are done with canonical maps.

Example 4.2.6. In the situation of Example 4.2.5 suppose further that we are given a DG S-module N. We may ask what is the canonical map ${}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$? Consider the natural sequence of isomorphisms:

$$\begin{aligned}\mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN, \mathsf{RHom}_S({}_SZ_R, {}_SN))\\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(S)}({}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN), {}_SN),\end{aligned}$$

where the first isomorphism is again given by Remark 4.2.4. The canonical map is again the image of the identity map.

4.2.2 An example computation

Computations involving canonical maps in the derived category such as verifying the commutativity of a diagram need to be carried out in the homotopy category. In this section we provide a detailed example of such a computation.

Remark 4.2.7. Before carrying out such a computation we make the following observation: given a DGA R over the ground ring $\mathbb{K}$ one can replace R with a quasi-isomorphic DGA which is K-flat over $\mathbb{K}$, see [39]. Hence, without loss of generality, we may assume that our DGAs are K-flat over the ground ring $\mathbb{K}$. Now suppose we have two DGAs which are each K-flat over the ground ring $\mathbb{K}$, then a K-projective object of $\mathscr{D}(R\text{-}S^{\mathsf{op}})$, the category of DG R-S-bimodules, is also a K-projective object of $\mathscr{D}(R)$ and $\mathscr{D}(S^{\mathsf{op}})$, forgetting the right S-structure and left R-structure, respectively.

Let R and S be DGAs and, as above, assume that they are K-flat over the ground ring $\mathbb{K}$ (replacing them with quasi-isomorphic K-flat DGAs if necessary). Let M be a DG R-S-bimodule which is finitely built from S in $\mathscr{D}(S^{\mathsf{op}})$ and let $Z = \mathsf{RHom}_{S^{\mathsf{op}}}(M, S)$ be an S-dual of M, as in Remark 4.2.4. Suppose also that N is a DG S-module. The aim of this section is to verify the commutativity of the following diagram in $\mathscr{D}(S)$ which appears in the proof of the implication $(1) \implies (2)$ in Proposition 4.3.4.

$$\begin{tikzcd}
Z \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N) \arrow[r, "\sim", "\text{assoc.}"'] \arrow[d, "\beta_N"'] & (Z \overset{\mathsf{L}}{\otimes}_R M) \overset{\mathsf{L}}{\otimes}_S N \arrow[d, "\alpha\otimes 1_N"] \\
N & S \overset{\mathsf{L}}{\otimes}_S N. \arrow[l, "\sim"', "\text{cancellation}"]
\end{tikzcd} \tag{4.4}$$

In diagram (4.4) the maps $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ and $\beta_N : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$ are canonical. The general technique will be to isolate a representative of the canonical map in the homotopy category. The first such canonical map which it is useful to isolate is that arising as the canonical isomorphism in Remark 4.2.4.

For a DG S-module N, Remark 4.2.4 gives a canonical isomorphism ${}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN \to \mathsf{RHom}_S({}_SZ_R, {}_SN)$. We take the following K-projective resolution:

- $\pi : {}_RP_S \to {}_RM_S$, K-projective resolution in $\mathscr{D}(R\text{-}S^{\mathsf{op}})$.

Note that Remark 4.2.7 implies that $\pi : P \to M$ is a K-projective resolution of M both as a DG R-module and as a DG S^{op}-module. With the above resolution, the sequence of isomorphims in Remark 4.2.4 becomes:

$$\begin{aligned}
{}_RP_S \otimes_S {}_SN \ &\xrightarrow{\sim}\ {}_RP_S \otimes_S \mathsf{Hom}_S({}_SS_S, {}_SN) \\
&\xrightarrow{\sim}\ \mathsf{Hom}_S(\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S), {}_SN)
\end{aligned}$$

in $\mathscr{K}(R)$, where $\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S) \simeq {}_SZ_R$ in $\mathscr{K}(S\text{-}R^{\mathsf{op}})$.

Remark 4.2.8. Note that $\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S)$ is a K-projective object of $\mathscr{K}(S)$ because we have the following isomorphisms:

$$\begin{aligned}\mathsf{Hom}_S(\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S), -) &\cong {}_RP_S \otimes_S \mathsf{Hom}_S({}_SS_S, -)\\ &\cong {}_RP_S \otimes_S -,\end{aligned}$$

where the first isomorphism is due to the fact that ${}_RP_S$ is finitely built from S_S in $\mathscr{K}(S^{\mathsf{op}})$ and Theorem 4.1.6. Now ${}_RP_S$ is K-projective in $\mathscr{K}(R\text{-}S^{\mathsf{op}})$, therefore, it is also K-flat, so the functor ${}_RP_S \otimes_S -$ preserves quasi-isomorphisms, hence $\mathsf{Hom}_S(\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S), -)$ preserves quasi-isomorphisms. Thus, $\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S)$ is K-projective in $\mathscr{K}(S)$; see Proposition 3.6.3 and Definition 3.6.4.

The canonical map ${}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN \to \mathsf{RHom}_S({}_SZ_R, {}_SN)$ is now represented by the map

$$\begin{aligned}{}_RP_S \otimes_S {}_SN &\xrightarrow{\sim} \mathsf{Hom}_S(\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S), {}_SN)\\ p \otimes n &\longmapsto \zeta \mapsto (-1)^{|\zeta(p)||n|}\zeta(p)n\end{aligned} \tag{4.5}$$

in $\mathscr{K}(R)$.

We next obtain a representative in $\mathscr{K}(S\text{-}S^{\mathsf{op}})$ for the canonical map ${}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$. In addition to the resolution taken above, we take the following K-projective and K-injective resolutions:

- $\theta : {}_SQ \to {}_SN$, K-projective resolution in $\mathscr{D}(S)$;
- $\iota : {}_SS_S \to {}_SI_S$, K-injective resolution in $\mathscr{D}(S\text{-}S^{\mathsf{op}})$.

Note that we only need ι to be a K-injective resolution of S as a DG S-module, which is true by Remark 4.2.7. With the resolutions as above, the sequence of isomorphisms in Example 4.2.5 becomes:

$$\begin{aligned}\mathsf{Hom}_{\mathscr{K}(R)}({}_RP_S, {}_RP_S) &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(R)}({}_RP_S, {}_RP_S \otimes_S {}_SI_S)\\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(R)}({}_RP_S, \mathsf{Hom}_S(\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S), {}_SI_S))\\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(S)}(\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S) \otimes_R {}_RP_S, {}_SI_S).\end{aligned}$$

The canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is now represented by the map

$$\begin{aligned}\tilde{\alpha} : \mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S) \otimes_R {}_RP_S &\longrightarrow {}_SI_S\\ \zeta \otimes p &\longmapsto \zeta(p)\iota(1)\end{aligned} \tag{4.6}$$

in $\mathscr{K}(S)$.

We now turn our attention to the canonical map ${}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$. Using the three K-projective and K-injective resolutions above, the sequence of isomorphisms in Example 4.2.6 becomes:

$$\begin{aligned}\mathsf{Hom}_{\mathscr{K}(R)}(P \otimes_S Q, P \otimes_S Q) &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(R)}(P \otimes_S Q, \mathsf{Hom}_S(\mathsf{Hom}_{S^{\mathsf{op}}}(P, S), Q)) \\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(S)}(\mathsf{Hom}_{S^{\mathsf{op}}}(P, S) \otimes_R (P \otimes_S Q), Q).\end{aligned}$$

The canonical map $\beta_N : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$ is represented by the map

$$\begin{aligned}\tilde{\beta}_N : \mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S) \otimes_R ({}_RP_S \otimes_S {}_SQ) &\longrightarrow {}_SQ \\ \zeta \otimes (p \otimes q) &\longmapsto \zeta(p)q\end{aligned} \tag{4.7}$$

in $\mathscr{K}(S)$.

Now diagram (4.4) is represented by the diagram,

$$\begin{tikzcd}
\mathsf{Hom}_{S^{\mathsf{op}}}(P, S) \otimes_R (P \otimes_S Q) \arrow[rr, "\sim", "\text{assoc.}"'] \arrow[d, "\tilde{\beta}_N"'] & & (\mathsf{Hom}_{S^{\mathsf{op}}}(P, S) \otimes_R P) \otimes_S Q \arrow[d, "\tilde{\alpha} \otimes 1_Q"] \\
Q \arrow[r, "\sim", "\text{cancel.}"'] & S \otimes_S Q \arrow[r, "\sim", "\iota \otimes 1"'] & I \otimes_S Q,
\end{tikzcd} \tag{4.8}$$

in the homotopy category $\mathscr{K}(S)$.

Remark 4.2.9. Note that $\mathsf{Hom}_{S^{\mathsf{op}}}({}_RP_S, {}_SS_S) \otimes_R ({}_RP_S \otimes_S {}_SQ) \simeq {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN)$ in $\mathscr{D}(S)$ because we have

$$\mathsf{Hom}_R({}_RP_S \otimes_S {}_SQ, -) \simeq \mathsf{Hom}_S({}_SQ, \mathsf{Hom}_R({}_RP_S, -)).$$

Since ${}_RP_S$ is K-projective in $\mathscr{D}(R)$ and ${}_SQ$ is K-projective in $\mathscr{D}(S)$, it follows that the the functor on the right hand side preserves quasi-isomorphisms, hence the functor on the left hand side preserves quasi-isomorphisms and, therefore, ${}_RP_S \otimes_S {}_SQ$ is K-projective in $\mathscr{D}(R)$; see Propositon 3.6.3. C.f. Remark 4.2.8 also.

Now with the canonical maps obtained in (4.6), (4.7) and (4.8), it is easy to see that diagram (4.8) commutes. Hence diagram (4.4) commutes, as we desired.

4.2.3 Another example computation

In this section we give a second example computation. Let R and S be DGAs. In this case, however, we shall assume that the DG R-S-bimodule M is finitely built from R in

$\mathscr{D}(R)$. The aim of this section is to verify the commutativity of the diagram,

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "\sigma"'] & \mathsf{RHom}_R(M, M \overset{\mathsf{L}}{\otimes}_S S) \arrow[d] \\ & \mathsf{RHom}_R(M, M), \end{tikzcd} \tag{4.9}$$

which appears in the proof of the implication $(1) \implies (2)$ in Proposition 4.3.7.

In this case we need only take one K-projective resolution:

- $\pi : {}_RP_S \to {}_RM_S$, K-projective resolution in $\mathscr{D}(R\text{-}S^{\mathsf{op}})$.

As in section 4.2.2, we isolate representatives of the three canonical maps involved in the appropriate homotopy categories.

We first turn our attention to the canonical map η_S. Note that, in the derived category, the canonical map is η_S, where η is the unit of the adjunction

$$\begin{tikzcd} \mathscr{D}(R) \arrow[r, shift right, "\mathsf{RHom}_R({}_RM_S, -)"'] & \mathscr{D}(S). \arrow[l, shift right, "{}_RM_S\overset{\mathsf{L}}{\otimes}_S -"'] \end{tikzcd}$$

Thus the canonical map η_S is represented by the map $\tilde{\eta}_S$ in $\mathscr{K}(S)$, where $\tilde{\eta}$ is the unit of the adjunction

$$\begin{tikzcd} \mathscr{K}(R) \arrow[r, shift right, "\mathsf{Hom}_R({}_RP_S, -)"'] & \mathscr{K}(S). \arrow[l, shift right, "{}_RP_S\otimes_S -"'] \end{tikzcd}$$

Now, $\tilde{\eta}_S$ is easy to compute and we obtain that $\tilde{\eta}_S(s)(p) = (-1)^{|s||p|}p \otimes s$.

Consider the canonical map $\sigma : {}_SS_S \to \mathsf{RHom}_R({}_RM_S, {}_RM_S)$. As in Examples 4.2.1 and 4.2.3, we have the following sequence of isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S, {}_RM_S) &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(R)}({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SS_S, {}_RM_S) \\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{D}(S)}({}_SS_S, \mathsf{RHom}_R({}_RM_S, {}_RM_S)). \end{aligned}$$

Using the K-projective resolution above, we obtain

$$\begin{aligned} \mathsf{Hom}_{\mathscr{K}(R)}({}_RP_S, {}_RP_S) &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(R)}({}_RP_S \otimes_S {}_SS_S, {}_RP_S) \\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathscr{K}(S)}({}_SS_S, \mathsf{Hom}_R({}_RP_S, {}_RP_S)). \end{aligned}$$

Note that, as in Remark 4.2.9, ${}_RP_S \otimes_S {}_SS_S$ is K-projective in $\mathscr{K}(R)$. We thus obtain the map

$$\begin{aligned} \tilde{\sigma} : {}_SS_S &\longrightarrow \mathsf{Hom}_R({}_RP_S, {}_RP_S) \\ s &\longmapsto p \mapsto (-1)^{|s||p|}ps. \end{aligned}$$

It is now easy to verify that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\tilde{\eta_S}"] \arrow[dr, "\tilde{\sigma}"'] & \mathsf{Hom}_R(P, P \otimes_S S) \arrow[d] \\
& \mathsf{Hom}_R(P, P).
\end{tikzcd}$$

Hence, diagram (4.9) commutes, as required.

4.3 Characterisation for the restriction of scalars functor

Before characterising when the restriction of scalars functor is a full embedding, we first recall the definition of a full embedding and state a well-known categorical result.

Definition 4.3.1. Suppose $F : \mathscr{A} \to \mathscr{B}$ is a functor between two categories $\mathscr{A}$ and $\mathscr{B}$. If

$$\mathsf{Hom}_{\mathscr{A}}(A, A') \cong \mathsf{Hom}_{\mathscr{B}}(FA, FA')$$

for all objects A, A' of $\mathscr{A}$, then we say that F is *fully faithful* (or *full and faithful*). The functor F will sometimes be called a *full embedding* of $\mathscr{A}$ into $\mathscr{B}$.

Lemma 4.3.2 ([43], Theorem IV.3.1). *Let $\mathscr{A}$ and $\mathscr{B}$ be categories and suppose*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, shift right, "G"'] & \mathscr{B} \arrow[l, shift right, "F"']
\end{tikzcd}$$

is an adjunction with F left adjoint to G. Then the unit of the adjunction is an isomomorphism if and only if F is fully faithful.

In the following, each condition generalises a condition of [24, Theorem 4.4].

Proposition 4.3.3. *Let R and S be DGAs and suppose that M is a DG R-S-bimodule. Then the following conditions are equivalent:*

(1) For all DG S-modules N the canonical map

$$\eta_N : {}_SN \to \mathsf{RHom}_R({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN)$$

is an isomorphism.

(2) For all DG S-modules N and N' the canonical map

$$\xi_{N,N'} : \mathsf{RHom}_S({}_SN, {}_SN') \to \mathsf{RHom}_R({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN')$$

is an isomorphism.

(3) The functor

$$ {}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R) $$

is a full embedding of derived categories.

Proof: Condition (2) is just a reformulation of what it means for the functor ${}_RM_S \overset{\mathsf{L}}{\otimes}_S -$ to be fully faithful, noting that

$$ H^0(\mathsf{RHom}_R(X, Y)) \cong \mathsf{Hom}_{\mathscr{D}(R)}(X, Y). $$

See Chapter 3 for details.

The functor ${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ is left adjoint to

$$ \mathsf{RHom}_R({}_RM_S, -) : \mathscr{D}(R) \to \mathscr{D}(S). $$

The unit of this adjunction is

$$ \eta_N : {}_SN \to \text{RHom}_R({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN). $$

By Lemma 4.3.2, the unit of an adjunction is an isomorphism if and only if the left adjoint is a full embedding. This gives the equivalence $(1) \Leftrightarrow (3)$. □

Proposition 4.3.4. *Let R and S be DGAs and suppose that M is a DG R-S-bimodule which is finitely built from S_S in $\mathscr{D}(S^{\mathsf{op}})$. Let $Z = \mathsf{RHom}_{S^{\mathsf{op}}}({}_RM_S, {}_SS_S)$ so that Z obtains the structure ${}_SZ_R$. Then the following conditions are equivalent:*

(1) The canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is an isomorphism.

(2) For all DG S-modules N the canonical map $\beta_N : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$ is an isomorphism.

(3) For all DG S^{op}-modules N and DG S-modules N' the canonical map

$$ \gamma_{N,N'} : (N_S \overset{\mathsf{L}}{\otimes}_S {}_SZ_R) \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN') \to N_S \overset{\mathsf{L}}{\otimes}_S {}_SN' $$

is an isomorphism.

Proof: $(1) \implies (2)$. Suppose the canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is an isomorphism. Then, for any DG S-module N we obtain:

$$\begin{tikzcd}
Z \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N) \arrow[r, "\sim", "\text{assoc.}"'] \arrow[d, "\beta_N"'] & (Z \overset{\mathsf{L}}{\otimes}_R M) \overset{\mathsf{L}}{\otimes}_S N \arrow[d, "\sim"', "\alpha\otimes 1_N"] \\
N & S \overset{\mathsf{L}}{\otimes}_S N \arrow[l, "\sim", "\text{cancellation}"']
\end{tikzcd}$$

The diagram above is diagram (4.4) from section 4.2.2. In section 4.2.2, we showed that this diagram commutes, hence the canonical map $\beta_N : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$ is an isomorphism.

$(2) \implies (3)$. Suppose that the canonical map $\beta_N : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN') \to {}_SN'$ is an isomorphism for all DG S-modules N'. Then for any DG S^{op}-module N and DG S-module N', we get:

$$\begin{tikzcd}
(N \overset{\mathsf{L}}{\otimes}_S Z) \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N') \arrow[r, "\sim", "\text{assoc.}"'] \arrow[dr, "\gamma_{N,N'}"'] & N \overset{\mathsf{L}}{\otimes}_S (Z \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N')) \arrow[d, "\sim"', "1_N\otimes\beta_{N'}"] \\
& N \overset{\mathsf{L}}{\otimes}_S N',
\end{tikzcd} \tag{4.10}$$

and the canonical map $\gamma_{N,N'} : (N \overset{\mathsf{L}}{\otimes}_S Z) \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N') \to N \overset{\mathsf{L}}{\otimes}_S N'$ is defined as the composite of $(1_N \otimes \beta_{N'})$ with the isomorphism induced by the associativity of the derived tensor product. Hence, the canonical map $\gamma_{N,N'}$ is an isomorphism.

$(3) \implies (1)$. Suppose that the canonical map, as obtained in (4.10),

$$\gamma_{N,N'} : (N_S \overset{\mathsf{L}}{\otimes}_S {}_SZ_R) \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN') \to N_S \overset{\mathsf{L}}{\otimes}_S {}_SN'$$

is an isomorphism for all DG S^{op}-modules N and DG S-modules N'. Setting $N = N' = S$ we obtain:

$$\begin{tikzcd}
Z \overset{\mathsf{L}}{\otimes}_R M \arrow[r, "\sim", "\text{cancel.}"'] \arrow[d, "\alpha"'] & (S \overset{\mathsf{L}}{\otimes}_S Z) \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S S) \arrow[d, "\sim"', "\gamma_{S,S}"] \\
S & S \overset{\mathsf{L}}{\otimes}_S S, \arrow[l, "\sim"', "\text{cancellation}"]
\end{tikzcd} \tag{4.11}$$

We claim that the diagram above commutes. As in section 4.2.2, we take the following K-projective and K-injective resolutions:

- $\pi : {}_RP_S \to {}_RM_S$, K-projective resolution in $\mathscr{D}(R\text{-}S^{\mathsf{op}})$;
- $\iota : {}_SS_S \to {}_SI_S$, K-injective resolution in $\mathscr{D}(S\text{-}S^{\mathsf{op}})$.

Note that the DG S-S-bimodule S is K-projective in $\mathscr{D}(S\text{-}S^{\mathsf{op}})$. Hence, in order to verify the commutativity of (4.11), it suffices to show that the following diagram commutes:

$$\begin{tikzcd}
Z \otimes_R P \arrow[r, "\sim", "\text{cancel.}"'] \arrow[d, "\tilde{\alpha}"'] & (S \otimes_S Z) \otimes_R (P \otimes_S I) \arrow[d, "\sim"', "\tilde{\gamma}_{S,S}"] \\
I & S \otimes_S I. \arrow[l, "\sim"', "\text{multiplication}"]
\end{tikzcd} \tag{4.12}$$

Note that ${}_RP_S \otimes_S {}_SS_S$ is K-projective in $\mathscr{K}(R)$ as in Remark 4.2.9. The map $\tilde{\gamma}_{S,S}$ is obtained via the composition

$$\begin{tikzcd}
(S \otimes_S Z) \otimes_R (P \otimes_S S) \arrow[r, "\sim", "\text{cancel.}"'] \arrow[dr, "\tilde{\gamma}_{S,S}"'] & S \otimes_S (Z \otimes_R (P \otimes_S S)) \arrow[d, "\sim"', "1_S \otimes \tilde{\beta}_S"] \\
& S \otimes_S I,
\end{tikzcd}$$

where $\tilde{\beta}_S$ is the canonical map representing β_S, obtained in the same manner as (4.8). We can now see that the canonical map $\gamma_{S,S}$ is represented by the map

$$\begin{array}{rcl}
\tilde{\gamma}_{S,S} : (S \otimes Z) \otimes (P \otimes S) & \xrightarrow{\sim} & S \otimes I \\
(s \otimes \zeta) \otimes (p \otimes t) & \longmapsto & s \otimes ((\zeta(p)\iota(t)
\end{array}$$

in $\mathscr{K}(S\text{-}S^{\mathsf{op}})$. It is now clear that (4.12) commutes, and hence (4.11) commutes. Thus the canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is an isomorphism, as required. □

Remark 4.3.5. Note that in the proof of Proposition 4.3.4, the assumption that M is finitely built from R in $\mathscr{D}(R)$ appears implicitly in the computations determining the canonical maps; see section 4.2.2 and Remark 4.2.4.

Using Remark 4.2.4 we can glue the conditions of Propositions 4.3.3 and 4.3.4 together to obtain the following generalisation of Theorem 4.4 in [24].

Theorem 4.3.6. *Let R and S be DGAs and suppose M is a DG R-S-bimodule which is finitely built from S_S in $\mathscr{D}(S^{\mathsf{op}})$. Let $Z = \mathsf{RHom}_{S^{\mathsf{op}}}({}_RM_S, {}_SS_S)$ so that Z obtains the structure ${}_SZ_R$. Then the following conditions are equivalent:*

(1) The canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is an isomorphism.

(2) For all DG S-modules N the canonical map $\beta_N : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) \to {}_SN$ is an isomorphism.

(3) For all DG S^{op}-modules N and DG S-modules N' the canonical map

$$\gamma_{N,N'} : (N_S \overset{\mathsf{L}}{\otimes}_S {}_SZ_R) \overset{\mathsf{L}}{\otimes}_R ({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN') \to N_S \overset{\mathsf{L}}{\otimes}_S {}_SN'$$

is an isomorphism.

(4) For all DG S-modules N the canonical map

$$\eta_N : {}_SN \to \mathsf{RHom}_R({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN)$$

is an isomorphism.

(5) *For all DG S-modules N and N' the canonical map*

$$\xi_{N,N'} : \mathsf{RHom}_S({}_SN, {}_SN') \to \mathsf{RHom}_R({}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN')$$

is an isomorphism.

(6) *The functor ${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories.*

Proof: The first three conditions are equivalent by Proposition 4.3.4 and the last three conditions are equivalent by Proposition 4.3.3. We show that $(1) \implies (4)$ and $(5) \implies (2)$, thereby giving the result.

$(1) \implies (4)$. Suppose the canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is an isomorphism. Then for all DG S-modules N we have the following diagram:

$$\begin{tikzcd}
N \arrow[r, "\text{evaluation}", "\sim"'] \arrow[dd, "\eta_N"'] & \mathsf{RHom}_S(S,N) \arrow[d, "\sim"', "\alpha^*"] \\
 & \mathsf{RHom}_S(Z \overset{\mathsf{L}}{\otimes}_R M, N) \arrow[d, "\sim"', "\text{adjunction}"] \\
\mathsf{RHom}_R(M, M \overset{\mathsf{L}}{\otimes}_S N) \arrow[r, "\sim", "(\mu_N)_*"'] & \mathsf{RHom}_R(M, \mathsf{RHom}_S(Z,N)),
\end{tikzcd} \tag{4.13}$$

where $(\mu_N)_*$ is the canonical map induced by the canonical map $\mu_N : {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN \to \mathsf{RHom}_S({}_SZ_R, {}_SN)$ from Remark 4.2.4. We claim that diagram (4.13) commutes, and as such the canonical map η_N is an isomorphism. In order to verify the commutativity of (4.13), we shall take the following K-projective and K-injective resolutions (c.f. sections 4.2.2 and 4.2.3):

- $\pi : {}_RP_S \to {}_RM_S$, K-projective resolution in $\mathscr{D}(R\text{-}S^{\mathsf{op}})$;
- $\varepsilon : {}_SN \to {}_SE$, K-injective resolution in $\mathscr{D}(S)$;
- $\iota : {}_SS_S \to {}_SI_S$, K-injective resolution in $\mathscr{D}(S\text{-}S^{\mathsf{op}})$.

With these resolutions, in order to verify the commutativity of diagram (4.13), it suffices to show that the following diagram commutes in $\mathscr{K}(S)$:

$$\begin{tikzcd}
E \arrow[dd, "\tilde{\eta_E}"'] & \mathsf{Hom}_S(S, E) \arrow[l, "\text{evaluation}", "\sim"'] & \mathsf{Hom}_S(I, E) \arrow[l, "\iota^*", "\sim"'] \arrow[d, "\tilde{\alpha}^*", "\sim"'] \\
 & & \mathsf{Hom}_S(Z \otimes_R P, E) \arrow[d, "\text{adjunction}", "\sim"'] \\
\mathsf{Hom}_R(P, P \otimes_S E) \arrow[rr, "\sim", "(\tilde{\mu_E})_*"'] & & \mathsf{Hom}_R(P, \mathsf{Hom}_S(Z, E)).
\end{tikzcd} \tag{4.14}$$

To check that this commutes we first calculate the composite of $\tilde{\alpha}^*$ with the adjunction. Let $f \in \mathsf{Hom}_S(I, E)$, then the composite is given by

$$\begin{array}{ccccc}
\mathsf{Hom}_S(I, E) & \xrightarrow{\tilde{\alpha}^*} & \mathsf{Hom}_S(Z \otimes_R P, E) & \xrightarrow{\text{adj}} & \mathsf{Hom}_R(P, \mathsf{Hom}_S(Z, E)) \\
f & \longmapsto & f \circ \tilde{\alpha} & \longmapsto & p \mapsto \left[\zeta \mapsto (-1)^{|p||\zeta|} f \circ \tilde{\alpha}(\zeta \otimes p)\right].
\end{array}$$

But, in section 4.2.2 we obtained the map $\tilde{\alpha}$ as the map $\zeta \otimes p \mapsto \zeta(p)\iota(1)$, hence the composite above gives the map

$$f \longmapsto \left\{p \mapsto \left[\zeta \mapsto (-1)^{|\zeta(p)|} f(\zeta(p)\iota(1))\right]\right\}.$$

Similarly, the composite of ι^* with the evaluation isomorphism is given by the assignment $f \longmapsto f(\iota(1))$, and the composite $(\tilde{\mu_N})_* \circ \tilde{\eta_N}$ is given by the assignment: $e \longmapsto p \mapsto (\zeta \mapsto (-1)^{|\zeta(p)||e|}\zeta(p)e)$. Putting these together gives the composite $\mathsf{Hom}_S(I, E) \to \mathsf{Hom}_R(P, \mathsf{Hom}_S(Z, E))$ obtained by going around diagram (4.14) anticlockwise, which is given by the assignment:

$$f \longmapsto \left\{p \mapsto \left[\zeta \mapsto (-1)^{|\zeta(p)||f(\iota(1))|} \zeta(p) f(\iota(1))\right]\right\}.$$

Since $|f(\iota(1))| = |f|$ and $f \in \mathsf{Hom}_S(I, E)$ is S-linear, it follows that the diagram commutes. Hence diagram (4.13) commutes, as desired.

$(5) \implies (2)$. Suppose that, for all DG S-modules N and N', the canonical map $\xi_{N,N'} : \mathsf{RHom}_S(N, N') \to \mathsf{RHom}_R(M \overset{\mathsf{L}}{\otimes}_R N, M \overset{\mathsf{L}}{\otimes}_R N)$ is an isomorphism. We claim that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{RHom}_S(N, N') \arrow[r, "\beta_N^*"] \arrow[d, "\xi_{N,N'}"', "\sim"] & \mathsf{RHom}_S(Z \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N), N') \\
\mathsf{RHom}_R(M \overset{\mathsf{L}}{\otimes}_S N, M \overset{\mathsf{L}}{\otimes}_S N') \arrow[r, "\sim", "(\mu_{N'})_*"'] & \mathsf{RHom}_R(M \overset{\mathsf{L}}{\otimes}_S N, \mathsf{RHom}_S(Z, N')) \arrow[u, "\sim", "\text{adjunction}"']
\end{tikzcd} \tag{4.15}$$

where β_N^* is the map induced by the canonical map $\beta_N : Z \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N) \to N$. If diagram (4.15) is commutative then it follows that β_N^* is an isomorphism for all DG S-modules N and N'. Fixing N we see that

$$H_0(\beta_N^*) = \mathsf{Hom}_{\mathscr{D}(S)}(\beta_N, N') : \mathsf{Hom}_{\mathscr{D}(S)}(N, N') \to \mathsf{Hom}_{\mathscr{D}(S)}(Z \overset{\mathsf{L}}{\otimes}_R (M \overset{\mathsf{L}}{\otimes}_S N), N')$$

is an isomorphism for all DG S-modules N'. Hence, β_N is an isomorphism for all DG S-modules N.

Thus it suffices to show that diagram (4.15) commutes, and then we have the result. Take the following K-projective and K-injective resolutions:

- $\pi : {}_RP_S \to {}_RM_S$, K-projective resolution in $\mathscr{D}(R\text{-}S^{\mathsf{op}})$;
- $\theta : {}_SQ \to {}_SN$, K-projective resolution in $\mathscr{D}(S)$;
- $\varepsilon : {}_SN \to {}_SE$, K-injective resolution in $\mathscr{D}(S)$;
- $\iota : {}_SN' \to {}_SI$, K-injective resolution in $\mathscr{D}(S)$.

Now, to verify that diagram (4.15) commutes we only need to show that the following diagram using these resolutions commutes:

$$\begin{tikzcd}
\mathsf{Hom}_S(E, I) \arrow[r, "\tilde{\beta}^*"] \arrow[d, "(\varepsilon\circ\theta)^*"', "\sim"] & \mathsf{Hom}_S(Z \otimes_R (P \otimes_S Q), I) \arrow[dd, "\sim"', "\text{adjunction}"] \\
\mathsf{Hom}_S(Q, I) \arrow[d, "\tilde{\xi}_{Q,I}"', "\sim"] & \\
\mathsf{Hom}_R(P \otimes_S Q, P \otimes_S I) \arrow[r, "\sim", "(\widetilde{\mu_E})_*"'] & \mathsf{Hom}_R(P \otimes_S Q, \mathsf{Hom}_S(Z, I)).
\end{tikzcd} \tag{4.16}$$

Note that, as in Remark 4.2.9, $P \otimes_S Q$ is K-projective in $\mathscr{K}(R)$. Let $f \in \mathsf{Hom}_S(E, I)$, the clockwise composition acts on f via the assignment

$$f \longmapsto (p \otimes q \mapsto \zeta \mapsto (-1)^{(|p|+|q|)|\zeta|} f(\zeta(p)(\varepsilon \circ \theta(q)))).$$

Again, for $f \in \mathsf{Hom}_S(E, I)$, the anticlockwise composition acts on f via the assignment

$$f \longmapsto (p \otimes q \mapsto \zeta \mapsto (-1)^{|\zeta|(|p|+|q|)} \zeta(p) f(\varepsilon \circ \theta(q))).$$

Since f is S-linear and $\zeta(p) \in S$, it follows that the clockwise and anticlockwise compositions agree, hence diagram (4.16) commutes. Therefore, diagram (4.15) commutes, completing the proof of the theorem. □

Proposition 4.3.7. *Let R and S be DGAs and M be a DG R-S-bimodule which is finitely built from ${}_RR$ in $\mathscr{D}(R)$. Then the following conditions are equivalent:*

(1) The canonical map $\sigma : {}_SS_S \to \mathsf{RHom}_R({}_RM_S, {}_RM_S)$ is an isomorphism.

(2) The functor ${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories.

Proof: Firstly, suppose that the functor ${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories. By Proposition 4.3.3 the canonical map $\eta_N : {}_SN \to \mathsf{RHom}_R({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN)$ is an isomorphism for all DG S-modules N. Setting $N = S$, we obtain

$$\begin{tikzcd}
{}_SS_S \arrow[r, "\eta_S", "\sim"'] \arrow[dr, "\sigma"'] & \mathsf{RHom}_R({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SS_S) \arrow[d, "\text{cancellation}", "\sim"'] \\
 & \mathsf{RHom}_R({}_RM_S, {}_RM_S).
\end{tikzcd}$$

The proof that this diagram commutes was the subject of section 4.2.3. Hence the canonical map $\sigma : {}_SS_S \to \mathsf{RHom}_R({}_RM_S, {}_RM_S)$ is an isomorphism, proving the implication.

Conversely, suppose that the canonical map $\sigma : {}_SS_S \to \mathsf{RHom}_R({}_RM_S, {}_RM_S)$ is an isomorphism. Then for any DG S-module N we have the following diagram:

$$\begin{tikzcd}
{}_SN \arrow[r, "\sim", "\text{cancellation}"'] \arrow[d, "\eta_N"'] & {}_SS_S \overset{\mathsf{L}}{\otimes}_S {}_SN \arrow[d, "\sigma\otimes 1_N", "\sim"'] \\
\mathsf{RHom}_R({}_RM_S, {}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SN) & \mathsf{RHom}_R({}_RM_S, {}_RM_S) \overset{\mathsf{L}}{\otimes}_S {}_SN \arrow[l, "\sim", "\nu"']
\end{tikzcd} \tag{4.17}$$

where ν is the canonical isomorphism arising from Theorem 4.1.6 due to the fact that M is finitely built from R in $\mathscr{D}(R)$ (c.f. [34]). Again, we need to verify that this diagram commutes. We take the following K-projective resolutions:

- $\pi : {}_RP_S \to {}_RM_S$, K-projective resolution in $\mathscr{D}(R\text{-}S^{\mathsf{op}})$;
- $\theta : {}_SQ \to {}_SN$, K-projective resolution in $\mathscr{D}(S)$.

Note that, as in Remark 4.2.9, ${}_RP_S \otimes_S {}_SQ$ is K-projective in $\mathscr{K}(R)$. It is now sufficient to show that the following diagram commutes:

$$\begin{tikzcd}
{}_SQ \arrow[r, "\sim", "\text{cancellation}"'] \arrow[d, "\tilde{\eta}_Q"'] & {}_SS_S \otimes_S {}_SQ \arrow[d, "\tilde{\sigma}\otimes 1_Q", "\sim"'] \\
\mathsf{Hom}_R({}_RP_S, {}_RP_S \otimes_S {}_SQ) & \mathsf{Hom}_R({}_RP_S, {}_RP_S) \otimes_S {}_SQ. \arrow[l, "\sim", "\tilde{\nu}"']
\end{tikzcd}$$

It is easy to check that the diagram above commutes, and hence that diagram (4.17) commutes. It follows that the canonical map $\eta_N : N \to \mathsf{RHom}_R(M, M \overset{\mathsf{L}}{\otimes}_S N)$ is an isomorphism for all DG S-modules N. Therefore, by Proposition 4.3.3 the functor ${}_RM_S \overset{\mathsf{L}}{\otimes}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories. □

We may obtain opposite statements of Propositions 4.3.3 and 4.3.4 for DG S^{op}-modules instead of DG left S-modules. Writing $M = \mathsf{RHom}_S({}_SZ_R, {}_SS_S)$ and arguing as in Remark 4.2.4 we see that when ${}_SZ_R$ is finitely built from ${}_SS$ in $\mathscr{D}(S)$ we have

$$\mathsf{RHom}_{S^{\mathsf{op}}}({}_RM_S, -) \cong - \overset{\mathsf{L}}{\otimes}_S {}_SZ_R.$$

Thus combining the opposite statements of Propositions 4.3.3 and 4.3.4 we obtain the opposite statement of Theorem 4.3.6 for DG S^{op}-modules.

Remark 4.3.8. Note that in Theorem 4.3.6 and its statement for DG S^{op}-modules alluded to above the first condition that the canonical map $\alpha : {}_SZ_R \overset{\mathsf{L}}{\otimes}_R {}_RM_S \to {}_SS_S$ is an isomorphism is left-right symmetric. Hence the equivalent conditions of the opposite statement of Theorem 4.3.6 are equivalent to those of Theorem 4.3.6.

4.3.1 Homological epimorphisms of DGAs

We obtain the following result from Theorem 4.3.6 by setting $M = S$. Again, the opposite statements for DG S^{op}-modules are equivalent to the statements for DG S-modules because of the left-right symmetry in statement (1); see Remark 4.3.8.

Theorem 4.3.9. *Let R and S be DGAs and let $\varphi : R \to S$ be a morphism of DGAs. The following conditions are equivalent:*

(1) The canonical map ${}_SS_R \overset{\mathsf{L}}{\otimes}_R {}_RS_S \to {}_SS_S$ is an isomorphism.

(2) For all DG S-modules M the canonical map ${}_SS_R \overset{\mathsf{L}}{\otimes}_R {}_RM \to {}_SM$ is an isomorphism.

(3) For all DG S^{op}-modules M and all DG S-modules N the canonical map

$$M_R \overset{\mathsf{L}}{\otimes}_R {}_RN \to M_S \overset{\mathsf{L}}{\otimes}_S {}_SN$$

is an isomorphism.

(4) For all DG S-modules M the canonical map ${}_SM \to \mathsf{RHom}_R({}_RS_S, {}_RM)$ is an isomorphism.

(5) For all DG S-modules M, M' the canonical map

$$\mathsf{RHom}_S({}_SM, {}_SM') \to \mathsf{RHom}_R({}_RM, {}_RM')$$

is an isomorphism.

(6) The induced functor $\mathscr{D}(\varphi_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories.

Following Geigle and Lenzing, [24], we make the following definition.

Definition 4.3.10. We call a morphism of DGAs $\varphi : R \to S$ a *homological epimorphism of DGAs* if it satisfies the equivalent conditions of Theorem 4.3.9.

Remark 4.3.11. The notion of a homological epimorphism of DGAs has recently been generalised to a homological epimorphism of DG categories by Nicholás and Saorín in [47].

For a morphism $\varphi : R \to S$ of DGAs, if S is finitely built from R as a DG R-module, then we can add an extra condition equivalent to the conditions in Theorem 4.3.9.

Proposition 4.3.12. *Let $\varphi : R \to S$ be a morphism of DGAs. Suppose that S is finitely built from R as a DG R-module. Then the following conditions are equivalent:*

(1) The canonical map ${}_SS_S \to \mathsf{RHom}_R({}_RS_S, {}_RS_S)$ is an isomorphism.

(2) The morphism $\varphi : R \to S$ is a homological epimorphism of DGAs.

Proof: Follows from Proposition 4.3.7. □

Remark 4.3.13. In Theorem 4.3.9 a homological epimorphism of DGAs is characterised by the canonical map $S \overset{\mathsf{L}}{\otimes}_R S \to S$ being an isomorphism. However, the strengthened condition that S is finitely built from ${}_RR$ in $\mathscr{D}(R)$ is required to prove that a homological epimorphism of DGAs can also be characterised by the canonical map $S \to \mathsf{RHom}_R(S, S)$ being an isomorphism. This apparent lack of symmetry arises because the characterisation of a homological epimorphism of DGAs involving left tensor requires the trivial fact that S is finitely built from S_S in $\mathscr{D}(S^{\mathsf{op}})$; whereas, the characterisation with

$\mathsf{RHom}(-,-)$ requires that S is finitely built from ${}_R R$ in $\mathscr{D}(R)$ (c.f. Theorem 4.3.6 and Proposition 4.3.7).

The next proposition gives an example of a homological epimorphism of DGAs.

Proposition 4.3.14. *Suppose $\varphi : R \to S$ is a surjective morphism of DGAs and S is K-flat over R. Then φ is a homological epimorphism of DGAs.*

Proof: Since S is K-flat over R we have

$$ {}_S S_R \overset{L}{\otimes}_R {}_R S_S \cong {}_S S_R \otimes_R {}_R S_S $$

We claim that the multiplication map ${}_S S_R \otimes_R {}_R S_S \to {}_S S_S$ is an isomorphism. To see this we use the proof of [53, Proposition 1] with module replaced by DG module.

Let M be a DG S-S-bimodule and note that M acquires the structure ${}_R M_R$ via S viewed through φ. We shall denote the action of $r \in R$ on $m \in M$ by $r.m$ and use the usual multiplicative notation sm for $s \in S$ and $m \in M$. We will show that the following sets of elements of M coincide:

$$ \{m \in M : r.m = m.r \text{ for all } r \in R\} = \{m \in M : sm = ms \text{ for all } s \in S\}. $$

Suppose, for a contradiction, that the sets do not coincide, that is: there exists $m \in M$ such that $r.m = m.r$ for all $r \in R$ but there is some $s \in S$ such that $sm \neq ms$. Define a DGA $S \oplus M$ with multiplication $(s_1, m_1)(s_2, m_2) := (s_1 s_2, s_1 m_2 + m_1 s_2)$. Let $\psi_1, \psi_2 : S \to S \oplus M$ be given by

$$ s \mapsto (s, 0) \text{ and } s \mapsto (s, sm - ms), $$

respectively. Then $\psi_1 \varphi = \psi_2 \varphi$, but $\psi_1 \neq \psi_2$ and thus φ is not surjective, giving a contradiction. Hence the two sets do coincide.

Take $M = {}_S S_R \otimes_R {}_R S_S$. The element $1 \otimes 1$ satisfies $r.(1 \otimes 1) = (1 \otimes 1).r$ so we get $1 \otimes s = s \otimes 1$, and it follows that the multiplication map ${}_S S_R \otimes_R {}_R S_S \to {}_S S_S$ is an isomorphism, as claimed.

One can show that the following diagram commutes

$$\begin{tikzcd}
{}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R S_S \arrow[r, "\sim"] \arrow[dr, "\text{canonical map}"'] & {}_S S_R \otimes_R {}_R S_S \arrow[d, "\sim"', "\text{multiplication}"] \\
 & {}_S S_S
\end{tikzcd}$$

and thus that the canonical map ${}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R S_S \to {}_S S_S$ is an isomorphism. Hence by Theorem 4.3.9, φ is a homological epimorphism of DGAs. □

4.3.2 Versions for the bounded and finite derived categories

Let R and S be DGAs. Recall the definitions of the bounded and finite derived categories, $\mathscr{D}^b(R)$ and $\mathscr{D}^f(R)$ from Chapter 3 (Definitions 3.5.3 and 3.5.4). In this section we make specialisations of Theorem 4.3.9 to the bounded and finite derived category. The following constitutes a proper generalisation of [24, Theorem 4.4].

Proposition 4.3.15. *Let $\varphi : R \to S$ be a morphism of DGAs. Suppose also that S is bounded as a DG S-module. The following conditions are equivalent:*

(1) The canonical map ${}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R S_S \to {}_S S_S$ is an isomorphism.

(2) For all bounded DG S-modules M the canonical map ${}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R M \to {}_S M$ is an isomorphism.

(3) For all bounded DG S^{op}-modules M and all bounded DG S-modules N the canonical map $M_R \overset{\mathsf{L}}{\otimes}_R {}_R N \to M_S \overset{\mathsf{L}}{\otimes}_S {}_S N$ is an isomorphism.

(4) For all bounded DG S-modules M the canonical map

$$ {}_S M \to \mathsf{RHom}_R({}_R S_S, {}_R M)$$

is an isomorphism.

(5) For all bounded DG S-modules M, M' the canonical map

$$\mathsf{RHom}_S({}_S M, {}_S M') \to \mathsf{RHom}_R({}_R M, {}_R M')$$

is an isomorphism.

(6) The induced functor $\mathscr{D}^b(\varphi_*) : \mathscr{D}^b(S) \to \mathscr{D}^b(R)$ is a full embedding of bounded derived categories.

(7) The morphism $\varphi : R \to S$ is a homological epimorphism of DGAs.

Proof: The proof of Theorem 4.3.6 applies here with the appropriate modifications. Note that the hypothesis that S is homologically bounded as a DG S-module is required in the proof of the implication $(3) \implies (1)$. □

Before we can state a version of Theorem 4.3.9 for the finite derived category we require a lemma. Recall the definition of a homologically finitely presented DG R-module and the finite derived category $\mathscr{D}^f(R)$ from Definition 3.5.4.

Lemma 4.3.16. *Let A and B be rings and suppose $\varphi : A \to B$ is a homomorphism of rings. Suppose B is finitely presented as a left A-module. Then, if M is finitely presented as a left B-module, M is also finitely presented as a left A-module.*

Proof: Suppose M is finitely presented as a left B-module. Then there is an exact sequence $B^n \to B^m \to M \to 0$. Since B is finitely presented as a left A-module there is an exact sequence $A^k \to A^l \to B \to 0$. It follows that there are exact sequences $A^c \to A^a \to B^n \to 0$ and $A^d \to A^b \to B^m \to 0$, where a, c and b, d are suitably chosen cardinals corresponding to n and m, respectively. These then give the following diagram:

$$\begin{tikzcd}
 & B^n \arrow[rr] & & B^m \arrow[r, two heads] & M \arrow[r] & 0 \\
 & A^a \arrow[u, two heads] \arrow[rr, dashed] & & A^b \arrow[u, two heads] \arrow[ur, dashed] & & \\
K \arrow[ur, hook] \arrow[rr, dashed] & & L \arrow[ur, hook] & & & \\
 & A^c \arrow[ul, two heads] \arrow[uu] \arrow[rr, dashed] & & A^d \arrow[ul, two heads] \arrow[uu] & &
\end{tikzcd}$$

where the broken arrows are given by projectivity. We can now deduce the following exact sequence by a diagram chase $A^a \oplus A^d \to A^b \to M \to 0$, so that M is finitely presented as a left A-module. □

Corollary 4.3.17. *Let R and S be DGAs and suppose $\varphi : R \to S$ is a morphism of DGAs. Suppose S is homologically finitely presented as a DG R-module, that is, an object of the finite derived category $\mathscr{D}^f(R)$. Then, if M is finitely presented as a DG S-module, M is also finitely presented as a DG R-module.*

Proof: Suppose that M is finitely presented as a DG left S-module. Then $H_i(M)$ is finitely presented as a $H_0(S)$-module for each $i \in \mathbb{Z}$. Since S is finitely presented as a DG left R-module, $H_i(S)$ is finitely presented as a $H_0(R)$-module. The result now follows from Lemma 4.3.16. □

Proposition 4.3.18. *Let R and S be DGAs and let $\varphi : R \to S$ be a morphism of DGAs. Suppose also that S is homologically bounded and finitely presented as a DG R-module, that is, an object of the finite derived category $\mathscr{D}^f(R)$. The following conditions are equivalent:*

(1) The canonical map ${}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R S_S \to {}_S S_S$ is an isomorphism.

(2) For all bounded, finitely presented DG S-modules M the canonical map ${}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R M \to {}_S M$ is an isomorphism.

(3) For all bounded, finitely presented DG S^{op}-modules M and all bounded, finitely presented DG S-modules N the canonical map $M_R \overset{\mathsf{L}}{\otimes}_R {}_R N \to M_S \overset{\mathsf{L}}{\otimes}_S {}_S N$ is an isomorphism.

(4) For all bounded, finitely presented DG S-modules M the canonical map ${}_S M \to \mathsf{RHom}_R({}_R S_S, {}_R M)$ is an isomorphism.

(5) For all bounded, finitely presented DG S-modules M, M' the canonical map

$$\mathsf{RHom}_S({}_S M, {}_S M') \to \mathsf{RHom}_R({}_R M, {}_R M')$$

is an isomorphism.

(6) The induced functor $\mathcal{D}^f(\varphi_*) : \mathcal{D}^f(S) \to \mathcal{D}^f(R)$ is a full embedding of finite derived categories.

(7) The morphism $\varphi : R \to S$ is a homological epimorphism of DGAs.

Proof: The proof of Theorem 4.3.6 again goes through with the appropriate modifications. Note that in Corollary 4.3.17 we only needed to obtain that $H_0(S)$ was finitely presented as a left $H_0(R)$-module; this is again required in the proof of the implication $(3) \implies (1)$. $\square$

4.4 Examples

We shall now consider two examples of the results in Section 4.3. The first example is in relation to Dwyer and Greenlees's Morita theory, and the second example is in relation to Geigle and Lenzing's homological epimorphisms of rings.

4.4.1 Dwyer and Greenlees' Morita theory

The following is taken from Jørgensen [34] which in turn is taken from Dwyer and Greenlees [19] with the trivial alteration of replacing ring by DGA.

Setup 4.4.1. Let R be a DGA and suppose M is a K-projective DG R-module which is finitely built from R in $\mathscr{D}(R)$. Let $\mathscr{F} = \mathsf{Hom}_R({}_RM, {}_RM)$ be the endomorphism DGA of M, and note that M acquires a left $\mathscr{F}$-structure compatible with its left R-structure, ${}_{R,\mathscr{F}}M$. DG right R-modules can be canonically identified with DG left R^{op}-modules (see Chapter 3), so, similarly, a DG left R-module can be canonically identified with a DG right R^{op}-module. Hence, the left $\mathscr{F}$-structure of M can be regarded as a right $\mathscr{F}^{\mathsf{op}}$-structure, giving M a DG R-$\mathscr{F}^{\mathsf{op}}$-structure, ${}_RM_{\mathscr{F}^{\mathsf{op}}}$. Note also that, by comparing multiplications, ${}_{\mathscr{F}^{\mathsf{op}}}\mathscr{F}_{\mathscr{F}^{\mathsf{op}}}$ and ${}_{\mathscr{F}^{\mathsf{op}}}(\mathscr{F}^{\mathsf{op}})_{\mathscr{F}^{\mathsf{op}}}$ are isomorphic as DG $\mathscr{F}^{\mathsf{op}}$-$\mathscr{F}^{\mathsf{op}}$-bimodules.

In the notation of Theorem 4.3.6, the opposite endomorphism DGA $\mathscr{F}^{\mathsf{op}}$ plays the role of S while the DGA R and the DG R-$\mathscr{F}^{\mathsf{op}}$-bimodule M above play the same roles as the DGA R and the DG R-S-bimodule M in Theorem 4.3.6.

Observe that

$$\begin{aligned}\mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, -) &\cong \mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, {}_RR_R \overset{\mathsf{L}}{\otimes}_R -)\\ &\cong \mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, {}_RR_R) \overset{\mathsf{L}}{\otimes}_R -,\end{aligned}$$

where the second isomorphism follows by the fact that M is finitely built from R in $\mathscr{D}(R)$ (see Proposition 4.1.6 and Corollary 4.2.4). Setting ${}_{\mathscr{F}^{\mathsf{op}}}Z_R = \mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, {}_RR_R)$ we notice that

$$\mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, -) \cong {}_{\mathscr{F}^{\mathsf{op}}}Z_R \overset{\mathsf{L}}{\otimes}_R -. \tag{4.18}$$

This gives the following picture:

$$\begin{tikzcd}
\mathscr{D}(R) \arrow[r, "j^*"] & \mathscr{D}(\mathscr{F}^{\mathsf{op}}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

where the functors are given by

$$\begin{aligned}j_!(-) &= {}_RM_{\mathscr{F}^{\mathsf{op}}} \overset{\mathsf{L}}{\otimes}_{\mathscr{F}^{\mathsf{op}}} -\\ j^*(-) &= \mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, -) \cong {}_{\mathscr{F}^{\mathsf{op}}}Z_R \overset{\mathsf{L}}{\otimes}_R -\\ j_*(-) &= \mathsf{RHom}_{\mathscr{F}^{\mathsf{op}}}({}_{\mathscr{F}^{\mathsf{op}}}Z_R, -),\end{aligned}$$

where $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs (the right adjoint written on the right in each ordered pair).

The following result was established in [19] and an alternative proof can be found in [34]. Here we see the result is an immediate consequence of Proposition 4.3.7.

Proposition 4.4.2. *In the setup above the functor* $j_!(-) : \mathscr{D}(\mathscr{F}^{\mathsf{op}}) \to \mathscr{D}(R)$ *is a full embedding of derived categories.*

Proof: Since M is a K-projective DG left R-module we have that

$$_{\mathscr{F}^{\mathsf{op}}}(\mathscr{F}^{\mathsf{op}})_{\mathscr{F}^{\mathsf{op}}} = \mathsf{Hom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, {}_RM_{\mathscr{F}^{\mathsf{op}}}) \xrightarrow{\sim} \mathsf{RHom}_R({}_RM_{\mathscr{F}^{\mathsf{op}}}, {}_RM_{\mathscr{F}^{\mathsf{op}}}).$$

Since M is finitely built from R in $\mathscr{D}(R)$ it follows by Proposition 4.3.7 that the functor $j_!(-) = {}_RM_{\mathscr{F}^{\mathsf{op}}} \overset{\mathsf{L}}{\otimes}_{\mathscr{F}^{\mathsf{op}}} - : \mathscr{D}(\mathscr{F}^{\mathsf{op}}) \to \mathscr{D}(R)$ is a full embedding of derived categories. It is clear that the composite is canonical. □

4.4.2 Homological epimorphisms of rings

We noted in section 4.3 that Theorem 4.3.6 was a generalisation of Geigle and Lenzing [24, Theorem 4.4]. Below we describe how to obtain [24, Theorem 4.4] from Theorem 4.3.6.

Let R be a ring and let $\mathscr{D}(\mathsf{Mod}(R))$ be the derived category of $\mathsf{Mod}(R)$. Recall from Theorem 3.3.8 and Definition 3.5.7 that

$$\begin{aligned} H^i(\mathsf{RHom}_R(X,Y)) &\cong \mathsf{Ext}^i_R(X,Y), \\ H_i(X \overset{\mathsf{L}}{\otimes}_R Y) &\cong \mathsf{Tor}^R_i(X,Y) \end{aligned}$$

for two complexes X and Y of R-modules. Moreover, when X and Y are complexes of R-modules concentrated in degree zero the i^{th}-hyperext (resp. i^{th}-hypertor) coincides with the usual Ext (resp. Tor).

Remark 4.4.3. A ring R can be considered as a DGA concentrated in degree zero, which we will also denote by R. Moreover, a DG module over R as a DGA is just a complex of R-modules, thus the derived category $\mathscr{D}(\mathsf{Mod}(R))$ of $\mathsf{Mod}(R)$ with R viewed as a ring and the derived category $\mathscr{D}(R)$ with R viewed as a DGA concentrated in degree zero coincide; c.f. Examples 3.4.2 and 3.4.7.

Setup 4.4.4. Let R and S be rings and suppose that $\varphi : R \to S$ is a homomorphism of rings. Regard R and S as DGAs concentrated in degree zero; φ then becomes a morphism of DGAs. Each DG S-module can be viewed as a DG R-module via φ. In particular, set ${}_RM_S = {}_RS_S$. Note that ${}_RS_S$ is a DG R-S-bimodule which is finitely built from S_S in $\mathscr{D}(S^{\mathrm{o}})$. In the notation of Theorem 4.3.6,

$${}_SZ_R = \mathsf{RHom}_{S^{\mathrm{op}}}({}_RS_S, {}_SS_S) \cong {}_SS_R,$$

so, for instance, in this setup, condition (1) of Theorem 4.3.6 now says that the canonical map $S \overset{\mathsf{L}}{\otimes}_R S \to S$ is an isomorphism, c.f. Theorem 4.3.9

We would like to apply Theorem 4.3.6 to the situation of Setup 4.4.4. In the following lemmas we consider the translations of the conditions appearing in Theorem 4.3.9 to those of [24, Theorem 4.4]. The lemmas are stated in the order in which the conditions appear in the theorems.

Lemma 4.4.5. *Suppose we are in the situation of Setup 4.4.4. The following conditions are equivalent:*

- *(i) The canonical map $S \overset{\mathsf{L}}{\otimes}_R S \to S$ is an isomorphism.*
- *(ii) The multiplication map $S \otimes_R S \to S$ is an isomorphism and $\mathsf{Tor}_i^R(S,S) = 0$ for all $i \geqslant 1$.*

Proof: The implication (i) $\Longrightarrow$ (ii) is proved analogously to the corresponding implication in Lemma 4.4.6 below. Conversely, suppose the multiplication map $S \otimes_R S \to S$ is an isomorphism and $\mathrm{Tor}_i^R(S,S) = 0$ for $i \geqslant 1$. Since S is concentrated in degree zero, $S \overset{\mathsf{L}}{\otimes}_R S$ satisfies $H_i(S \overset{\mathsf{L}}{\otimes}_R S) = 0$ for all $i \leqslant -1$. For $i \geqslant 1$ we have:

$$H_i(S \overset{\mathsf{L}}{\otimes}_R S) \cong \mathrm{Tor}_i^R(S,S) = 0 = H_i(S).$$

For $i = 0$, we obtain

$$H_0(S \overset{\mathsf{L}}{\otimes}_R S) \cong S \otimes_R S \cong S \cong H_0(S).$$

Thus $S \overset{\mathsf{L}}{\otimes}_R S \to S$ is a quasi-isomorphism. $S \overset{\mathsf{L}}{\otimes}_R S$ and S are objects in a derived category, thus $S \overset{\mathsf{L}}{\otimes}_R S \to S$ is an isomorphism. One can again check that this is the canonical map. $\square$

Lemma 4.4.6. *Suppose we are in the situation of Setup 4.4.4. The following conditions are equivalent:*

(i) For all bounded DG S-modules M the canonical map $S \overset{\mathsf{L}}{\otimes}_R M \to M$ is an isomorphism.

(ii) For all S-modules M the multiplication map $S \otimes_R M \to M$ is an isomorphism and $\mathsf{Tor}_i^R(S, M) = 0$ for all $i \geqslant 1$.

Proof: Suppose that condition (i) holds. Let M be an S-module. Then, by Remark 4.4.3, M can be considered as a DG S-module concentrated in degree zero (where we are also considering S as a DGA concentrated in degree zero). Hence the canonical map $S \overset{\mathsf{L}}{\otimes}_R M \to M$ is an isomorphism. We have $H_i(S \overset{\mathsf{L}}{\otimes}_R M) \cong \mathsf{Tor}_i^R(S, M)$. Since M is concentrated in degree zero, we have,

$$\mathsf{Tor}_i^R(S, M) \cong H_i(S \overset{\mathsf{L}}{\otimes}_R M) \cong H_i(M) = 0$$

for $i \neq 0$. For $i = 0$ we have

$$S \otimes_R M \cong H_0(S \overset{\mathsf{L}}{\otimes}_R M) \cong H_0(M) = M.$$

Using the ad hoc methods outlined in sections 4.2.2 and 4.2.3, one can verify that the composition arising from the canonical map yields the multiplication map.

Conversely, suppose that for all S-modules M the multiplication map $S \otimes_R M \to M$ is an isomorphism and $\mathsf{Tor}_i^R(S, M) = 0$ for $i \geqslant 1$. Let M be a bounded DG S-module, then, by Remark 4.4.3, M is a bounded complex of S-modules. For each S-module M_j occuring in the complex M we have $\mathsf{Tor}_i^R(S, M_j) = 0$ for $i \geqslant 1$. Hence, M is a complex of modules which are acyclic with respect to the functor ${}_S S_R \otimes_R -$. Since M is bounded as a DG S-module it is a right bounded complex of acylic modules with respect to ${}_S S_R \otimes_R -$, as such it follows that

$${}_S S_R \overset{\mathsf{L}}{\otimes}_R {}_R M \cong {}_S S_R \otimes_R {}_R M \cong {}_S M,$$

where the second isomorphism comes from a degreewise application of the first condition that the multiplication map $S \otimes_R M \to M$ is an isomorphism for all S-modules M. The first isomorphism comes from [26]. Hence condition (ii) implies condition (i). □

Lemma 4.4.7. *Suppose we are in the situation of Setup 4.4.4. The following conditions are equivalent:*

(i) For all bounded DG S^{op}-modules M and bounded DG S-modules N the canonical map $M \overset{\mathsf{L}}{\otimes}_R N \to M \overset{\mathsf{L}}{\otimes}_S N$ is an isomorphism.

(ii) For all S^{op}-modules M and S-modules N the natural map

$$\mathsf{Tor}_i^R(M, N) \to \mathsf{Tor}_i^S(M, N)$$

is an isomorphism for all $i \geqslant 0$.

Proof: The proof is analogous to the proof of Lemma 4.4.9 given below. □

Lemma 4.4.8. *Suppose we are in the situation of Setup 4.4.4. The following conditions are equivalent:*

(i) For all bounded DG S-modules M the canonical map $M \to \mathsf{RHom}_R(S, M)$ is an isomorphism.

(ii) For all S-modules M the natural map $M \to \mathsf{Hom}_R(S, M)$ is an isomorphism and $\mathsf{Ext}_R^i(S, M) = 0$ for all $i \geqslant 1$.

Proof: The proof is analogous to the proof of Lemma 4.4.6 given above. □

Lemma 4.4.9. *Suppose we are in the situation of Setup 4.4.4. The following conditions are equivalent:*

(i) For all bounded DG S-modules M and M' the canonical map $\mathsf{RHom}_S(M, M') \to \mathsf{RHom}_R(M, M')$ is an isomorphism.

(ii) For all S-modules M and M' the natural map $\mathsf{Ext}_S^i(M, M') \to \mathsf{Ext}_R^i(M, M')$ is an isomorphism for all $i \geqslant 0$.

Proof: For left S-modules M and M' we have the following commutative diagram:

$$\begin{tikzcd}
H^i(\mathsf{RHom}_S(M, M')) \arrow[r, "\sim"] \arrow[d, "\sim"'] & H^i(\mathsf{RHom}_R(M, M')) \arrow[d, "\sim"'] \\
\mathsf{Ext}_i^S(M, M') \arrow[r, "\text{natural}", "\sim"'] & \mathsf{Ext}_i^R(M, M').
\end{tikzcd}$$

Thus it follows that if the canonical map $\mathsf{RHom}_S(M, M') \to \mathsf{RHom}_R(M, M')$ is an isomorphism then so is the natural map $\mathsf{Ext}_i^S(M, M') \to \mathsf{Ext}_i^R(M, M')$.

Conversely, suppose that for all S-modules M and M' the natural map $\mathsf{Ext}^i_S(M, M') \to \mathsf{Ext}^i_R(M, M')$ is an isomorphism for all $i \geqslant 0$. In particular, for all S-modules the natural map $\mathsf{Hom}_S(M, M') \to \mathsf{Hom}_R(M, M')$ is an isomorphism. Now suppose we have two bounded DG S-modules M and M'. These are just complexes of S-modules; without loss of generality we may assume that they are concentrated in non-negative degree (and bounded to the right):

$$\begin{aligned} M: &\qquad \cdots \to 0 \to M_0 \to M_1 \to M_2 \to M_3 \to \cdots , \\ M': &\qquad \cdots \to 0 \to M'_0 \to M'_1 \to M'_2 \to M'_3 \to \cdots , \end{aligned}$$

where M_i and M'_j are S-modules for all i and j. By assumption $\mathsf{Hom}_S(M_i, M'_j) \cong \mathsf{Hom}_R(M_i, M'_j)$ for all i and j. Now the hom complex $\mathsf{Hom}_S(M, M')$ for M and M' as DG S-modules is given by

$$(\mathsf{Hom}_S(M, M'))_p = \prod_{i \geqslant 0} \mathsf{Hom}_S(M_i, M'_{i+p}),$$

where the hom groups on the right hand side are of M_i and M'_{i+p} as S-modules (see Definition 3.3.5 for details on how to define the hom complex and its differential). It follows that

$$(\mathsf{Hom}_S(M, M'))_p = \prod_{i \geqslant 0} \mathsf{Hom}_S(M_i, M'_{i+p}) \cong \prod_{i \geqslant 0} \mathsf{Hom}_R(M_i, M'_{i+p}) = (\mathsf{Hom}_R(M, M'))_p,$$

where the hom groups in the central two terms are of M_i and M'_{i+p} as S-modules and R-modules, respectively. The isomorphism above holds because M and M' are bounded complexes of S-modules (resp. R-modules). It is easy to check that the resulting diagram of hom complexes is commutative and therefore we obtain an isomorphism of complexes,

$$\mathsf{Hom}_S(M, M') \cong \mathsf{Hom}_R(M, M').$$

It then follows that $\mathsf{RHom}_S(M, M') \cong \mathsf{RHom}_R(M, M')$; one can then check that it is the canonical map, completing the proof. □

In the situation of Setup 4.4.4, using these translation lemmas, we obtain:

Theorem 4.4.10 ([24], Theorem 4.4)**.** *For a homomorphism of rings $\varphi : R \to S$ the following conditions are equivalent:*

(1) The multiplication map ${}_SS_R \otimes_R {}_RS_S \to {}_SS_S$ is an isomorphism and $\mathsf{Tor}_i^R(S,S) = 0$ for all $i \geqslant 1$.

(2) For all left S-modules M the multiplication map ${}_SS_R \otimes_R {}_RM \to {}_SM$ is an isomorphism and $\mathsf{Tor}_i^R(S,M) = 0$ for all $i \geqslant 1$.

(3) For all right S-modules M and all left S-modules N the natural map

$$\mathsf{Tor}_i^R(M,N) \to \mathsf{Tor}_i^S(M,N)$$

is an isomorphism for all $i \geqslant 0$.

(4) For all left S-modules M the natural map ${}_SM \to \mathsf{Hom}_R({}_RS_S, {}_RM)$ is an isomorphism and $\mathsf{Ext}_R^i(S,M) = 0$ for all $i \geqslant 1$.

(5) For all left S-modules M and M' the natural map

$$\mathsf{Ext}_S^i({}_SM, {}_SM') \to \mathsf{Ext}_R^i({}_RM, {}_RM')$$

is an isomorphism for all $i \geqslant 0$.

(6) The induced functor $\mathscr{D}(\varphi_*) : \mathscr{D}(\mathsf{Mod}(S)) \to \mathscr{D}(\mathsf{Mod}(R))$ is a full embedding of derived categories.

Proof: By applying Lemmas 4.4.5, 4.4.6, 4.4.7, 4.4.8 and 4.4.9 to conditions (1) to (5), respectively, we obtain conditions (1) to (5) of Proposition 4.3.15. Now, S is considered as a DGA concentrated in degree zero, thus Proposition 4.3.15 applies, hence φ is a homological epimorphisms of DGAs (concentrated in degree zero). Now, by Theorem 4.3.9, it follows that the induced functor $\mathscr{D}(\varphi_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is a full embedding of derived categories. Note, by Remark 4.4.3, since S and R are rings considered as DGAs concentrated in degree zero, the derived categories $\mathscr{D}(S)$ and $\mathscr{D}(\mathsf{Mod}(S))$ and $\mathscr{D}(R)$ and $\mathscr{D}(\mathsf{Mod}(R))$ coincide, respectively. Hence we obtain the equivalence of condition (6) above with conditions (1) to (5). $\square$

It is clear that the corresponding theorem holds for right S-modules, and the symmetric nature of condition (1) shows that the statements for right S-modules are equivalent to those for left S-modules.

Remark 4.4.11. Theorem 4.4.10 as stated above is a strengthening of [24, Theorem 4.4]. Geigle and Lenzing's theorem is stated in terms of the bounded derived category whereas we are able to obtain the result for the full derived category. The abstract machinery used in Theorem 4.3.6 allows us to work with the full derived category and specialise to Theorem 4.3.9. A further specialisation to bounded DGAs in Proposition 4.3.15 constitutes a proper generalisation of Geigle and Lenzing's characterisation, indeed, when R and S are the trivial DGAs consisting of rings concentrated in degree zero, the translation lemmas above show that the conditions of [24, Theorem 4.4] and Proposition 4.3.15 are the same. Condition (1) then allows the extension to the full derived category.

4.4.3 Characterisation for the co-induction functor

In section 4.3, we saw how the characterisation of homological epimorphisms of DGAs is in fact a characterisation of when a derived restriction of scalars functor yields a full embedding of derived categories. Analogous questions may be asked of the extension of scalars functor and the co-induction functor. The following proposition constitutes a characterisation of when the co-induction functor induces a full embedding of derived categories and is an easy consequence of Theorem 4.3.6.

Proposition 4.4.12. *Let R and S be DGAs and M a DG R-S-bimodule. Suppose also that M is finitely built from R in $\mathscr{D}(R)$ and let $Z = \mathsf{RHom}_R(M, R)$ be an R-dual of M so that Z acquires the structure ${}_SZ_R$. Then the following conditions are equivalent:*

(1) The canonical map ${}_RM_S \overset{\mathsf{L}}{\otimes}_S {}_SZ_R \to {}_RR_R$ is an isomorphism.

(2) For all DG R-modules N the canonical map ${}_RM_S \overset{\mathsf{L}}{\otimes}_S \mathsf{RHom}_R({}_RM_S, {}_RN) \to {}_RN$ is an isomorphism.

(3) For all DG R^{op}-modules N and DG R-modules N' the canonical map

$$\mathsf{RHom}_{R^{\mathsf{op}}}({}_SZ_R, N_R) \overset{\mathsf{L}}{\otimes}_S \mathsf{RHom}_R({}_RM_S, {}_RN') \to N_R \overset{\mathsf{L}}{\otimes}_R {}_RN'$$

is an isomorphism.

(4) For all DG R-modules N the canonical map

$$\mathsf{RHom}_S(\mathsf{RHom}_R({}_RM_S, {}_RR_R), \mathsf{RHom}_R({}_RM_S, {}_RN)) \to {}_RN$$

is an isomorphism.

(5) For all DG R-modules N and N' the canonical map

$$\mathsf{RHom}_R({}_RN, {}_RN') \to \mathsf{RHom}_S(\mathsf{RHom}_R({}_RM_S, {}_RN), \mathsf{RHom}_R({}_RM_S, {}_RN'))$$

is an isomorphism.

(6) The functor $\mathsf{RHom}_R({}_RM_S, -) : \mathscr{D}(R) \to \mathscr{D}(S)$ is a full embedding of derived categories.

Proof: Above, the roles of R and S are interchanged. However since M is finitely built from R in $\mathscr{D}(R)$, by Remark 4.2.4 we can rewrite the the co-induction functor as

$$\mathsf{RHom}_R({}_RM_S, -) \cong - \overset{\mathsf{L}}{\otimes}_S {}_SZ_R.$$

Applying this to conditions (1) to (6) above yields the opposite version of Theorem 4.3.6 and thus the characterisation follows. □

Chapter 5

Corigid Objects in Triangulated Categories and Co-t-structures

Suppose $\mathscr{T}$ is a triangulated category with set indexed coproducts and suspension functor $\Sigma : \mathscr{T} \to \mathscr{T}$ (see Chapter 3). In [31] a canonical t-structure (Definition 5.1.4) is induced on $\mathscr{T}$ by a compact rigid object S of $\mathscr{T}$ which satisfies that $\{\Sigma^i S \,|\, i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$. A detailed explanation of the terminology can be found in sections 5.1 and 5.2. The two halves of this induced t-structure are given by

$$\begin{aligned}\mathscr{X} &= \{X \in \mathscr{T} \,|\, \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\},\\ \mathscr{Y} &= \{X \in \mathscr{T} \,|\, \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}.\end{aligned}$$

This situation bears resemblance to the example of a chain DGA in its derived category. Recall that a DGA R is called a *chain DGA* if $H^i(R) = 0$ for all $i > 0$. Moreover, if M is a DG R-module then

$$H^i(M) \cong \mathsf{Hom}_{\mathscr{D}(R)}(R, \Sigma^i M) \text{ for all } i \in \mathbb{Z}.$$

Hence, the rigid object S considered by Hoshino, Kato and Miyachi in [31] behaves in a manner analogous to a chain DGA (see Definition 5.2.4 and Remark 5.2.5); the two halves of the natural t-structure which is induced are analogous to the full subcategories of $\mathscr{D}(R)$ which consist of DG R-modules whose cohomology vanishes in positive and negative degrees, respectively.

In the theory of DGAs, when one has a construction for chain DGAs it is natural to ask: what is the dual construction for cochain DGAs? As such, it would also be natural to ask what is the dual structure analogous to the construction in [31] when one considers an object which behaves like a cochain DGA.

Unfortunately, it is well-known in the theory of DGAs that constructing a dual theory for cochain DGAs is often difficult: at present, the construction of a viable theory for cochain DGAs often requires the additional assumption that R is simply connected in the sense that $H^0(R)$ is a division ring and $H^1(R) = 0$. The terminology of simply connectedness here is motivated by algebraic topology. With this lack of symmetry taken into account, we consider the case of an object of a triangulated category which behaves like a simply connected cochain DGA.

The structure which we obtain from such an object of $\mathscr{T}$ is not a t-structure on $\mathscr{T}$ but a structure which is almost dual to that of a t-structure, which we call a co-t-structure (Definition 5.1.5). Both t-structures and co-t-structures provide examples of torsion theories in triangulated categories in the sense of Iyama and Yoshino, [32]. Co-t-structures have also been introduced by Bondarko in [11] where they are called weight structures. In [11] they are studied in connection with the theory of motives and stable homotopy theory.

In the next section we introduce some of the basic definitions, terminology and notation which we shall use throughout this chapter. Throughout this chapter $\mathscr{T}$ will be a triangulated category with set indexed coproducts and suspension functor $\Sigma : \mathscr{T} \to \mathscr{T}$. See Chapter 3 for the definition of a triangulated category and the basic properties of triangulated categories.

The results appearing in this chapter are an expansion of those presented in the paper [48].

5.1 The definition of a co-t-structure

We start by recalling the concept of a preenvelope and the notation of perpendicular categories; then we give the definition of a t-structure on a triangulated category and introduce the definition of a co-t-structure and some of its basic properties.

5.1.1 Preenvelopes and perpendicular categories

We recall the following definitions from [20].

Definition 5.1.1. Let $\mathscr{F}$ be a full subcategory of a category $\mathscr{C}$ and suppose X is an object of $\mathscr{C}$. A morphism $\varphi : X \to F$ with $F \in \mathscr{F}$ is called an *$\mathscr{F}$-preenvelope* if for each morphism $X \to F'$ with $F' \in \mathscr{F}$ there exists a morphism $F \to F'$ making the following triangle commute.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr] & F \arrow[d, dashed, "\exists"] \\
& F'
\end{tikzcd}$$

Definition 5.1.2. Let $\mathscr{F}$ be a full subcategory of a category $\mathscr{C}$ and suppose X is an object of $\mathscr{C}$. A morphism $\varphi : F \to X$ with $F \in \mathscr{F}$ is called an *$\mathscr{F}$-precover* if for each morphism $F' \to X$ with $F' \in \mathscr{F}$ there exists a morphism $F' \to F$ making the following triangle commute.

$$\begin{tikzcd}
F \arrow[r, "\varphi"] & X \\
F' \arrow[u, dashed, "\exists"] \arrow[ur] &
\end{tikzcd}$$

In [3] and [4], the terms $\mathscr{F}$-preenvelopes and $\mathscr{F}$-precovers are referred to as *left $\mathscr{F}$-approximations* and *right $\mathscr{F}$-approximations*, respectively.

Definition 5.1.3. Let $\mathscr{S}$ be a set of objects of a triangulated category $\mathscr{T}$. The *subcategory right n-perpendicular* to $\mathscr{S}$, denoted by $\mathscr{S}^{\perp_n}$, is given by:

$$\mathscr{S}^{\perp_n} := \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for all } S \in \mathscr{S} \text{ and for all } i = 1, \dots, n\}.$$

The *subcategory right ∞-perpendicular* to $\mathscr{S}$, denoted by $\mathscr{S}^{\perp_\infty}$, is given by:

$$\mathscr{S}^{\perp_\infty} := \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for all } S \in \mathscr{S} \text{ and for all } i > 0\}.$$

If $\mathscr{S} = \{S\}$ is a set consisting of just one object of $\mathscr{T}$, we write $S^{\perp_n}$ and $S^{\perp_\infty}$ for the subcategories right n-perpendicular and right ∞-perpendicular to $\{S\}$, respectively.

For a subcategory $\mathscr{S}$ of $\mathscr{T}$, we define:

$$\begin{aligned}
\mathscr{S}^{\perp} &:= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, X) = 0 \text{ for all } S \in \mathscr{S}\}, \\
{}^{\perp}\mathscr{S} &:= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(X, S) = 0 \text{ for all } S \in \mathscr{S}\}.
\end{aligned}$$

Compare the defintions of n-perpendicular categories to the classical definition of perpendicular category given in [24].

5.1.2 t-structures and co-t-structures

The concept of a t-structure on a triangulated category $\mathscr{T}$ was first introduced by Beilinson, Bernstein and Deligne in [7]. The basic theory of t-structures can be found in [7] and [36].

Definition 5.1.4. Let $\mathscr{T}$ be a triangulated category. A pair of full subcategories of $\mathscr{T}$, $(\mathscr{X}, \mathscr{Y})$, is called a *$t$-structure* on $\mathscr{T}$ if it satisfies the following properties:

(1) $\mathscr{X} \subseteq \Sigma^{-1}\mathscr{X}$ and $\Sigma^{-1}\mathscr{Y} \subseteq \mathscr{Y}$;

(2) $\mathsf{Hom}_{\mathscr{T}}(\mathscr{X}, \Sigma^{-1}\mathscr{Y}) = 0$;

(3) For any object Z of $\mathscr{T}$ there exists a distinguished triangle

$$X \to Z \to \Sigma^{-1}Y \to \Sigma X$$

with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$.

The full subcategories $\mathscr{X}$ and $\mathscr{Y}$ are often denoted by $\mathscr{T}^{t\leqslant 0}$ and $\mathscr{T}^{t\geqslant 0}$, or simply by $\mathscr{T}^{\leqslant 0}$ and $\mathscr{T}^{\geqslant 0}$, respectively; see [11] and [31].

The notion of a t-structure has become widespread in the study of triangulated categories and lends itself particularly well to induction arguments in this setting, see for example [9].

We next introduce the almost dual notion of a co-t-structure. We note that co-t-structures have also recently been introduced by Bondarko in [11] where they are called *weight structures* and are studied in connection with the theory of motives.

Definition 5.1.5. Let $\mathscr{T}$ be a triangulated category. A pair of full subcategories of $\mathscr{T}$, $(\mathscr{A}, \mathscr{B})$, will be called a *co-$t$-structure* on $\mathscr{T}$ if it satisfies the following properties:

(0) $\mathscr{A}$ and $\mathscr{B}$ are closed under direct summands;

(1) $\Sigma^{-1}\mathscr{A} \subseteq \mathscr{A}$ and $\mathscr{B} \subseteq \Sigma^{-1}\mathscr{B}$;

(2) $\mathsf{Hom}_{\mathscr{T}}(\Sigma^{-1}\mathscr{A}, \mathscr{B}) = 0$;

(3) For any object X of $\mathscr{T}$ there exists a distinguished triangle

$$\Sigma^{-1}A \to X \to B \to A$$

with $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

In [11] the full subcategories $\mathscr{A}$ and $\mathscr{B}$ are denoted by $\mathscr{T}^{w\geqslant 0}$ and $\mathscr{T}^{w\leqslant 0}$, respectively.

It is easy to see that $\mathscr{A}$ is closed under direct summands if and only if $\Sigma^{-1}\mathscr{A}$ is closed under direct summands; similarly for $\mathscr{B}$.

One can see that conditions (1), (2) and (3) of Definition 5.1.5 are dual to the corresponding conditions in Defintion 5.1.4. The inclusion of condition (0) in the definition of a co-t-structure is the reason why a co-t-structure is almost dual to the notion of a t-structure rather than simply being its dual. We also note that properties (2) and (3) in Definitions 5.1.4 and 5.1.5 make $(\mathscr{X}, \Sigma^{-1}\mathscr{Y})$ and $(\Sigma^{-1}\mathscr{A}, \mathscr{B})$ into examples of torsion theories in the sense of [32].

The notion of a non-degenerate co-t-structure can be defined in a manner analogous to that of a non-degenerate t-structure.

Definition 5.1.6. A co-t-structure $(\mathscr{A}, \mathscr{B})$ on a triangulated category $\mathscr{T}$ will be called *non-degenerate* if we have

$$\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathscr{A} = \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathscr{B} = \{0\}.$$

Definition 5.1.7. A full subcategory $\mathscr{X}$ of a triangulated category $\mathscr{T}$ is said to be *closed under extensions* if, whenever we have a distinguished triangle

$$X' \to X \to X'' \to \Sigma X'$$

with X' and X'' objects of $\mathscr{X}$, then X is also an object of $\mathscr{X}$.

The following proposition contains some elementary properties of co-t-structures.

Proposition 5.1.8. *Let $\mathscr{T}$ be a triangulated category and suppose $(\mathscr{A}, \mathscr{B})$ is a co-t-structure on $\mathscr{T}$. We have:*

(i) For all objects X of $\mathscr{T}$ there exists a $\Sigma^{-1}\mathscr{A}$-precover $\alpha : \Sigma^{-1}A \to X$.

(ii) For all objects X of $\mathscr{T}$ there exists a $\mathscr{B}$-preenvelope $\beta : X \to B$.

(iii) We have $\Sigma^{-1}\mathscr{A} = {}^{\perp}\mathscr{B}$ and $\mathscr{B} = (\Sigma^{-1}\mathscr{A})^{\perp}$.

(iv) $\mathscr{A}$ is closed under extensions.

(v) $\mathscr{B}$ is closed under extensions.

Proof: (i) and (ii). These properties are immediate consequences of the definition of a co-t-structure: the $\Sigma^{-1}\mathscr{A}$-precover $\alpha : \Sigma^{-1}A \to X$ and the $\mathscr{B}$-preenvelope $\beta : X \to B$ are just the first and second morphisms in the distinguished triangle given by property (3) of Definition 5.1.5.

(iii). This follows by virtue of $(\Sigma^{-1}\mathscr{A}, \mathscr{B})$ being a torsion theory; see [32].

(iv). In order to show (iv) it is sufficient to show $\Sigma^{-1}\mathscr{A}$ is closed under extensions. Suppose we have a distinguished triangle

$$X' \to X \to X'' \to \Sigma X' \tag{5.1}$$

with $X', X'' \in \Sigma^{-1}\mathscr{A}$. Since $(\mathscr{A}, \mathscr{B})$ is a co-t-structure on $\mathscr{T}$ there is a distinguished triangle

$$\Sigma^{-1}A \to X \to B \to A \tag{5.2}$$

with $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Apply the functor $\mathsf{Hom}_{\mathscr{T}}(-, B)$ to distinguished triangle (5.1):

$$\mathsf{Hom}_{\mathscr{T}}(X'', B) \to \mathsf{Hom}_{\mathscr{T}}(X, B) \to \mathsf{Hom}_{\mathscr{T}}(X', B).$$

Now $\mathsf{Hom}_{\mathscr{T}}(X, B) = 0$ because $\mathsf{Hom}_{\mathscr{T}}(X', B) = \mathsf{Hom}_{\mathscr{T}}(X'', B) = 0$. Hence, the map $X \to B$ in distinguished triangle (5.2) is zero. Hence X is a direct summand of $\Sigma^{-1}A$. Since $\Sigma^{-1}\mathscr{A}$ is closed under direct summands, it follows that $X \in \Sigma^{-1}\mathscr{A}$. Hence $\Sigma^{-1}\mathscr{A}$ is closed under extensions.

(v). Is proved in a similar manner to (iv). $\Box$

One sees in Proposition 5.1.8 that preenvelopes and precovers replace the truncation functors associated with t-structures. In order to obtain the equalities of property (iii), and the fact that both halves of the co-t-structure are closed under extensions, we need to assume condition (0) of Definition 5.1.5 which says that both halves of a co-t-structure are closed under direct summands.

Let $(\mathscr{X}, \mathscr{Y})$ be a t-structure on a triangulated category $\mathscr{T}$. The intersection $\mathscr{H} = \mathscr{X} \cap \mathscr{Y}$ of both halves of the t-structure is called the *heart* of the t-structure. It has the useful property that it is an abelian subcategory of $\mathscr{T}$. In particular, the hearts of t-structures provide a means of obtaining abelian categories from triangulated categories. We define an analogous notion for co-t-structures.

Definition 5.1.9. Let $(\mathscr{A}, \mathscr{B})$ be a co-t-structure for a triangulated category $\mathscr{T}$. The intersection $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ will be called the *coheart* of the co-t-structure.

It would be hoped that the coheart of a co-t-structure on $\mathscr{T}$ would be an abelian subcategory of $\mathscr{T}$. Unfortunately, this is not the case, see [11]. However, in this chapter we present an example in which the coheart does turn out to be an abelian category.

5.2 Some examples of t-structures and co-t-structures

5.2.1 A canonical example

The following example is taken from [11]. Let $\mathscr{C}$ be an additive category and let $\mathscr{K}(\mathscr{C})$ be its homotopy category. We claim that the following pair of full subcategories of $\mathscr{K}(\mathscr{C})$ forms a co-t-structure on $\mathscr{T} = \mathscr{K}(\mathscr{C})$. Let

$$\begin{aligned} \mathscr{A} &= \{\text{complexes in } \mathscr{K}(\mathscr{C}) \text{ isomorphic to complexes } C \text{ such that } C^i = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{\text{complexes in } \mathscr{K}(\mathscr{C}) \text{ isomorphic to complexes } C \text{ such that } C^i = 0 \text{ for } i > 0\}. \end{aligned}$$

One can show that $\mathscr{A}$ and $\mathscr{B}$ are closed under direct summands, and that $\Sigma^{-1}\mathscr{A} \subseteq \mathscr{A}$ and $\mathscr{B} \subseteq \Sigma^{-1}\mathscr{B}$. It is also clear that $\mathsf{Hom}_{\mathscr{T}}(\Sigma^{-1}\mathscr{A}, \mathscr{B}) = 0$. We need to show that property (3) of Definition 5.1.5 holds. Suppose X is an object of $\mathscr{K}(\mathscr{C})$:

$$X: \qquad \cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots .$$

Recall from [54, section 0.5] that a short exact sequence of complexes $0 \to A \to B \to C \to 0$ is called *semi-split* if it is split in each degree. (For the definition of a split short exact sequence see [2] or [4], for example.) We obtain the following semi-split short exact sequence of complexes:

$$\begin{tikzcd}
\Sigma^{-1}A: \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d, equal] & X^2 \arrow[r] \arrow[d, equal] & \cdots \\
X: \arrow[d] & \cdots \arrow[r] & X^{-2} \arrow[r] \arrow[d, equal] & X^{-1} \arrow[r] \arrow[d, equal] & X^0 \arrow[r] \arrow[d, equal] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d] & \cdots \\
B: & \cdots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r] & X^0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which gives us a distinguished triangle (see [18, Proposition 1.1.23] for instance),

$$\Sigma^{-1}A \to X \to B \to A$$

in $\mathscr{K}(\mathscr{C})$. Hence $(\mathscr{A},\mathscr{B})$ is a co-t-structure on $\mathscr{T} = \mathscr{K}(\mathscr{C})$. The coheart of this co-t-structures lies in the idempotent completion of $\mathscr{C}$, see [11].

5.2.2 Chain and cochain DGAs

Recall that a DGA R is called a *chain DGA* if $H^i(R) = 0$ for all $i > 0$; similarly, a DGA R is called a *cochain DGA* if $H^i(R) = 0$ for all $i < 0$. A cochain DGA R is called *simply connected* if, in addition, $H^0(R)$ is a division ring and $H^1(R) = 0$.

Remark 5.2.1. This is a slight abuse of terminology: normally a chain DGA is a DGA R for which $R^i = 0$ for $i > 0$. Here we only insist that $H^i(R) = 0$ for $i > 0$. We commit a similar abuse of terminology with the term cochain DGA.

Example 5.2.2. Let R be a chain DGA. Let $\mathscr{D}(R)$ be the derived category of DG R-modules, see [10], and define a pair of subcategories of $\mathscr{D}(R)$ as follows:

$$\begin{aligned} \mathscr{X} &= \{M \in \mathscr{D}(R) \mid H^i(M) = 0 \text{ for } i > 0\}, \\ \mathscr{Y} &= \{M \in \mathscr{D}(R) \mid H^i(M) = 0 \text{ for } i < 0\}. \end{aligned}$$

It is easy to show that the pair $(\mathscr{X},\mathscr{Y})$ forms a t-structure on $\mathscr{D}(R)$. Again, this t-structure is non-degenerate and its heart consists of DG R-modules whose cohomology is concentrated in degree zero.

Example 5.2.3. Let R be a simply connected cochain DGA. Let $\mathscr{D}(R)$ be the derived category of DG R-modules and define a pair of subcategories of $\mathscr{D}(R)$ as follows:

$$\begin{aligned} \mathscr{A} &= \{M \in \mathscr{D}(R) \mid H^i(M) = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{M \in \mathscr{D}(R) \mid H^i(M) = 0 \text{ for } i > 0\}. \end{aligned}$$

It is easy to show that the pair $(\mathscr{A},\mathscr{B})$ forms a co-t-structure on $\mathscr{D}(R)$. As in Example 5.2.2, this co-t-structure is non-degenerate and its coheart consists of DG R-modules whose cohomology sits in degree zero.

Simply connected cochain DGAs arise naturally in algebraic topology as the cochain algebras of simply connected CW-complexes, see [21] and [52], for example.

5.2.3 A t-structure obtained from a rigid object

Example 5.2.2 can be generalised to an arbitrary triangulated category by looking at objects behaving like chain DGAs: such an object is said to be rigid.

Definition 5.2.4. Let $\mathscr{T}$ be an arbitrary triangulated category with set indexed coproducts. An object S of $\mathscr{T}$ is called *rigid* if we have $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i S) = 0$ for $i > 0$.

Remark 5.2.5. Let R be a chain DGA, and recall from the introduction that, given a DG R-module M we have

$$H^i(M) = \mathsf{Hom}_{\mathscr{D}(R)}(R, \Sigma^i M) \text{ for } i \in \mathbb{Z}.$$

Thus, R being a chain DGA means that $\mathsf{Hom}_{\mathscr{D}(R)}(R, \Sigma^i R) = 0$ for $i > 0$. Hence, a DGA R is a chain DGA if an only if it is a rigid object in its derived category $\mathscr{D}(R)$. Note the abuse of terminology mentioned in Remark 5.2.1.

If we replace the chain DGA R with some suitably nice rigid object S of $\mathscr{T}$, the following is a candidate for a t-structure on $\mathscr{T}$:

$$\begin{aligned} \mathscr{X} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}, \\ \mathscr{Y} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}. \end{aligned}$$

The suitably nice conditions we must place on S to obtain this t-structure are that S must be a compact object of $\mathscr{T}$ and the set $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ must be a generating set for $\mathscr{T}$.

Recall the definition of a compact object from Definition 4.1.1. Before stating the theorem in full, we recall the notion of a generating set.

Definition 5.2.6. A set of objects $\mathscr{G}$ in a triangulated category $\mathscr{T}$ is called a *generating set* for $\mathscr{T}$ if given any object X of $\mathscr{T}$ with $\mathsf{Hom}_{\mathscr{T}}(G, X) = 0$ for all objects G of $\mathscr{G}$, we have $X = 0$.

Example 5.2.2 is a special case of the following theorem of Hoshino, Kato and Miyachi, which appears in [31].

Theorem 5.2.7 ([31], Theorem 1.3)**.** *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S is a compact rigid object of $\mathscr{T}$ and assume that $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is* *a generating set for $\mathscr{T}$. Then the following forms a non-degenerate* t*-structure on $\mathscr{T}$:*

$$\begin{aligned} \mathscr{X} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}, \\ \mathscr{Y} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}. \end{aligned}$$

Moreover, its heart $\mathscr{H} = \mathscr{X} \cap \mathscr{Y}$ is an admissible abelian subcategory of $\mathscr{T}$ in the sense of [7], and the functor

$$\mathsf{Hom}_{\mathscr{T}}(S, -) : \mathscr{H} \to \mathrm{Mod}(\mathrm{End}(S)^{\mathsf{op}})$$

is an equivalence of categories.

The t-structure is induced as follows: suppose T is an object of $\mathscr{T}$, a morphism $\alpha : T \to Y$ with $Y \in \mathscr{Y}$ is constructed such that, given any other object $Y' \in \mathscr{Y}$ and a morphism $T \to Y'$, then this morphism factors uniquely through $\alpha : T \to Y$,

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[dr] & Y \arrow[d, dashed, "\exists!"] \\ & Y'. \end{tikzcd}$$

5.3 A co-t-structure obtained from a corigid object

In this chapter we shall look at the structure which is induced by an object behaving like a cochain DGA.

We now make precise what we mean by an object of a triangulated category behaving like a cochain DGA. Following the definition of an n-rigid object in a triangulated category of Iyama and Yoshino in [32], we make the following definitions of an n-corigid object and a corigid object.

Definition 5.3.1. An object S of $\mathscr{T}$ will be called *n-corigid* if we have

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma^i S, S) = 0 \text{ for } 0 < i < n.$$

An object S of $\mathscr{T}$ will be called *corigid* if we have

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma^i S, S) = 0 \text{ for } i > 0.$$

Note that a DGA R is a cochain DGA if and only if it is a corigid object in its derived category $\mathscr{D}(R)$; c.f. Remark 5.2.5. Note Remark 5.2.1.

Definition 5.3.2. Let S be an object of $\mathscr{T}$. We shall call S a *simply connected corigid object* of $\mathscr{T}$ if it satisfies the following assumptions:

(1) S is corigid, that is, $\mathsf{Hom}_{\mathscr{T}}(\Sigma^i S, S) = 0$ for $i > 0$;

(2) $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma S) = 0$;

(3) $\mathsf{End}(S)$ is a division ring.

Note that a DGA R is a simply connected cochain DGA if and only if it is a simply connected corigid object in its derived category $\mathscr{D}(R)$. Note Remark 5.2.1 once again.

The subsequent sections of this chapter are devoted to proving the following theorem, which is the cochain analogue, or dual, of Theorem 5.2.7.

Theorem 5.3.3. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S is a compact simply connected corigid object of $\mathscr{T}$ and assume that $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$. Then the following forms a non-degenerate co-t-structure on $\mathscr{T}$:*

$$\begin{aligned} \mathscr{A} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}. \end{aligned}$$

Moreover, its coheart $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ is an abelian subcategory of $\mathscr{T}$, and the functor

$$\mathsf{Hom}_{\mathscr{T}}(S, -) : \mathscr{C} \to \mathrm{Mod}(\mathrm{End}(S)^{\mathsf{op}})$$

is an equivalence of categories.

Proof: The proof of this theorem will be divided into two parts: Theorem 5.3.12 and Theorem 5.4.5. □

Theorem 5.3.3 is the natural generalisation of Example 5.2.3 in the same way that Theorem 5.2.7 is the natural generalisation of Example 5.2.2.

In section 5.3.1 we shall show that given any object M of $\mathscr{T}$, there exists a morphism $\mu : M \to \bar{M}$ with $\bar{M} \in S^{\perp_\infty}$ such that, given any other object $N \in S^{\perp_\infty}$ and a morphism $M \to N$, then this morphism factors through $\mu : M \to \bar{M}$,

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr] & \bar{M} \arrow[d, dashed, "\exists"] \\ & N. \end{tikzcd}$$

However, the factorisation is not necessarily unique. Thus we obtain an $S^{\perp_\infty}$-preenvelope. Note that in Theorem 5.3.3 above, $\mathscr{B} = S^{\perp_\infty}$. In section 5.3.4 we show that this $S^{\perp_\infty}$-preenvelope induces a non-degenerate co-t-structure on $\mathscr{T}$, while section 5.4 is dedicated to proving that the coheart of this non-degenerate co-t-structure is equivalent to the module category $\mathrm{Mod}(\mathrm{End}(S)^{\mathsf{op}})$, and hence abelian.

Remark 5.3.4. For technical reasons, in the cochain analogue of Theorem 5.2.7 we must also insist that S is simply connected in the sense of Definition 5.3.2. This is due to the lack of symmetry in the theory of chain and cochain DGAs mentioned in the introduction: one is able to construct a theory for chain DGAs, but in order to construct a viable dual theory for cochain DGAs one has to introduce the assumption of simply connectedness; see, for example, [5].

5.3.1 Existence of an $S^{\perp_n}$-preenvelope

In order to obtain an $S^{\perp_\infty}$-preenvelope, we first show how to construct an $S^{\perp_n}$-preenvelope for each $n \in \mathbb{N}$. It is useful to refer to a simply connected n-corigid object of a triangulated category:

Definition 5.3.5. Let S be an object of $\mathscr{T}$. We shall call S a *simply connected n-corigid object* of $\mathscr{T}$ if it satisfies the following assumptions:

(1) S is n-corigid, that is, $\mathsf{Hom}_{\mathscr{T}}(\Sigma^i S, S) = 0$ for $0 < i < n$;

(2) $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma S) = 0$;

(3) $\mathsf{End}(S)$ is a division ring.

Proposition 5.3.6. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S is a compact simply connected $(n+1)$-corigid object of $\mathscr{T}$. Then, for each object M of $\mathscr{T}$ there exists an $S^{\perp_n}$-preenvelope $\mu : M \to \bar{M}$.*

Proof: Let M be an arbitrary object of $\mathscr{T}$. We first construct a chain of objects and morphisms,

$$M = M_0 \xrightarrow{\mu_0} M_1 \xrightarrow{\mu_1} M_2 \xrightarrow{\mu_2} M_3 \xrightarrow{\mu_3} \cdots \xrightarrow{\mu_{n-1}} M_n = \bar{M}$$

with $M_k \in S^{\perp_k}$ for each $k \geqslant 1$, inductively using distinguished triangles. Secondly, we verify that the composite of these maps is an $S^{\perp_n}$-preenvelope.

Write $M = M_0$. Let $n = 1$; we construct an object M_1 and a morphism $\mu_0 : M_0 \to M_1$ such that $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma M_1) = 0$. If $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma M_0) = 0$ then set $M_1 = M_0$ and $\mu_0 = 1_{M_0}$, the identity map on M_0. If not, we can choose a, possibly infinite, coproduct $S^{(m_1)}$ of copies of S and a nonzero morphism $S^{(m_1)} \to \Sigma M_0$ which becomes a surjection under the functor $\mathsf{Hom}_{\mathscr{T}}(S, -)$. Since the endomorphism ring $\mathsf{End}(S)$ is a division ring we can, moreover, choose m_1 so that this morphism becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S, -)$. We now extend this morphism to a distinguished triangle:

$$S^{(m_1)} \to \Sigma M_0 \to \Sigma M_1 \to \Sigma S^{(m_1)}. \tag{5.3}$$

Applying $\mathsf{Hom}_{\mathscr{T}}(S, -)$ to (5.3) gives the exact sequence:

$$\mathsf{Hom}_{\mathscr{T}}(S, S^{(m_1)}) \xrightarrow{\sim} \mathsf{Hom}_{\mathscr{T}}(S, \Sigma M_0) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma M_1) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma S^{(m_1)}).$$

Since $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma S^{(m_1)}) = 0$, we get $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma M_1) = 0$.

Now suppose $k \geqslant 1$ and suppose we have constructed a chain of objects and morphisms

$$M = M_0 \xrightarrow{\mu_0} M_1 \xrightarrow{\mu_1} M_2 \xrightarrow{\mu_2} M_3 \xrightarrow{\mu_3} \cdots \xrightarrow{\mu_{k-1}} M_k$$

with $M_i \in S^{\perp_i}$ for $1 \leqslant i \leqslant k$, and where $\mu_i : M_i \to M_{i+1}$ is either the identity map or sits in a distinguished triangle

$$\Sigma^{-(i+1)} S^{(m_{i+1})} \to M_i \xrightarrow{\mu_i} M_{i+1} \to \Sigma^{-i} S^{(m_{i+1})}.$$

If $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1} M_{k+1}) = 0$ then set $M_{k+1} = M_k$ and take $\mu_k : M_k \to M_{k+1}$ to be the identity map 1_{M_k}. If not, we can choose a, possibly infinite, coproduct $S^{(m_{k+1})}$ of copies of S and a nonzero morphism $S^{(m_{k+1})} \to \Sigma^{k+1} M_k$ which becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S, -)$, and then extend it to a distinguished triangle:

$$S^{(m_{k+1})} \to \Sigma^{k+1} M_k \to \Sigma^{k+1} M_{k+1} \to \Sigma S^{(m_{k+1})}. \tag{5.4}$$

As above, apply the functor $\mathsf{Hom}_{\mathscr{T}}(S, -)$ to (5.4) to yield the long exact sequence:

$$\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1-j} M_k) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1-j} M_{k+1}) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{-j+1} S^{(m_{k+1})}).$$

Since $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1-j}M_k) = 0$ and $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{-j+1}S^{(m_{k+1})}) = 0$ for $j = 2, \ldots, k+1$, it follows that $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i M_{k+1}) = 0$ for $i = 1, \ldots, k-1$. For the case $i = k$, the morphism $S^{(m_{k+1})} \to \Sigma^{k+1}M_k$ is injective under $\mathsf{Hom}_{\mathscr{T}}(S, -)$, so the exact sequence

$$0 \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^k M_{k+1}) \to \mathsf{Hom}_{\mathscr{T}}(S, S^{(m_{k+1})}) \hookrightarrow \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1}M_k)$$

gives $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^k M_{k+1}) = 0$. Similarly, $S^{(m_{k+1})} \to \Sigma^{k+1}M_k$ is also surjective under the functor $\mathsf{Hom}_{\mathscr{T}}(S, -)$, so the exact sequence

$$\mathsf{Hom}_{\mathscr{T}}(S, S^{(m_{k+1})}) \twoheadrightarrow \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1}M_k) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1}M_{k+1}) \to 0$$

gives $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{k+1}M_{k+1}) = 0$. Thus, we obtain

$$\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i M_{k+1}) = 0 \text{ for } i = 1, \ldots, k+1.$$

Hence, inductively we obtain a chain of objects and morphisms of $\mathscr{T}$,

$$M = M_0 \xrightarrow{\mu_0} M_1 \xrightarrow{\mu_1} M_2 \xrightarrow{\mu_2} M_3 \xrightarrow{\mu_3} \cdots \xrightarrow{\mu_{n-1}} M_n, \tag{5.5}$$

where each map $\mu_k : M_k \to M_{k+1}$ is either the identity map or sits in a distinguished triangle

$$\Sigma^{-(k+1)}S^{(m_{k+1})} \to M_k \xrightarrow{\mu_k} M_{k+1} \to \Sigma^{-k}S^{(m_{k+1})}. \tag{5.6}$$

To see that the composite $\mu = \mu_{n-1} \circ \cdots \circ \mu_1 \circ \mu_0$ from (5.5) is an $S^{\perp_n}$-preenvelope, we shall show that for each $X \in S^{\perp_n}$ the map $\mathsf{Hom}_{\mathscr{T}}(M_{k+1}, X) \to \mathsf{Hom}_{\mathscr{T}}(M_k, X)$ induced by μ_k is a surjection. Without loss of generality we may assume that each map μ_k sits in a distinguished triangle (5.6) above, because if $\mu_k = 1_{M_k}$, then the map $\mathsf{Hom}_{\mathscr{T}}(M_{k+1}, X) \to \mathsf{Hom}_{\mathscr{T}}(M_k, X)$ is trivially an isomorphism for all $X \in \mathscr{T}$.

Let $X \in S^{\perp_n}$; applying $\mathsf{Hom}_{\mathscr{T}}(-, X)$ to distinguished triangle (5.6), we get the long exact sequence of Hom-sets below:

$$(\Sigma^{-k}S^{(m_{k+1})}, X) \to (M_{k+1}, X) \to (M_k, X) \to (\Sigma^{-(k+1)}S^{(m_{k+1})}, X),$$

where we have written (A, B) as a shorthand for $\mathsf{Hom}_{\mathscr{T}}(A, B)$. Now since we have $\mathsf{Hom}_{\mathscr{T}}(S^{(m_{k+1})}, \Sigma^k X) = \mathsf{Hom}_{\mathscr{T}}(S^{(m_{k+1})}, \Sigma^{(k+1)}X) = 0$ for $k = 1, \ldots, n-1$, the map

$$\mathsf{Hom}_{\mathscr{T}}(M_{k+1}, X) \to \mathsf{Hom}_{\mathscr{T}}(M_k, X)$$

induced by μ_k is an isomorphism for $k = 1, \ldots, n-1$ and a surjection for $k = 0$. Hence, writing $\bar{M} = M_n$, the composite $\mu : M \to \bar{M}$ is an $S^{\perp_n}$-preenvelope. $\square$

Lemma 5.3.7. *Suppose further that S is a simply connected corigid object of $\mathscr{T}$. Then, for the $S^{\perp_n}$-preenvelope, $\mu : M \to \bar{M}$, obtained in Proposition 5.3.6, we have that*

$$\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i \mu) : \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i M) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i \bar{M})$$

is an isomorphism for all $i < 1$.

Proof: Applying the functor $\mathsf{Hom}_{\mathscr{T}}(S, -)$ to distinguished triangle (5.6),

$$\Sigma^{-(k+1)} S^{(m_{k+1})} \to M_k \xrightarrow{\mu_k} M_{k+1} \to \Sigma^{-k} S^{(m_{k+1})},$$

for $0 \leqslant k < n$ in the proof of Proposition 5.3.6 shows that

$$\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i \mu) : \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i M_k) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i M_{k+1})$$

is an isomorphism for all $i < k + 1$. The isomorphism for $i = k$ follows by the fact that the morphism $S^{(m_{k+1})} \to \Sigma^{k+1} M_k$ in (5.6) is constructed to be an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S, -)$. Hence the composite $\mu = \mu_{n-1} \circ \cdots \circ \mu_1 \circ \mu_0$ is an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i -)$ for all $i < 1$. □

5.3.2 Homotopy colimits

In order to obtain an $S^{\perp_\infty}$-preenvelope we need to introduce a key tool, which is called the homotopy colimit. The following definition is taken from [44]. A more general treatment can be found in [46].

Definition 5.3.8. Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$

be a sequence of objects and morphisms in $\mathscr{T}$. The *homotopy colimit* $\mathsf{hocolim}\, X_i$ is constructed by extending the map

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X_i$$

to a distinguished triangle:

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} X_i \longrightarrow \mathsf{hocolim}\, X_i \longrightarrow \Sigma \coprod_{i=0}^{\infty} X_i.$$

In Definition 5.3.8 above, the map $1 - \mathrm{shift}$ is given by the infinite matrix:

$$\begin{pmatrix} 1_{X_0} & 0 & 0 & 0 & \cdots \\ -f_0 & 1_{X_1} & 0 & 0 & \cdots \\ 0 & -f_1 & 1_{X_2} & 0 & \cdots \\ 0 & 0 & -f_2 & 1_{X_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Elementary properties of homotopy colimits can be found in [46].

Remark 5.3.9. Note that the existence of homotopy colimits in $\mathscr{T}$ is guaranteed by the existence of set-indexed coproducts in $\mathscr{T}$, see [46, Definition 1.6.4].

To pass from the $S^{\perp_n}$-preenvelope which we obtained in Proposition 5.3.6 to the $S^{\perp_\infty}$-preenvelope which we desire, we will need the following relation between the homotopy colimit and the usual categorical colimit. For the definition of the usual categorical colimit see [12], [43] or [55] for example.

Lemma 5.3.10 ([44], Lemma 2.8)**.** *Suppose S is a compact object of a triangulated category $\mathscr{T}$ and we have a sequence of objects and morphisms of $\mathscr{T}$:*

$$X_0 \to X_1 \to X_2 \to X_3 \to \cdots$$

then $\mathsf{colim}\,\mathsf{Hom}_{\mathscr{T}}(S, X_n) \cong \mathsf{Hom}_{\mathscr{T}}(S, \mathsf{hocolim}\, X_n)$.

5.3.3 Existence of an $S^{\perp_\infty}$-preenvelope

Let $\mathscr{I}$ be the directed category consisting of a sequence of objects and morphisms:

$$1 \to 2 \to 3 \to 4 \to \cdots .$$

Let $\mathscr{C}$ be a category and $F : \mathscr{I} \to \mathscr{C}$ be a functor taking n to X_n. Then we get a sequence of objects and morphisms in $\mathscr{C}$:

$$X_1 \to X_2 \to X_3 \to X_4 \to \cdots ,$$

and write $\mathsf{colim}\, X_n$ instead of $\mathsf{colim}\, F$. Then there is a commutative diagram:

$$\begin{tikzcd} & X_j \arrow[dr] & \\ X_i \arrow[ur] \arrow[rr] & & \mathsf{colim}\, X_n, \end{tikzcd} \tag{5.7}$$

for $j > i$. In particular, if $X_j = 0$ for sufficiently large j, it follows that $\operatorname{colim} X_n = 0$; see [43], for example. We shall use this fact in the proof of the next proposition.

Proposition 5.3.11. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S is a compact simply connected corigid object of $\mathscr{T}$. Then, for each object M of $\mathscr{T}$ there exists an $S^{\perp_\infty}$-preenvelope $\mu : M \to \bar{M}$.*

Proof: Let M be an object of $\mathscr{T}$ and write $M = M_0$. Let $X \in S^{\perp_\infty}$ and suppose we have a morphism $\alpha_0 : M_0 \to X$. By the argument of Proposition 5.3.6 we can construct the following commutative diagram:

$$\begin{tikzcd}
 & X & & & & \\
M_0 \arrow[r, "\mu_0"'] \arrow[ur, "\alpha_0"] & M_1 \arrow[r, "\mu_1"'] \arrow[u, "\alpha_1"] & M_2 \arrow[r, "\mu_2"'] \arrow[ul, "\alpha_2"] & \cdots \arrow[r] & M_n \arrow[r, "\mu_n"'] \arrow[ulll, "\alpha_n"'] & \cdots \, .
\end{tikzcd}$$

with $M_n \in S^{\perp_n}$ for each $n \geqslant 1$. We now construct the homotopy colimit, $\operatorname{hocolim} M_i$. By construction, the composite

$$\coprod_{i=0}^{\infty} M_i \xrightarrow{1-\text{shift}} \coprod_{i=0}^{\infty} M_i \xrightarrow{\langle \alpha_i \rangle} X$$

is zero, so that we have the following commutative diagram:

$$\begin{tikzcd}
\coprod_{i=0}^{\infty} M_i \arrow[r, "1-\text{shift}"] \arrow[dr, "0"'] & \coprod_{i=0}^{\infty} M_i \arrow[r] \arrow[d, "\langle \alpha_i \rangle"] & \operatorname{hocolim} M_i \arrow[r] \arrow[dl, dashed, "\exists"] & \Sigma \coprod M_i \, . \\
 & X & &
\end{tikzcd}$$

That is, every morphism $M \to X$ factors through $\operatorname{hocolim} M_i \to X$.

Now we have:

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{T}}(S, \Sigma^j \operatorname{hocolim} M_i) &\cong \operatorname{Hom}_{\mathscr{T}}(S, \operatorname{hocolim} \Sigma^j M_i) \\
&\cong \operatorname{colim} \operatorname{Hom}_{\mathscr{T}}(S, \Sigma^j M_i) \\
&= 0
\end{aligned}$$

for $j \geqslant 1$. We obtain the first isomorphism because the homotopy colimit commutes with the suspension functor and the second isomorphism by Lemma 5.3.10. The final equality is a consequence of the fact that $\operatorname{Hom}_{\mathscr{T}}(S, \Sigma^j M_i) = 0$ for i sufficiently large and $j \geqslant 1$ (see (5.7) above). Hence we have $\operatorname{hocolim} M_i \in S^{\perp_\infty}$. Therefore, setting $\bar{M} = \operatorname{hocolim} M_i$, we obtain an $S^{\perp_\infty}$-preenvelope $\mu : M \to \bar{M}$. □

5.3.4 The induced co-t-structure

The aim of this section is to give a proof of the following theorem, which is the first half of Theorem 5.3.3.

Theorem 5.3.12. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S is a compact simply connected corigid object of $\mathscr{T}$. Further assume that $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set in $\mathscr{T}$. Then the following forms a non-degenerate co-t-structure on $\mathscr{T}$:*

$$\begin{aligned} \mathscr{A} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}. \end{aligned}$$

Note that in Theorem 5.3.12 we have $\mathscr{B} = S^{\perp_\infty}$. In order to prove Theorem 5.3.12 we need a lemma analogous to Lemma 5.3.7.

Recall that a category $\mathscr{C}$ is called *cocomplete* if for any functor $F : \mathscr{I} \to \mathscr{C}$ with $\mathscr{I}$ a small category (i.e. a category whose class of objects is a set) has a colimit $\mathsf{colim}\, F$; see [55], for instance.

Lemma 5.3.13. *Suppose we have two sequences in a cocomplete abelian category $\mathscr{C}$:*

$$\begin{tikzcd}
X_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"'] & X_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & X_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & X_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & \cdots \\
Y_0 \arrow[r, "\beta_0"'] & Y_1 \arrow[r, "\beta_1"'] & Y_2 \arrow[r, "\beta_2"'] & Y_3 \arrow[r, "\beta_3"'] & \cdots
\end{tikzcd}$$

such that $f_n : X_n \to Y_n$ is an isomorphism and $f_{n+1} \circ \alpha_n = \beta_n \circ f_n$ for each n. Then $\mathsf{colim}\, X_n \cong \mathsf{colim}\, Y_n$.

In the next two lemmas note that we are assuming that $\mathscr{T}$ has set-indexed coproducts and therefore homotopy colimits exist in $\mathscr{T}$, see Remark 5.3.9.

Lemma 5.3.14 ([46], Lemma 1.1.6)**.** *Let X be an object of a triangulated category $\mathscr{T}$, and let*

$$X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} X \xrightarrow{1} \cdots$$

be the sequence in which all the maps are the identity on X. Then $\mathsf{hocolim}\, X$ *is canonically isomorphic to X.*

Lemma 5.3.15. *Let S be a compact object of a triangulated category $\mathscr{T}$ and suppose we have a sequence of objects and morphisms*

$$X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \cdots$$

such that $\mathsf{Hom}_{\mathscr{T}}(S,\alpha_n) : \mathsf{Hom}_{\mathscr{T}}(S,X_n) \to \mathsf{Hom}_{\mathscr{T}}(S,X_{n+1})$ *is an isomorphism for each* $n \geqslant 0$. *Then* $\mathsf{Hom}_{\mathscr{T}}(S,X_0) \cong \mathsf{Hom}_{\mathscr{T}}(S,\mathsf{hocolim}\, X_n)$.

Proof: Consider the following commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "1"] \arrow[d, "1"'] & X_0 \arrow[r, "1"] \arrow[d, "\alpha_0"'] & X_0 \arrow[r, "1"] \arrow[d, "\alpha_1\alpha_0"'] & X_0 \arrow[r, "1"] \arrow[d, "\alpha_2\alpha_1\alpha_0"'] & \cdots \\
X_0 \arrow[r, "\alpha_0"'] & X_1 \arrow[r, "\alpha_1"'] & X_2 \arrow[r, "\alpha_2"'] & X_3 \arrow[r, "\alpha_3"'] & \cdots .
\end{tikzcd}$$

Since $\alpha_n^* = \mathsf{Hom}(C,\alpha_n) : \mathsf{Hom}(C,X_n) \to \mathsf{Hom}(C,X_{n+1})$ is an isomorphism for each n, then $\beta_n^* = \alpha_n^* \cdots \alpha_0^* : \mathsf{Hom}(C,X_0) \to \mathsf{Hom}(C,X_{n+1})$ is an isomorphism for each n. Hence we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Hom}(C,X_0) \arrow[r, "1"] \arrow[d, "1"'] & \mathsf{Hom}(C,X_0) \arrow[r, "1"] \arrow[d, "\beta_0^*"'] & \mathsf{Hom}(C,X_0) \arrow[r, "1"] \arrow[d, "\beta_1^*"'] & \cdots \\
\mathsf{Hom}(C,X_0) \arrow[r, "\alpha_0^*"'] & \mathsf{Hom}(C,X_1) \arrow[r, "\alpha_1^*"'] & \mathsf{Hom}(C,X_2) \arrow[r, "\alpha_2^*"'] & \cdots .
\end{tikzcd}$$

Hence, we have $\mathsf{colim}\,(\mathsf{Hom}(C,X_0)) \cong \mathsf{colim}\,(\mathsf{Hom}(C,X_n))$. It follows by Lemma 5.3.10 that $\mathsf{colim}\,\mathsf{Hom}(C,X_n) \cong \mathsf{Hom}(C,\mathsf{hocolim}\,X_n)$. By Lemma 5.3.14, we know that $\mathsf{hocolim}\,X_0 \cong X_0$, canonically, so that

$$\mathsf{colim}\,(\mathsf{Hom}(C,X_0)) \cong \mathsf{Hom}(C,\mathsf{hocolim}\,X_0) \cong \mathsf{Hom}(C,X_0),$$

and thus we obtain that $\mathsf{Hom}(C,X_0) \cong \mathsf{Hom}(C,\mathsf{hocolim}\,X_n)$, as required. □

Lemma 5.3.16. *Under the assumptions of Proposition 5.3.11 we have that*

$$\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i\mu) : \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i M) \to \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i \bar{M})$$

is an isomorphism for $i < 1$.

Proof: In Lemma 5.3.7, $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i\mu) : \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i M_k) \to \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i M_{k+1})$ is an isomorphism for $i < 1$. Now apply Lemma 5.3.15. □

Proof of Theorem 5.3.12: Conditions (0) and (1) of the definition of a co-t-structure are clear.

In order to show (2) assume $X \in \Sigma^{-1}\mathscr{A}$ and $Y \in \mathscr{B}$. Recall that $\mathscr{B} = S^{\perp_\infty}$. By Proposition 5.3.11 there exists an $S^{\perp_\infty}$-preenvelope $\mu : X \to \bar{X}$, that is, we have a surjection of Hom spaces

$$\mathsf{Hom}_{\mathscr{T}}(\bar{X}, Y) \twoheadrightarrow \mathsf{Hom}_{\mathscr{T}}(X, Y).$$

It is therefore sufficient to show $\mathsf{Hom}_{\mathscr{T}}(\bar{X}, Y) = 0$.

We have the following isomorphism of Hom-spaces and trivial Hom-spaces:

$$\begin{aligned}
\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) &\cong \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i \bar{X}) \text{ for all } i < 1 \quad \text{(Lemma 5.3.16)} \\
\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) &= 0 \text{ for all } i < 1 \quad (\text{since } X \in \Sigma^{-1}\mathscr{A}) \\
\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i \bar{X}) &= 0 \text{ for all } i > 0 \quad (\text{since } \bar{X} \in \mathscr{B}).
\end{aligned}$$

It follows that $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i \bar{X}) = 0$ for all $i \in \mathbb{Z}$. The assumption that $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$ implies that $\bar{X} = 0$. Thus $\mathsf{Hom}_{\mathscr{T}}(\bar{X}, Y) = 0$ and we see that $\mathsf{Hom}_{\mathscr{T}}(X, Y) = 0$. Hence $\mathsf{Hom}_{\mathscr{T}}(\Sigma^{-1}\mathscr{A}, \mathscr{B}) = 0$.

We next show condition (3). Suppose X is an object of $\mathscr{T}$. By Proposition 5.3.11 there is an $S^{\perp_\infty}$-preenvelope $\mu : X \to \bar{X}$. Write $B = \bar{X}$ and extend the morphism $\mu : X \to B$ to a distinguished triangle:

$$\Sigma^{-1} A \to X \to B \to A. \tag{5.8}$$

We claim that $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i A) = 0$ for $i < 0$. Consider the following long exact sequence obtained from (5.8):

$$\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^{i-1} A) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i B) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i A).$$

Now, by Lemma 5.3.16, we see that $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) \to \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i B)$ is an isomorphism for all $i < 1$. Hence $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i A) = 0$ for all $i < 0$ and $A \in \mathscr{A}$. Hence the distinguished triangle in (5.8) above gives us the required distinguished triangle.

It is clear that $\cap_{n\in\mathbb{Z}} \Sigma^n \mathscr{A} = \cap_{n\in\mathbb{Z}} \Sigma^n \mathscr{B} = \{0\}$ because $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$. Hence $(\mathscr{A}, \mathscr{B})$ is a non-degenerate co-t-structure on $\mathscr{T}$. □

Remark 5.3.17. In [11], a co-t-structure $(\mathscr{A}, \mathscr{B})$ is called *right adjacent* to a t-structure $(\mathscr{X}, \mathscr{Y})$ if $\mathscr{A} = \mathscr{Y}$. By [8], the full subcategory $\mathscr{A}$ of $\mathscr{T}$ occurs in a t-structure $(\mathscr{X}, \mathscr{A})$

on $\mathscr{T}$, where

$$\mathscr{X} = {}^{\perp}\mathscr{A} := \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(X, A) = 0 \text{ for all } A \in \mathscr{A}\}.$$

Therefore, the co-t-structure on $\mathscr{T}$ obtained in Theorem 5.3.12 is right adjacent to the t-structure $(\mathscr{X}, \mathscr{A})$.

5.4 The coheart of the co-t-structure of Theorem 5.3.12

In Theorem 5.2.7, Hoshino, Kato and Miyachi not only obtain a non-degenerate t-structure on a triangulated category $\mathscr{T}$, but they also show that its heart, which is admissible abelian, is equivalent to the module category $\mathsf{Mod}(\mathsf{End}(S)^{\mathsf{op}})$. We shall show that the coheart of the co-t-structure obtained in Theorem 5.3.12 is equivalent to $\mathsf{Mod}(\mathsf{End}(S)^{\mathsf{op}})$, where S is the object from Theorem 5.3.12 and where $\mathsf{End}(S)^{\mathsf{op}}$ indicates that this is the category of right $\mathsf{End}(S)$-modules. This is the second half of Theorem 5.3.3, and will complete the proof of the cochain analogue of Theorem 5.2.7.

Setup 5.4.1. Throughout this section, we shall consider the co-t-structure of Theorem 5.3.12: let $\mathscr{T}$ be a triangulated category with set indexed coproducts, suppose S is a compact simply connected corigid object of $\mathscr{T}$. Furthermore, assume that $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$. Then, by Theorem 5.3.12, the following is a non-degenerate co-t-structure on $\mathscr{T}$:

$$\begin{aligned} \mathscr{A} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}. \end{aligned}$$

Let $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ be the coheart of this co-t-structure.

Lemma 5.4.2. *Under the hypotheses of Setup 5.4.1 the functor*

$$\mathsf{Hom}_{\mathscr{T}}(S, -) : \mathscr{C} \to \mathrm{Mod}(\mathbb{K}^{\mathsf{op}}),$$

where $\mathbb{K} = \mathsf{End}(S)$*, is dense.*

Proof: Consider the object S of $\mathscr{T}$. Since $(\mathscr{A}, \mathscr{B})$ forms a co-t-structure on $\mathscr{T}$, there is a distinguished triangle

$$\Sigma^{-1}A \xrightarrow{\alpha} S \longrightarrow B \longrightarrow A \tag{5.9}$$

with $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Applying the functor $\mathsf{Hom}_{\mathscr{T}}(S,-)$ to (5.9) gives the following long exact sequence:

$$\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i S) \to \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i B) \to \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i A). \tag{5.10}$$

In (5.10) we have $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i S) = \mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i A) = 0$ for all $i < 0$ since $A \in \mathscr{A}$, so that $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i B) = 0$ for all $i < 0$. We know that $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i B) = 0$ for all $i > 0$ since $B \in \mathscr{B}$. Therefore, $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i B) = 0$ for all $i \neq 0$. Thus B is in the coheart $\mathscr{C}$. By Lemma 5.3.16, $\mathsf{Hom}_{\mathscr{T}}(S,S) \to \mathsf{Hom}_{\mathscr{T}}(S,B)$ is an isomorphism. Hence we have:

$$\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i B^{(m)}) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{K}^{(m)} & \text{if } i = 0. \end{cases}$$

where $\mathbb{K} = \mathsf{End}(S)$. Hence $\mathsf{Hom}_{\mathscr{T}}(S,-) : \mathscr{C} \to \mathrm{Mod}(\mathbb{K}^{\mathsf{op}})$ is dense. □

The fact $\mathsf{Hom}_{\mathscr{T}}(S,S) \to \mathsf{Hom}_{\mathscr{T}}(S,B)$ is an isomorphism forces $\mathsf{Hom}_{\mathscr{T}}(S,A) = 0$. Therefore, A is an object of $\Sigma^{-1}\mathscr{A}$ rather than just an object of $\mathscr{A}$.

The object B introduced in triangle (5.9) above has the useful property that any object in the coheart $\mathscr{C}$ can be described in terms of it; we show this in the next lemma.

Lemma 5.4.3. *Under the hypotheses of Setup 5.4.1 we have that each $M \in \mathscr{C}$ is isomorphic to $B^{(m)}$ for some m.*

Proof: Consider the distinguished triangle (5.9) from Lemma 5.4.2:

$$\Sigma^{-1}A \to S \to B \to A$$

with $A \in \Sigma^{-1}\mathscr{A}$ and $B \in \mathscr{C}$. Let $M \in \mathscr{C}$; since $\mathbb{K} = \mathsf{Hom}_{\mathscr{T}}(S,S)$ is a division ring, we can choose m such that $S^{(m)} \to M$ becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S,-)$. Again, by Lemma 5.3.16, the morphism $S \to B$ becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S,-)$.

We may now apply the functor $\mathsf{Hom}_{\mathscr{T}}(-,M)$ to (5.9) and from the long exact sequence notice that the morphism $\mathsf{Hom}_{\mathscr{T}}(B,M) \to \mathsf{Hom}_{\mathscr{T}}(S,M)$ is an isomorphism since $A \in \Sigma^{-1}\mathscr{A}$. So we obtain the commutative diagram:

$$\begin{tikzcd}
\Sigma^{-1}A^{(m)} \arrow[r] \arrow[dr, "0"'] & S^{(m)} \arrow[r] \arrow[d] & B^{(m)} \arrow[r] \arrow[dl, dashed, "\exists!"] & A^{(m)} \\
 & M & &
\end{tikzcd}$$

where both $S^{(m)} \to M$ and $S^{(m)} \to B^{(m)}$ are isomorphisms under $\mathsf{Hom}_{\mathscr{T}}(S,-)$. Hence the unique map $B^{(m)} \to M$ making the diagram above commute becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S,-)$.

Now extend this unique map $B^{(m)} \to M$ to a distinguished triangle

$$B^{(m)} \to M \to Z \to \Sigma B^{(m)} \tag{5.11}$$

and apply the functor $\mathsf{Hom}_{\mathscr{T}}(S,-)$ to give a long exact sequence. One easily sees from this long exact sequence that $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i Z) = 0$ for $i \neq 0$. The fact that the morphism $B^{(m)} \to M$ becomes the isomorphism $\mathsf{Hom}_{\mathscr{T}}(S,B^{(m)}) \xrightarrow{\sim} \mathsf{Hom}_{\mathscr{T}}(S,M)$ forces $\mathsf{Hom}_{\mathscr{T}}(S,Z) = 0$ so that $\mathsf{Hom}_{\mathscr{T}}(S,\Sigma^i Z) = 0$ for all $i \in \mathbb{Z}$. Since $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$, it follows that $Z = 0$. Hence $B^{(m)} \to M$ is an isomorphism. □

Proposition 5.4.4. *Under the hypotheses of Setup 5.4.1, the functor*

$$\mathsf{Hom}_{\mathscr{T}}(S,-) : \mathscr{C} \to \mathrm{Mod}(\mathbb{K}^{\mathsf{op}})$$

is full and faithful.

Proof: We first show that $\mathsf{Hom}_{\mathscr{T}}(S,-)$ is faithful. Consider distinguished triangle (5.9):

$$\Sigma^{-1}A \to S \to B \to A.$$

By Lemma 5.4.3, any object $M \in \mathscr{C}$ is isomorphic to some coproduct $B^{(I)}$, where I is an indexing set and $B^{(I)}$ denotes the, possibly infinite, coproduct $\coprod_{i \in I} B$. Hence, to show fidelity we can consider a morphism $B^{(I)} \to B^{(J)}$ which becomes zero under $\mathsf{Hom}_{\mathscr{T}}(S,-)$ and show that it is itself necessarily zero. It is sufficient to show that the composite $B \hookrightarrow B^{(I)} \to B^{(J)}$ is zero, where the morphism $B \hookrightarrow B^{(I)}$ is just the coproduct inclusion into the i^{th}-summand for each $i \in I$. This puts us in the following situation:

$$\begin{tikzcd}
\Sigma^{-1}A \arrow[r] & S \arrow[r] \arrow[ddr, "0"'] & B \arrow[r] \arrow[d, hook] & A \arrow[ddl, dashed, "\exists"] \\
 & & B^{(I)} \arrow[d] & \\
 & & B^{(J)}. &
\end{tikzcd}$$

But, $A \in \Sigma^{-1}\mathscr{A}$ and $B^{(J)} \in \mathscr{C} = \mathscr{A} \cap \mathscr{B}$, so that $\mathsf{Hom}_{\mathscr{T}}(A, B^{(J)}) = 0$. Therefore, the broken arrow above is necessarily zero. Hence the composite $B \hookrightarrow B^{(I)} \to B^{(J)}$ is zero, showing that $\mathsf{Hom}_{\mathscr{T}}(S,-)$ is faithful.

We must also show that $\mathsf{Hom}_{\mathscr{T}}(S,-)$ is full. Suppose we have a morphism

$$\theta : \mathsf{Hom}_{\mathscr{T}}(S, B^{(I)}) \rightarrow \mathsf{Hom}_{\mathscr{T}}(S, B^{(J)}),$$

where I and J are again indexing sets. We must construct a morphism $B^{(I)} \rightarrow B^{(J)}$ which induces θ under $\mathsf{Hom}_{\mathscr{T}}(S,-)$. We recall distinguished triangle (5.9) again:

$$\Sigma^{-1}A \longrightarrow S \xrightarrow{\sigma} B \longrightarrow A.$$

Note that $\sigma : S \rightarrow B$ becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(-, B^{(I)})$ because $B^{(I)} \in \mathscr{C}$. Hence we get the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathscr{T}}(S, B^{(I)}) \arrow[r, "\theta"] & \mathsf{Hom}_{\mathscr{T}}(S, B^{(J)}) \\
\mathsf{Hom}_{\mathscr{T}}(B, B^{(I)}) \arrow[r, "\varphi"'] \arrow[u, "\mathsf{Hom}_{\mathscr{T}}(\sigma, B^{(I)})", "\sim"'] & \mathsf{Hom}_{\mathscr{T}}(B, B^{(J)}) \arrow[u, "\sim", "\mathsf{Hom}_{\mathscr{T}}(\sigma, B^{(J)})"']
\end{tikzcd} \tag{5.12}$$

where $\varphi = \mathsf{Hom}_{\mathscr{T}}(\sigma, B^{(J)})^{-1} \circ \theta \circ \mathsf{Hom}_{\mathscr{T}}(\sigma, B^{(I)})$. Let $q_i : B \hookrightarrow B^{(I)}$ be the i^{th}-inclusion of the coproduct and consider its image $\varphi(q_i) : B \rightarrow B^{(J)}$. By the universal property of the coproduct there exists a unique map $\langle\varphi(q_i)\rangle : B^{(I)} \rightarrow B^{(J)}$ such that the following diagram commutes for each $i \in I$:

$$\begin{tikzcd}
B \arrow[r, "q_i"] \arrow[dr, "\varphi(q_i)"'] & B^{(I)} \arrow[d, "\langle\varphi(q_i)\rangle"] \\
& B^{(J)}.
\end{tikzcd}$$

Let us show that $\langle\varphi(q_i)\rangle$ induces θ under $\mathsf{Hom}_{\mathscr{T}}(S,-)$. The map $\mathsf{Hom}_{\mathscr{T}}(S,\sigma) : \mathsf{Hom}_{\mathscr{T}}(S,S) \rightarrow \mathsf{Hom}_{\mathscr{T}}(S,B)$ is an isomorphism, therefore, it takes a set of generators for $\mathsf{Hom}_{\mathscr{T}}(S,S)$ to a set of generators for $\mathsf{Hom}_{\mathscr{T}}(S,B)$. The vector space $\mathsf{Hom}_{\mathscr{T}}(S,S)$ is one-dimensional and generated by the identity map on S, 1_S, whose image under $\mathsf{Hom}_{\mathscr{T}}(S,\sigma)$ is $\sigma : S \rightarrow B$. Hence $\mathsf{Hom}_{\mathscr{T}}(S,B)$ is generated by σ. By the compactness of S, we have

$$\mathsf{Hom}_{\mathscr{T}}(S, B^{(I)}) \cong \coprod_{I} \mathsf{Hom}_{\mathscr{T}}(S, B)$$

and $\mathsf{Hom}_{\mathscr{T}}(S, B^{(I)})$ is generated by $|I|$ copies of σ. It follows that $\mathsf{Hom}_{\mathscr{T}}(S, B^{(I)})$ is generated by the family $\{\sigma \circ q_i\}_{i\in I}$. Therefore, we now only need to check that θ and the map, $\mathsf{Hom}_{\mathscr{T}}(S, \langle\varphi(q_i)\rangle)$, induced by $\langle\varphi(q_i)\rangle$ coincide on this set of generators.

By the commutativity of diagram (5.12) we have:

$$\begin{aligned}
\theta(q_i \circ \sigma) &= \mathsf{Hom}_{\mathscr{T}}(\sigma, B^{(J)}) \circ \varphi(q_i) \\
&= \varphi(q_i) \circ \sigma \\
&= (\langle \varphi(q_j) \rangle \circ q_i) \circ \sigma \\
&= \langle \varphi(q_j) \rangle \circ (q_i \circ \sigma) \\
&= \mathsf{Hom}_{\mathscr{T}}(S, \langle \varphi(q_i) \rangle)(q_i \circ \sigma)
\end{aligned}$$

Hence, θ and $\mathsf{Hom}_{\mathscr{T}}(S, \langle \varphi(q_i) \rangle)$ coincide on a set of generators of $\mathsf{Hom}_{\mathscr{T}}(S, B^{(I)})$, thus

$$\theta = \mathsf{Hom}_{\mathscr{T}}(S, \langle \varphi(q_i) \rangle)$$

with $\langle \varphi(q_i) \rangle \in \mathsf{Hom}_{\mathscr{T}}(B^{(I)}, B^{(J)})$. Therefore, the functor $\mathsf{Hom}_{\mathscr{T}}(S, -)$ is full and faithful. □

Theorem 5.4.5. *Under the hypotheses of Setup 5.4.1, the functor*

$$\mathsf{Hom}_{\mathscr{T}}(S, -) : \mathscr{C} \to \mathrm{Mod}(\mathbb{K}^{\mathsf{op}}),$$

is an equivalence of categories, and hence, the coheart $\mathscr{C}$ of the non-degenerate co-t-structure obtained in Theorem 5.3.12 is an abelian category.

Proof: By Lemma 5.4.2 and Proposition 5.4.4, $\mathsf{Hom}_{\mathscr{T}}(S, -)$ is dense and fully faithful. Hence, by [43, Theorem IV.4.1], $\mathsf{Hom}_{\mathscr{T}}(S, -)$ is an equivalence of categories. □

Theorems 5.3.12 and 5.4.5 now combine to give Theorem 5.3.3.

Although it is known that the coheart of a co-t-structure is not always an abelian subcategory of $\mathscr{T}$, see [11], Theorem 5.4.5 leads us to pose the following question.

Question 5.4.6. *Under what circumstances is the coheart of a co-t-structure on a triangulated category $\mathscr{T}$ an abelian subcategory of $\mathscr{T}$?*

5.5 A version for more than one object

Theorem 5.3.3 can be further generalised to a version giving a co-t-structure arising from a set of objects in the triangulated category. The basic setup for this section is the following; compare with the definitions of simply connected corigid objects and simply connected $(n + 1)$-corigid objects given in Definitions 5.3.2 and 5.3.5.

Setup 5.5.1. Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Let $\mathscr{S} = \{S_i\}_{i\in I}$ be a set of compact objects of $\mathscr{T}$. Assume that $\mathscr{S}$ satisfies the following assumptions:

(1) $\mathsf{Hom}_{\mathscr{T}}(\Sigma^k S_i, S_j) = 0$ for all $i, j \in I$ and for $0 < k < n+1$;

(2) $\mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma S_j) = 0$ for all $i, j \in I$;

(3) $\mathsf{Hom}_{\mathscr{T}}(S_i, S_j) = 0$ for all $i \neq j \in I$;

(4) $\mathsf{End}(S_i)$ is a division ring for all $i \in I$.

Proposition 5.5.2. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose $\mathscr{S}$ is a set of objects of $\mathscr{T}$ satisfying the assumptions of Setup 5.5.1. Then for each object M of $\mathscr{T}$ there exists an $\mathscr{S}^{\perp_n}$-preenvelope $\mu : M \to \bar{M}$.*

Proof: The preenvelope is constructed by the same argument as in the proof of Proposition 5.3.6. Let M be an object of $\mathscr{T}$. We construct a chain of objects and morphisms,

$$M = M_0 \xrightarrow{\mu_0} M_1 \xrightarrow{\mu_1} M_2 \xrightarrow{\mu_2} M_3 \xrightarrow{\mu_3} \cdots \xrightarrow{\mu_{n-1}} M_n = \bar{M}, \tag{5.13}$$

with $M_k \in \mathscr{S}^{\perp_k}$ for each $k \geqslant 1$. The construction is now analogous to that in Proposition 5.3.6, the only difference being the slight change in the distinguished triangles used to obtain the morphisms μ_k. We include here a mention of the distinguished triangle used in the induction step of the construction.

Suppose we have constructed $M_k \in \mathscr{S}^{\perp_k}$. Without loss of generality, we may assume that $\mathsf{Hom}_{\mathscr{T}}(S_j, \Sigma^{k+1} M_k) \neq 0$ for some $j \in I$, otherwise set $M_{k+1} = M_k$. Since $\mathsf{End}(S_i)$ is a division ring for each $i \in I$, we may choose a (possibly infinite) cardinal $m_{i,k+1}$ and a morphism $\coprod_{i\in I} S_i^{(m_{i,k+1})} \to \Sigma^{k+1} M_k$ which becomes an isomorphism under $\mathsf{Hom}_{\mathscr{T}}(S_j, -)$ for each $j \in I$. Extending this morphism to a distinguished triangle yields

$$\Sigma^{-(k+1)} \coprod_{i\in I} S_i^{(m_{i,k+1})} \to M_k \xrightarrow{\mu_k} M_{k+1} \to \Sigma^{-k} \coprod_{i\in I} S_i^{(m_{i,k+1})}.$$

Now, arguing as in the proof of Proposition 5.3.6 shows that $M_{k+1} \in \mathscr{S}^{\perp_{k+1}}$, as desired. One argues the composite (5.13) which is obtained by this construction as in Proposition 5.3.6. □

We make the following alteration to Setup 5.5.1.

Setup 5.5.3. In Setup 5.5.1, replace assumption 1 with the following assumption on the set $\mathscr{S}$:

(1) $\mathsf{Hom}_{\mathscr{T}}(\Sigma^k S_i, S_j) = 0$ for all $i, j \in I$ and all $k > 0$.

We obtain a generalisation of Proposition 5.3.11 which serves as the extension of the $\mathscr{S}^{\perp_n}$-preenvelope obtained in Proposition 5.5.2 to an $\mathscr{S}^{\perp_\infty}$-preenvelope.

Proposition 5.5.4. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose $\mathscr{S}$ is a set of objects of $\mathscr{T}$ satisfying the modified assumptions of Setup 5.5.3. Then for each object M of $\mathscr{T}$ there exists an $\mathscr{S}^{\perp_\infty}$-preenvelope $\mu : M \to \bar{M}$.*

Proof: The argument of the proof of Proposition 5.3.11 applies here. □

We obtain a corollary analogous to Lemma 5.3.15.

Corollary 5.5.5. *Under the assumptions of Proposition 5.5.4 we have that*

$$\mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^k \mu) : \mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^k M) \to \mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^k \bar{M})$$

is an isomorphism for all $i \in I$ and all $k < 1$.

Using the $\mathscr{S}^{\perp_\infty}$-preenvelope obtained in Proposition 5.5.4, we obtain a generalisation of Theorem 5.3.12.

Theorem 5.5.6. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose $\mathscr{S}$ is a set of objects of $\mathscr{T}$ satisfying the modified assumptions of Setup 5.5.3. Further assume that $\{\Sigma^k S_i \mid i \in I \text{ and } k \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$. Then the following forms a non-degenerate co-t-structure on $\mathscr{T}$:*

$$\begin{aligned} \mathscr{A} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^n X) = 0 \text{ for all } i \in I \text{ and } n < 0\}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^n X) = 0 \text{ for all } i \in I \text{ and } n > 0\}. \end{aligned}$$

Again, we note that $\mathscr{B} = \mathscr{S}^{\perp_\infty}$.

Proof: The argument of the proof of Theorem 5.3.12 applies here with Corollary 5.5.5 playing the role of Lemma 5.3.15. □

Consider the following setup analogous to Definition 5.3.2.

Setup 5.5.7. Let S be an object of $\mathscr{T}$ satisfying the following assumptions:

(1) S is corigid, that is, $\mathsf{Hom}_{\mathscr{T}}(\Sigma^i S, S) = 0$ for $i > 0$;

(2) $\mathsf{Hom}_{\mathscr{T}}(S, \Sigma S) = 0$;

(3) $\mathsf{End}(S)$ is a simple Artinian ring

Let D be a ring with identity and $M_n(D)$ denote the ring of $n \times n$ matrices with entries in D. It is well known that if $R = M_n(D)$ then the matrix ring R and the ring D are Morita equivalent, that is $\mathsf{Mod}(R)$ and $\mathsf{Mod}(D)$ are equivalent categories. Recall that a ring is simple Artinian if and only if it can be written as a matrix ring over a division ring. As such, we make the following conjectures based on Theorem 5.3.3 and Theorem 5.5.6.

Conjecture 5.5.8. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose S is a compact object of $\mathscr{T}$ satisfying the assumptions of Setup 5.5.7 and assume that $\{\Sigma^i S \mid i \in \mathbb{Z}\}$ is a generating set for $\mathscr{T}$. Then the following forms a non-degenerate co-t-structure on $\mathscr{T}$:*

$$\begin{aligned} \mathscr{A} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i < 0\}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^i X) = 0 \text{ for } i > 0\}. \end{aligned}$$

Moreover, its coheart $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ is an abelian subcategory of $\mathscr{T}$, and the functor

$$\mathsf{Hom}_{\mathscr{T}}(S, -) : \mathscr{C} \to \mathrm{Mod}(\mathrm{End}(S)^{\mathsf{op}})$$

is an equivalence of categories.

Conjecture 5.5.9. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose $\mathscr{S}$ is a set of objects of $\mathscr{T}$ satisfying the assumptions of Setup 5.5.1 with the modification that* $\mathsf{End}(S_i)$ *is a simple Artinian ring instead of a division ring. Further assume* *that* $\{\Sigma^k S_i \,|\, i \in I \text{ and } k \in \mathbb{Z}\}$ *is a generating set for* $\mathscr{T}$*. Then the following forms a non-degenerate co-*t*-structure on* $\mathscr{T}$*:*

$$\begin{aligned}\mathscr{A} &= \{X \in \mathscr{T} \,|\, \mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^n X) = 0 \text{ for all } i \in I \text{ and } n < 0\},\\ \mathscr{B} &= \{X \in \mathscr{T} \,|\, \mathsf{Hom}_{\mathscr{T}}(S_i, \Sigma^n X) = 0 \text{ for all } i \in I \text{ and } n > 0\}.\end{aligned}$$

Again, we note that $\mathscr{B} = \mathscr{S}^{\perp_\infty}$.

In light of Theorem 5.5.6, we conjecture that if Conjectures 5.5.8 and 5.5.9 are true then Conjecture 5.5.8 can be generalised further to the case where S satisfies the assumptions of Conjecture 5.5.8 with the modification that the endomorphism ring $\mathsf{End}(S)$ is a semisimple Artinian ring instead of a simple Artinian ring.

One can consider Conjecture 5.5.9 to be a multiobject version of Conjecture 5.5.8 in the same way that Theorem 5.5.6 is a multiobject version of Theorem 5.3.12. Thus, the semisimple generalisation of Conjecture 5.5.8 is merely a version of Conjecture 5.5.9 with I being a finite set. We believe that the fact that all the modules of a semisimple Artinian ring are projective means that the arguments employed in section 5.4 can be adapted to yield the equivalence of the coheart this co-t-structure induces with the module category $\mathsf{Mod}(\mathsf{End}(S)^{\mathsf{op}})$.

5.6 Concluding remarks

In this section we make some remarks regarding the absence of the functoriality in the theory of co-t-structures compared with that which is present in the almost dual theory of t-structures.

5.6.1 Functoriality

The functoriality present in the theory of t-structures arises from a stable analogue of Wakamatsu's Lemma which gives connections between the existence of precovers and preenvelopes and the vanishing of $\mathsf{Ext}(-,-)$ functors in abelian categories; see [8, Chapter 2] for example. Let $\mathscr{T}$ be a triangulated category and $\mathscr{X}$ a full subcategory of $\mathscr{T}$, the stable analogue of Wakamatsu's Lemma, stated below, gives a connection between the existence of preenvelopes (resp. precovers) and a left adjoint (resp. right adjoint) to the inclusion functor. It is this connection which accounts for the underlying functoriality inherent in the theory of t-structures.

Lemma 5.6.1 ([8] Lemma II.2.3)**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be a pair of full subcategories of a triangulated category $\mathscr{T}$.*

(i) If $\Sigma\mathscr{X} \subseteq \mathscr{X}$, then the following conditions are equivalent:

  (a) The inclusion functor $i : \mathscr{X} \to \mathscr{T}$ has a right adjoint.

  (b) For any object T of $\mathscr{T}$ there exists a left triangle $\Sigma^{-1}T \to Y \to X \xrightarrow{f} T$, where f is an $\mathscr{X}$-precover and $Y \in \mathscr{X}^{\perp}$.

(ii) If $\mathscr{Y} \subseteq \Sigma\mathscr{Y}$, then the following conditions are equivalent:

  (a) The inclusion functor $j : \mathscr{Y} \to \mathscr{T}$ has a left adjoint.

  (b) For any object T of $\mathscr{T}$ there exists a right triangle $T \xrightarrow{g} Y \longrightarrow X \longrightarrow \Sigma T$, where g is a $\mathscr{Y}$-preenvelope and $X \in {}^{\perp}\mathscr{Y}$.

Remark 5.6.2. Lemma 5.6.1 is proved more generally for a pretriangulated category $\mathscr{T}$ in [8] and here the inverse of the suspension functor, Σ^{-1}, is actually the “loop functor”, usually denoted by Ω. Defintions of a pretriangulated category, the loop functor, left triangles and right triangles can be found in [8].

In the theory of co-t-structures such a connection between the existence of preenvelopes and precovers and the existence of left and right adjoints to inclusion functors doesn’t occur. This is a result of the inclusions satisfied by the two halves of a co-t-structure $(\mathscr{A}, \mathscr{B})$ on $\mathscr{T}$ in condition (1) of Definition 5.1.5. As such, we require additional hypotheses in order to work with co-t-structures, hence the subtle difference arising in the statements of Hoshino, Kato and Miyachi’s Theorem on a t-structure induced by a rigid compact object, Theorem 5.2.7, and the corresponding co-t-structure analogue regarding a co-t-structure obtained from a compact simply connected corigid object of $\mathscr{T}$, Theorem 5.3.3.

5.6.2 The full subcategory $\mathscr{B}$

In the work of Beligiannis and Reiten, [8], it is shown that for any set of compact objects $\mathscr{S}$ of a triangulated category $\mathscr{T}$ then there is always a t-structure $(\mathscr{X}, \mathscr{Y})$ given by

$$\begin{aligned} \mathscr{X} &= {}^{\perp}\mathscr{Y}, \\ \mathscr{Y} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^n X) = 0 \text{ for all } S \in \mathscr{S} \text{ and } n < 0\}. \end{aligned}$$

Thus, for a set of compact objects $\mathscr{S}$ of $\mathscr{T}$, the full subcategory $\mathscr{Y}$ above always occurs in a t-structure on $\mathscr{T}$. Unfortunately, the other half of the t-structure, $\mathscr{X}$, cannot always be obtained explicity. In order to do so, certain conditions need to be placed on the set of compact objects, $\mathscr{S}$. These are precisely the conditions stated in Hoshino, Kato and Miyachi's theorem in [31]. That is, for a set of compact objects $\mathscr{S}$ of $\mathscr{T}$, there is a t-structure $(\mathscr{X}, \mathscr{Y})$ on $\mathscr{T}$ with

$$\begin{aligned} \mathscr{X} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^n X) = 0 \text{ for all } S \in \mathscr{S} \text{ and } n > 0\}, \\ \mathscr{Y} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^n X) = 0 \text{ for all } S \in \mathscr{S} \text{ and } n < 0\}, \end{aligned}$$

if and only if each object of $\mathscr{S}$ is rigid and the set $\{\Sigma^i S \mid S \in \mathscr{S} \text{ and } n \in \mathbb{Z}\}$ is a generating set.

Since the full subcategory $\mathscr{Y}$ always occurs in a such a t-structure it is natural to question why the same cannot be said of the full subcategory $\mathscr{X}$? Indeed, the introduction of the notion of co-t-structure in this thesis seems to partially answer this question. Indeed, the full subcategory $\mathscr{X}$ occurs as the full subcategory $\mathscr{B}$ appearing in the co-t-structure obtained in Theorem 5.3.12. Based on analogy with Beligiannis and Reiten, [8], we conjecture that for any set of compact objects $\mathscr{S}$ of $\mathscr{T}$ satisfying suitably nice conditions, there is a co-t-structure $(\mathscr{A}, \mathscr{B})$ on $\mathscr{T}$ given by

$$\begin{aligned} \mathscr{A} &= {}^{\perp}\mathscr{B}, \\ \mathscr{B} &= \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^n X) = 0 \text{ for all } S \in \mathscr{S} \text{ and } n > 0\}. \end{aligned}$$

Then we conjecture that $\mathscr{A}$ can be explicity described as,

$$\mathscr{A} = \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(S, \Sigma^n X) = 0 \text{ for all } S \in \mathscr{S} \text{ and } n < 0\},$$

under the conditions of Theorem 5.3.12 or the generalisations in Conjectures 5.5.8 and 5.5.9. I believe the methods employed by Beligiannis and Reiten in [8] can be adapted to the co-t-structure case with the appropriate modifications made and alternative techniques used when one faces the difficulties arising from the loss of functoriality in the co-t-structure case caused by the absence of an analogue of Wakamatsu's Lemma highlighted in the previous subsection. In this light the theory of co-t-structures may help illuminate the theory of t-structures.

Chapter 6

Generalised Moore Spectra in a Triangulated Category

In this chapter we discuss the existence of “Moore spectra” in a triangulated category. This construction was first employed by Jørgensen in [35], where it is used as a tool in considering certain lifts of homological functors between the derived category of the integers $\mathscr{D}(\mathbb{Z})$ and an arbitrary triangulated category $\mathscr{T}$. However, the idea of Moore spectra in a triangulated category holds independent interest, in particular, they yield a new technique of embedding an abelian category in a triangulated category and obtaining a module category from a triangulated category in a nice way.

The construction given in this chapter is an extension and generalisation of Jørgensen’s construction, and the techniques used to prove the theorems are very different from those employed in [35], with more category theoretic language being used. The proof of the main theorem (Theorem 6.2.5) is constructive and consists of a long induction in which the action of the Moore spectra functor on objects is constructed via projective resolutions of modules and functoriality is obtained via a classical representability trick. Before embarking on the general construction we present a brief review of Jørgensen’s construction which appears in [35].

6.1 Jørgensen's construction

Let R be a ring. Recall that a triangulated category $\mathscr{T}$ is called *R-linear* if for any two objects X and Y of $\mathscr{T}$ the Hom-space $\mathsf{Hom}_{\mathscr{T}}(X, Y)$ is an R-module and the composition of morphisms is R-bilinear. A functor $F : \mathscr{T} \to \mathscr{T}'$ of R-linear triangulated categories is said to be R-linear if $F(r\alpha) = rF(\alpha)$ for all morphisms α of $\mathscr{T}$ and all elements $r \in R$; see [35, Definition 1.2].

In [35, Section 4], Jørgensen considers the following setup.

Setup 6.1.1 ([35], Setups 4.1 and 4.12)**.** *Let R be a principal ideal domain, let $\mathscr{T}$ be an R-linear triangulated category with set indexed coproducts and let C be a compact object of $\mathscr{T}$ which satisfies the following assumptions:*

(1) $\mathsf{Hom}_{\mathscr{T}}(C, C)$ *is a flat R-module;*

(2) $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1}C) = 0$*;*

(3) $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-2}C) = 0$.

The idea in [35] is to construct the best possible approximation of an R-module A in $\mathscr{T}$. This approximation of A in $\mathscr{T}$ is denoted by $M(A)$ and is called the *Moore spectrum* of A in $\mathscr{T}$. Jørgensen's construction depends on which object, C, of $\mathscr{T}$ is employed as the Moore spectrum of the ring itself.

Jørgensen introduces an auxilliary category $\mathscr{M}$, an analogue of which we shall also introduce in the general construction. Below is the definition of $\mathscr{M}$ in Jørgensen's setting; see [35, Definition 4.3].

Definition 6.1.2. Let $\mathscr{M}$ be the full subcategory of $\mathscr{T}$ consisting of all objects M of $\mathscr{T}$ which occur in distinguished triangles of the form

$$C \otimes F_1 \xrightarrow{1_C \otimes f} C \otimes F_0 \longrightarrow M \longrightarrow \Sigma C \otimes F_1,$$

when A is an R-module with free resolution

$$0 \longrightarrow F_1 \xrightarrow{f} F_0 \longrightarrow A \longrightarrow 0.$$

Note that in this definition, the tensor product is not the "usual" tensor product but is in fact a bifunctor $- \otimes - : \mathscr{T} \times \mathsf{Free}(R) \to \mathscr{T}$ which is R-linear, preserves set indexed coproducts and has $X \otimes R \cong X$ for each X in $\mathscr{T}$. Here, $\mathsf{Free}(R)$ denotes the full subcategory of $\mathsf{Mod}(R)$ consisting of all free R-modules. Indeed, the construction of $- \otimes -$, for each X in $\mathscr{T}$ and F in $\mathsf{Free}(R)$, identifies $X \otimes F$ with a coproduct $\coprod_{I_F} X$, where I_F is an indexing set for a basis of F. For full details of the construction see [35, Construction 1.4 and Lemma 1.5]. Jørgensen then obtains the following theorem.

Theorem 6.1.3 ([35], Proposition 4.7 and Theorem 4.9)**.** *Under the hypotheses of Setup 6.1.1, the functor*

$$\mathsf{Hom}_{\mathscr{T}}(C, -) : \mathscr{M} \to \mathsf{Mod}(R)$$

has a left adjoint

$$M : \mathsf{Mod}(R) \to \mathscr{M}.$$

If M is viewed as a functor $M : \mathsf{Mod}(R) \to \mathscr{T}$ by composition with the inclusion functor $i : \mathscr{M} \hookrightarrow \mathscr{T}$, then M is an R-linear functor, it has $M(R) \cong C$ and it preserves set indexed coproducts.

Jørgensen then continues to prove that the functor M constructed above is well behaved with respect to short exact sequences in $\mathsf{Mod}(R)$ and distinguished triangles in $\mathscr{T}$ as well as under the functor $\mathsf{Ext}(-, -)$.

Note that, in the proof of Theorem 6.1.3, the assumption that $\mathsf{Hom}_{\mathscr{T}}(C, C)$ is flat as an R-module is required for proving the injectivity of a certain map which is used in the construction, see [35, Lemma 4.5].

The main result of this chapter generalises Theorem 6.1.3 to arbitrary triangulated categories and dispenses with the requirement that R be a principal ideal domain. We are also able to prove that the generalised Moore spectra functor M is well behaved with respect to short exact sequences and distinguished triangles as well as under the functor $\mathsf{Ext}(-, -)$. The next section concerns the construction of generalised Moore spectra.

6.2 The general construction

The starting point of the general construction is the following generalisation of a well-known result. Before stating the theorem, we recall and fix some notation; see [2] or [4], for example.

Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. By $\mathsf{Add}(C)$ we denote the *(infinite) additive closure* of C in $\mathscr{T}$, that is the smallest full subcategory of $\mathscr{T}$ whose objects are direct summands of (possibly infinite) set indexed coproducts of C.

Let R be a ring and recall that $\mathsf{Mod}(R)$ denotes the category of left R-modules. We define the following full subcategories of $\mathsf{Mod}(R)$. By $\mathsf{Proj}(R)$ we denote the full subcategory of $\mathsf{Mod}(R)$ whose objects are all projective left R-modules. By $\mathsf{Proj}^k(R)$ we denote the full subcategory of $\mathsf{Mod}(R)$ whose objects are left R-modules with projective dimension at most k.

Proposition 6.2.1. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts and suppose C is a compact object of $\mathscr{T}$. Let $S = \mathsf{End}_{\mathscr{T}}(C)$. Then the functor*

$$\mathsf{Hom}_{\mathscr{T}}(C,-) : \mathsf{Add}(C) \to \mathsf{Proj}(S^{\mathsf{op}})$$

is an equivalence of categories.

Proof: See [4, Proposition II.2.1] for example. The compactness of C can be used to pass to the infinite additive closure and infinitely generated projective modules. □

Remark 6.2.2. The fact that $\mathsf{Hom}_{\mathscr{T}}(C,-) : \mathsf{Add}(C) \to \mathsf{Proj}(S^{\mathsf{op}})$ is an equivalence of categories means that it is part of an adjunction which is an equivalence of categories:

$$\begin{tikzcd}\mathsf{Add}(C) \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}(S^{\mathsf{op}}). \arrow[l, shift right, "M_0"']\end{tikzcd}$$

In particular, the unit of this adjunction is an isomorphism; see Lemma 4.3.2.

Throughout this chapter we shall use the following setup (c.f. Setup 6.1.1). Recall the definitions of global dimension and projective dimension from Chapter 2.

Setup 6.2.3. Let $\mathscr{T}$ be a triangulated category with set indexed coproducts and suppose C is an object of $\mathscr{T}$ satisfying the following assumptions:

(1) C is a compact object of $\mathscr{T}$;

(2) Its endomorphism alegbra $S^{\mathsf{op}} = \mathsf{End}_{\mathscr{T}}(C)^{\mathsf{op}}$ has finite global dimension n; and,

(3) We have $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i C) = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i} C) = 0$ for $i = 1, \dots, n+1$.

In [35], an auxilliary category $\mathscr{M}$, which is a certain full subcategory of $\mathscr{T}$, is introduced; see Definition 6.1.2. We define auxilliary categories $\mathscr{M}_k$ for $k \in \mathscr{N} \cup \{0\}$ with a view to arriving at an analogous definition of the auxilliary category $\mathscr{M}$.

Definition 6.2.4. We shall define full subcategories $\mathscr{M}_k$ of $\mathscr{T}$ as follows:

$$\mathscr{M}_k := \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i}X) = 0 \text{ for } i = 1, \dots, k\}$$

with the convention that $\mathscr{M}_0 = \mathscr{T}$.

Theorem 6.2.5. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose C is an object of $\mathscr{T}$ satisfying the assumptions of Setup 6.2.3. Let $\mathscr{M} = \mathscr{M}_{n+1}$. Then, the functor*

$$\mathsf{Hom}_{\mathscr{T}}(C, -) : \mathscr{M} \to \mathsf{Mod}(S^{\mathsf{op}})$$

has a left adjoint

$$M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{M}.$$

Moreover, the functor M is a full embedding of the module category $\mathsf{Mod}(S^{\mathsf{op}})$ *into the full subcategory $\mathscr{M}$ of $\mathscr{T}$.*

The proof of Theorem 6.2.5 consists of a large induction in which several related claims need to be proved. There are two analogies which are useful for viewing this construction. The first is to view the induction as a kind of "bootstrapping" procedure. The second analogy is to consider the construction of a building: here, Theorem 6.2.5 is the desired building and the related claims in the induction hypotheses are the scaffolding which is required to build it.

For ease of exposition, we break down the proof into several steps and constructions. At times we will make assumptions which will be later justified in the section devoted to the scaffolding. Before we start this long induction we explicitly state all the induction hypotheses.

6.2.1 The induction hypotheses and base step

Hypotheses 6.2.6. Under the assumptions of Setup 6.2.3, for $k \geqslant 0$ we have:

(1) There exists a functor $M_k : \mathsf{Proj}^k(S^{\mathsf{op}}) \to \mathscr{M}_k$.

(2) Let $\mathscr{N}_k = \{X \in \mathscr{T} \mid X \cong M_k(A) \text{ for some } A \in \mathsf{Proj}^k(S^{\mathsf{op}})\}$. We have $\mathscr{N}_k \subseteq \mathscr{M}_k$ so that $\mathscr{N}_k$ will become the essential image of M_k in $\mathscr{M}_k$. We also have a natural isomorphism

$$\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(A, \mathsf{Hom}_{\mathscr{T}}(C, X)) \simeq \mathsf{Hom}_{\mathscr{T}}(M_k(A), X)$$

which is natural for $A \in \mathsf{Proj}^k(S^{\mathsf{op}})$ and $X \in \mathscr{M}_k$. This restricts to an adjoint pair

$$\begin{tikzcd} \mathscr{N}_k \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^k(S^{\mathsf{op}}), \arrow[l, shift right, "M_k"'] \end{tikzcd} \tag{6.1}$$

where the left adjoint is written above the right adjoint.

(3) The unit of the adjunction in (6.1), $\eta : A \mapsto \mathsf{Hom}_{\mathscr{T}}(C, M_k(A))$, is an isomorphism.

(4) $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma M_k(A)) = \cdots = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{n+1-k} M_k(A)) = 0$ for all $A \in \mathsf{Proj}^k(S^{\mathsf{op}})$.

(5) $\mathsf{Hom}_{\mathscr{T}}(M_k(A), \Sigma^{-1} X) = 0$ for all $A \in \mathsf{Proj}^k(S^{\mathsf{op}})$ and $X \in \mathscr{M}_{k+1}$.

(6) $M_k(P) \in \mathsf{Add}(C)$ for any $P \in \mathsf{Proj}(S^{\mathsf{op}})$.

In condition (2) of Hypotheses 6.2.6 we take, as a convention, $\mathscr{N}_0 = \mathsf{Add}(C)$.

Remark 6.2.7. Note that the fact that $\mathscr{N}_k \subseteq \mathscr{M}_k$ means we have the following:

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1} M_k(A)) = \cdots = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-k} M_k(A)) = 0 \tag{6.2}$$

for all $A \in \mathsf{Proj}^k(S^{\mathsf{op}})$.

One can now think of the first three induction hypotheses as being the building which we are constructing in the proof of the theorem and the last three conditions as the scaffolding which we require to complete the construction.

A useful tool in the proof will be Lemma 4.3.2 in Chapter 4 which relates the full fidelity of an adjoint functor with the unit or counit of the adjunction being an isomorphism. Before embarking on the construction we first prove the base step of the induction.

Lemma 6.2.8. *Under the assumptions of Setup 6.2.3, Hypotheses 6.2.6 are true for* $k = 0$.

Proof: By convention, we have $\mathscr{N}_0 = \mathsf{Add}(C)$. From Proposition 6.2.1, we have an adjoint equivalence of categories:

$$\begin{tikzcd}\mathsf{Add}(C) = \mathscr{N}_0 \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}(S^{\mathsf{op}}). \arrow[l, shift right, "M_0"']\end{tikzcd}$$

Hence the unit of this adjunction is an isomorphism (see Lemma 4.3.2), and for $P \in \mathsf{Proj}(S^{\mathsf{op}})$ and $X \in \mathscr{M}_0 = \mathscr{T}$ we have a natural isomorphism

$$\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(P, \mathsf{Hom}_{\mathscr{T}}(C, X)) \simeq \mathsf{Hom}_{\mathscr{T}}(M_0(P), X).$$

(Note that this isomorphism is stronger than just adjointness in this case because $X \in \mathscr{T}$.) We also have $\mathscr{N}_0 = \text{Ess.Im}(M_0) = \mathsf{Add}(C) \subseteq \mathscr{M}_0$. This proves (1), (2) and (3) of Hypotheses 6.2.6 for $k = 0$.

Since $M_0(P) \in \mathsf{Add}(C)$ for $P \in \mathsf{Proj}(S^{\mathsf{op}})$, the assumption that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i C) = 0$ for $i = 1, \ldots, n+1$ forces

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma M_0(P)) = \cdots = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{n+1} M_0(P)) = 0.$$

Hence hypothesis (4) is satisfied for $k = 0$.

For $X \in \mathscr{M}_1$, we have $\mathsf{Hom}_{\mathscr{T}}(M_0(P), \Sigma^{-1} X) = 0$ since $M_0(P) \in \mathsf{Add}(C)$, so hypotheses (5) and (6) are also satisfied for $k = 0$. This completes the proof of the base step. □

6.2.2 The construction of the map M_{k+1}

We now turn our attention to the proof of the induction step, starting with the construction of a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$. Assume $1 \leqslant k \leqslant n$ and assume that Hypotheses 6.2.6 are satisfied for k. Given any $A \in \mathsf{Mod}(S^{\mathsf{op}})$ we shall choose a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

where n is the projective dimension of A, that is, so that we have chosen a projective resolution of A of minimal length. For $P \in \mathsf{Proj}(S^{\mathsf{op}})$ choose the obvious projective resolution

$$0 \to P \xrightarrow{\sim} P \to 0.$$

Note that the projective dimension of A is at most n because S^{op} has global dimension n.

By hypothesis (1) of Hypotheses 6.2.6, there exists a functor $M_k : \mathsf{Proj}^k(S^{\mathsf{op}}) \to \mathscr{M}_k$. Given the existence of this functor, we show how to construct a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$. In the subsequent section we shall explain how this becomes a functor.

Suppose $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ and consider the projective resolution chosen for A:

$$0 \longrightarrow P_{k+1} \xrightarrow{\pi_{k+1}} P_k \xrightarrow{\pi_k} \cdots \longrightarrow P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A \longrightarrow 0$$

where $P_i = 0$ whenever $i > \text{projdim}(A)$. This long exact sequence naturally splits into short exact sequences:

$$\begin{gathered} 0 \to P_{k+1} \to P_k \to K_{k-1} \to 0 \\ \vdots \\ 0 \to K_i \to P_i \to K_{i-1} \to 0 \\ \vdots \\ 0 \to K_0 \to P_0 \to A \to 0 \end{gathered} \tag{6.3}$$

where $K_i = \ker \pi_i$. Note that $K_0, P_0 \in \mathsf{Proj}^k(S^{\mathsf{op}})$, hence, we have $M_k(K_0), M_k(P_0) \in \mathscr{M}_k$. We can apply the functor M_k to the homomorphism $K_0 \to P_0$ in (6.3) and extend the resulting morphism in $\mathscr{T}$ to a distinguished triangle:

$$M_k(K_0) \to M_k(P_0) \to M \to \Sigma M_k(K_0). \tag{6.4}$$

Let $X, Y \in \mathscr{M}_{k+1}$ and apply $\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(-, \mathsf{Hom}_{\mathscr{T}}(C, X))$ and $\mathsf{Hom}_{\mathscr{T}}(-, X)$ to (6.3) and (6.4), respectively, to obtain the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[rr] & & (A,(C,X)) \arrow[r] \arrow[d, dashed, "\sim"] & (P_0,(C,X)) \arrow[r] \arrow[d, "\sim"] & (K_0,(C,X)) \arrow[r] \arrow[d, "\sim"] & {} \\ (M_k(K_0),\Sigma^{-1}X) \arrow[rr] & & (M,X) \arrow[r] & (M_k(P_0),X) \arrow[r] & (M_k(K_0),X) \arrow[r] & {} \end{tikzcd}$$

where, in the above diagram, we have used the notation $(-,(C,X))$ and $(-,X)$ as shorthand for the functors $\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(-, \mathsf{Hom}_{\mathscr{T}}(C, X))$ and $\mathsf{Hom}_{\mathscr{T}}(-, X)$, respectively. By hypothesis (5) of 6.2.6, $\mathsf{Hom}_{\mathscr{T}}(M_k(K_0), \Sigma^{-1}X) = 0$ for $X \in \mathscr{M}_{k+1}$, hence, the broken arrow exists and is an isomorphism.

Now the functors $\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(-, \mathsf{Hom}_{\mathscr{T}}(C, X))$, $\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(-, \mathsf{Hom}_{\mathscr{T}}(C, Y))$, $\mathsf{Hom}_{\mathscr{T}}(-, X)$ and $\mathsf{Hom}_{\mathscr{T}}(-, Y)$ applied to (6.3) and (6.4) for $X, Y \in \mathscr{M}_{k+1}$ yield the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[rr] & & (A,(C,Y)) \arrow[rr] \arrow[dd] & & (P_0,(C,Y)) \arrow[rr] \arrow[dd] & & (K_0,(C,Y)) \arrow[rr] \arrow[dd] & & \cdots \\
0 \arrow[rr] & & (A,(C,X)) \arrow[rr] \arrow[ur] \arrow[dd] & & (P_0,(C,X)) \arrow[rr] \arrow[ur] \arrow[dd] & & (K_0,(C,X)) \arrow[rr] \arrow[ur] \arrow[dd] & & \cdots & \\
& 0 \arrow[rr] & & (M,Y) \arrow[rr] & & (M_k(P_0),Y) \arrow[rr] & & (M_k(K_0),Y) \arrow[rr] & & \cdots \\
0 \arrow[rr] & & (M,X) \arrow[rr] \arrow[ur] & & (M_k(P_0),X) \arrow[rr] \arrow[ur] & & (M_k(K_0),X) \arrow[rr] \arrow[ur] & & \cdots &
\end{tikzcd}$$

and we obtain a natural isomorphism on $\mathscr{M}_{k+1}$:

$$\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(A, \mathsf{Hom}_{\mathscr{T}}(C,-)) \simeq \mathsf{Hom}_{\mathscr{T}}(M,-).$$

This construction defines a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{T}$ by setting $M_{k+1}(A) = M$. Now consider the full subcategory of $\mathscr{T}$ defined by (c.f. Hypotheses 6.2.6)

$$\mathscr{N}_{k+1} := \{X \in \mathscr{T} \mid X \cong M_{k+1}(A) \text{ for some } A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})\}.$$

We want to show that $\mathscr{N}_{k+1} \subseteq \mathscr{M}_{k+1}$, that is, that we have a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$. Let $X \in \mathscr{N}_{k+1}$; then there exists an S^{op}-module A of projective dimension at most $k+1$ such that $M_{k+1}(A) \cong X$. As before, we have a short exact sequence $0 \to K_0 \to P_0 \to A \to 0$ in which P_0 is projective and $K_0 \in \mathsf{Proj}^{k}(S^{\mathsf{op}})$. By construction, we have a distinguished triangle in $\mathscr{T}$:

$$M_k(K_0) \to M_k(P_0) \to M_{k+1}(A) \to \Sigma M_k(K_0).$$

Applying the functor $\mathsf{Hom}_{\mathscr{T}}(C,-)$ to this distinguished triangle gives the following long exact sequence:

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i} M_k(P_0)) \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i} M_{k+1}(A)) \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i+1} M_k(K_0)).$$

By Remark 6.2.7 (which follows from hypothesis (2) of 6.2.6), $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i} M_k(K_0)) = 0$ for $i = 1, \ldots, k$, and by the assumption $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-j} C) = 0$ for $j = 1, \ldots, n+1$. Note that here we have used our assumption that $k \leqslant n$ (see page 120). We also have $M_k(P_0) \in \mathsf{Add}(C)$ by hypothesis (6) of 6.2.6. Hence, we obtain

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i} M_{k+1}(A)) = 0 \text{ for } i = 2, \ldots, k+1.$$

By 6.2.6 hypothesis (2), we have an adjoint pair

$$\begin{tikzcd}
\mathscr{N}_k \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^k(S^{\mathsf{op}}) \arrow[l, shift right, "M_k"']
\end{tikzcd}$$

whose unit η is an isomorphism (hypothesis (3)). So we have the following commutative square:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathscr{T}}(C, M_k(K_0)) \arrow[r] & \mathsf{Hom}_{\mathscr{T}}(C, M_k(P_0)) \\
K_0 \arrow[u, "\eta_{K_0}", "\sim"'] \arrow[r, hook] & P_0 \arrow[u, "\eta_{P_0}", "\sim"']
\end{tikzcd}$$

Hence the map $\mathsf{Hom}_{\mathscr{T}}(C, M_k(K_0)) \to \mathsf{Hom}_{\mathscr{T}}(C, M_k(P_0))$ is an injection, which forces $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1}M_{k+1}(A)) = 0$. Thus $M_{k+1}(A) \in \mathscr{M}_{k+1}$ and we have $\mathscr{N}_{k+1} \subseteq \mathscr{M}_{k+1}$ and a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$.

Remark 6.2.9. So far, all we have proved is that we have a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$ and that $\mathscr{N}_{k+1} \subseteq M_{k+1}$. In order to prove this we have used all the hypotheses of Hypotheses 6.2.6 with the exception of (4).

The next step is to make M_{k+1} into a functor.

6.2.3 Functoriality of M_{k+1}

Recall that by hypothesis (1) of Hypotheses 6.2.6 there exists a functor $M_k : \mathsf{Proj}^k(S^{\mathsf{op}}) \to \mathscr{M}_k$. By the work of section 6.2.2 we have constructed a map $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$ using the functor M_k. We now need to get a functor $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$ from the map M_{k+1}. This can be done indirectly by noticing that we have a natural isomorphism,

$$\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(A, \mathsf{Hom}_{\mathscr{T}}(C, -)) \simeq \mathsf{Hom}_{\mathscr{T}}(M_{k+1}(A), -),$$

which is natural on $\mathscr{M}_{k+1}$. If the functor $\mathsf{Hom}_{\mathscr{T}}(C, -)$, when it acts on $\mathscr{N}_{k+1}$, takes values in $\mathsf{Proj}^{k+1}(S^{\mathsf{op}})$, then we obtain an adjunction,

$$\begin{tikzcd}
\mathscr{N}_{k+1} \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^{k+1}(S^{\mathsf{op}}), \arrow[l, shift right, "M_{k+1}"']
\end{tikzcd}$$

via the following classical representability result.

Lemma 6.2.10 ([43], Corollary IV.1.2)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $G : \mathscr{C} \to \mathscr{D}$ has a left adjoint if and only if for every object C of $\mathscr{C}$ there is a natural isomorphism*

$$\varphi : \mathsf{Hom}_{\mathscr{C}}(C, G(D)) \simeq \mathsf{Hom}_{\mathscr{D}}(F_0(C), D)$$

which is natural in $D \in \mathscr{D}$. Then F_0 is the object function of the left adjoint of G.

Composition of the functor $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{N}_{k+1}$ with the inclusion functor $\iota : \mathscr{N}_{k+1} \to \mathscr{M}_{k+1}$ gives the desired functor.

We now show that we do indeed have a functor $\mathsf{Hom}_{\mathscr{T}}(C,-) : \mathscr{N}_{k+1} \to \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$. Suppose $X \in \mathscr{N}_{k+1}$, then there is an S^{op}-module A of projective dimension at most $k+1$ such that $M_{k+1}(A) \cong X$. We get the usual short exact sequence $0 \to K_0 \to P_0 \to A \to 0$ and the usual distinguished triangle $M_k(K_0) \to M_k(P_0) \to M_{k+1}(A) \to \Sigma M_k(K_0)$. Applying the functor $\mathsf{Hom}_{\mathscr{T}}(C,-)$ to the usual distinguished triangle gives the following long exact sequence:

$$0 \to (C, M_k(K_0)) \to (C, M_k(P_0)) \to (C, M_{k+1}(A)) \to (C, \Sigma M_k(K_0)),$$

where we have used the shorthand described earlier to denote the Hom-spaces. The zero on the left hand side comes by the fact that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1} M_{k+1}(A)) = 0$ because $M_{k+1}(A) \in \mathscr{M}_{k+1}$ (hypothesis (5) of 6.2.6). By condition (4) of Hypotheses 6.2.6, we have $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma M_k(K_0)) = 0$. Hence, we have the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & (C, M_k(K_0)) \arrow[r] & (C, M_k(P_0)) \arrow[r] & (C, M_{k+1}(A)) \arrow[r] & 0. \\
& K_0 \arrow[u, "\eta_{K_0}", "\sim"'] \arrow[r, hook] & P_0 \arrow[u, "\eta_{P_0}", "\sim"'] & &
\end{tikzcd}$$

Therefore, $\mathsf{Hom}_{\mathscr{T}}(C, M_{k+1}(A)) \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$. Moreover, $\mathsf{Hom}_{\mathscr{T}}(C, M_{k+1}(A)) \cong A$ for all $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$. It follows that, $\mathsf{Hom}_{\mathscr{T}}(C, X) \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$.

Thus, we obtain a functor $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$. Furthermore, $\mathscr{N}_{k+1} \subseteq \mathscr{M}_{k+1}$ and there is a natural isomorphism,

$$\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(A, \mathsf{Hom}_{\mathscr{T}}(C, X)) \simeq \mathsf{Hom}_{\mathscr{T}}(M_{k+1}(A), X),$$

which is natural on $\mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ and $\mathscr{M}_{k+1}$. This restricts to an adjoint pair,

$$\begin{tikzcd}
\mathscr{N}_{k+1} \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^{k+1}(S^{\mathsf{op}}). \arrow[l, shift right, "M_{k+1}"']
\end{tikzcd}$$

We now summarise the work of the preceding sections in the following proposition.

Proposition 6.2.11. *Under the assumptions of Setup 6.2.3 and the hypotheses of Hypotheses 6.2.6, we have:*

(1) There exists a functor $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{M}_{k+1}$.

(2) Let $\mathscr{N}_{k+1} = \{X \in \mathscr{T} \mid X \cong M_{k+1}(A) \text{ for some } A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})\}$. *We have* $\mathscr{N}_{k+1} \subseteq \mathscr{M}_{k+1}$. *We also have a natural isomorphism*

$$\mathsf{Hom}_{\mathsf{Mod}(S^{\mathsf{op}})}(A, \mathsf{Hom}_{\mathscr{T}}(C, X)) \simeq \mathsf{Hom}_{\mathscr{T}}(M_{k+1}(A), X)$$

which is natural for $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ *and* $X \in \mathscr{M}_{k+1}$. *This restricts to an adjoint pair*

$$\begin{tikzcd} \mathscr{N}_{k+1} \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^{k+1}(S^{\mathsf{op}}), \arrow[l, shift right, "M_{k+1}"'] \end{tikzcd}$$

where the left adjoint is written above the right adjoint, as in Hypotheses 6.2.6.

Remark 6.2.12. In Proposition 6.2.11 we have accomplished the proof of the induction step for the first two hypotheses in Hypotheses 6.2.6. However, in obtaining this proof we have used all the hypotheses stated in Hypotheses 6.2.6. This demonstrates how the remaining hypotheses are analogous to scaffolding in the construction and the subsequent section is devoted to building the scaffolding.

6.2.4 The scaffolding

We start the construction of the scaffolding with the easier hypotheses, (4), (5) and (6) of Hypotheses 6.2.6.

Lemma 6.2.13. *Under the assumptions of Setup 6.2.3 and Hypotheses 6.2.6 we have:*

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma M_{k+1}(A)) = \cdots = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{n-k} M_{k+1}(A)) = 0$$

for all $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$. *(C.f. Hypotheses 6.2.6, condition* (4).*)*

Proof: Suppose $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$; then we have the usual short exact sequence $0 \to K_0 \to P_0 \to A \to 0$ from which we obtain the usual distinguished triangle $M_k(K_0) \to M_k(P_0) \to M_{k+1}(A) \to \Sigma M_k(K_0)$ with $K_0 \in \mathsf{Proj}^k(S^{\mathsf{op}})$ and P_0 a projective S^{op}-module. Now apply the functor $\mathsf{Hom}_{\mathscr{T}}(C, -)$ to the usual distinguished triangle to get the long exact sequence

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i M_k(P_0)) \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i M_{k+1}(A)) \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{i+1} M_k(K_0)).$$

By Hypotheses 6.2.6, $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i M_k(K_0)) = 0$ for $i = 1, \dots, n+1-k$, and $M_k(P_0) \in \mathsf{Add}(C)$ so that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i M_k(P_0)) = 0$ for $i = 1, \dots, n+1$. Condition (4) of Hypotheses 6.2.6 now follows for $k+1$. □

Lemma 6.2.14. *Under the assumptions of Setup 6.2.3 and Hypotheses 6.2.6 we have:*

$$\mathsf{Hom}_{\mathscr{T}}(M_{k+1}(A), \Sigma^{-1}X) = 0 \text{ for all } A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \text{ and } X \in \mathscr{M}_{k+2}.$$

(C.f. Hypotheses 6.2.6, condition (5)*.)*

Proof: Let $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ and $X \in \mathscr{M}_{k+2}$. Consider the projective resolution we have chosen for A:

$$0 \longrightarrow P_{k+1} \xrightarrow{\pi_{k+1}} P_k \xrightarrow{\pi_k} \cdots \longrightarrow P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} A \longrightarrow 0,$$

which breaks up into short exact sequences

$$\begin{gathered} 0 \to P_{k+1} \to P_k \to K_{k-1} \to 0 \\ \vdots \\ 0 \to K_i \to P_i \to K_{i-1} \to 0 \\ \vdots \\ 0 \to K_0 \to P_0 \to A \to 0 \end{gathered}$$

with $K_i = \ker \pi_i$. These short exact sequences in turn give rise to distinguished triangles:

$$\begin{gathered} M_k(P_{k+1}) \to M_k(P_k) \to M_k(K_{k-1}) \to \Sigma M_k(P_{k+1}) \\ \vdots \\ M_k(K_i) \to M_k(P_i) \to M_k(K_{i-1}) \to \Sigma M_k(K_i) \\ \vdots \\ M_k(K_0) \to M_k(P_0) \to M_{k+1}(A) \to \Sigma M_k(K_0) \end{gathered} \tag{6.5}$$

Consider the long exact sequence obtained by applying the functor $\mathsf{Hom}_{\mathscr{T}}(-, X)$ to distinguished triangle (6.5) with $X \in \mathscr{M}_{k+2}$. We get:

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma^2 M_k(K_0), X) \to \mathsf{Hom}_{\mathscr{T}}(\Sigma M_{k+1}(A), X) \to \mathsf{Hom}_{\mathscr{T}}(\Sigma M_k(P_0), X).$$

We know that $\mathsf{Hom}_{\mathscr{T}}(\Sigma M_k(P_0), X) = 0$ because $X \in \mathscr{M}_{k+2}$ and, by hypothesis (6) of 6.2.6, we have $M_k(P_0) \in \mathsf{Add}(C)$. So, we only need to see that $\mathsf{Hom}_{\mathscr{T}}(\Sigma^2 M_k(K_0), X) =$

0. By the same argument, $\mathsf{Hom}_{\mathscr{T}}(\Sigma^2 M_k(K_0), X) = 0$ if $\mathsf{Hom}_{\mathscr{T}}(\Sigma^3 M_k(K_1), X) = 0$. Applying the same argument inductively yields the following implication:

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma^{i+3} M_k(K_{i+1}), X) = 0 \implies \mathsf{Hom}_{\mathscr{T}}(\Sigma^{i+2} M_k(K_i), X) = 0.$$

Now, $X \in \mathscr{M}_{k+2}$ and $K_k = P_{k+1}$, so

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma^{k+2} M_k(K_k), X) = \mathsf{Hom}_{\mathscr{T}}(\Sigma^{k+2} M_k(P_{k+1}), X) = 0,$$

because $M_k(P_{k+1}) \in \mathsf{Add}(C)$, so it follows that $\mathsf{Hom}_{\mathscr{T}}(\Sigma^{k+1} M_k(K_{k-1}), X) = 0$, and eventually we see that

$$\mathsf{Hom}_{\mathscr{T}}(\Sigma M_{k+1}(A), X) = 0.$$

That is, for $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ and $X \in \mathscr{M}_{k+2}$, we have $\mathsf{Hom}_{\mathscr{T}}(M_{k+1}(A), \Sigma^{-1} X) = 0$, as required. □

Lemma 6.2.15. *Under the assumptions of Setup 6.2.3 and Hypotheses 6.2.6 then given any projective S^{op}-module P we have $M_{k+1}(P) \in \mathsf{Add}(C)$. (C.f. Hypotheses 6.2.6, condition* (6).*)*

Proof: Let P be a projective S^{op}-module and consider the short exact sequence coming from its projective resolution:

$$0 \longrightarrow 0 \longrightarrow P \xrightarrow{\sim} P \longrightarrow 0.$$

This gives a distinguished triangle:

$$0 \to M_k(P) \to M_{k+1}(P) \to \Sigma 0,$$

by construction. Hence $M_{k+1}(P) \cong M_k(P)$ and $M_k(P) \in \mathsf{Add}(C)$ by induction, so we have $M_{k+1}(P) \in \mathsf{Add}(C)$. □

We now turn our attention to the trickier third hypothesis in Hypotheses 6.2.6.

Proposition 6.2.16. *Under the assumptions of Setup 6.2.3 and Hypotheses 6.2.6 then the unit, η, of the adjunction obtained in Proposition 6.2.11*

$$\begin{tikzcd}\mathscr{N}_{k+1} \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \arrow[l, shift right, "M_{k+1}"']\end{tikzcd}$$

is an isomorphism. (C.f. Hypotheses 6.2.6, condition (3).*)*

Proof: The fact that the unit η is an isomorphism is equivalent, by Lemma 4.3.2, to the functor $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{N}_{k+1}$ being full and faithful. Hence we shall show that M_{k+1} is full and faithful and then deduce that its unit must be an isomorphism. By hypothesis (3) of 6.2.6, the unit of the adjunction

$$\begin{tikzcd}
\mathscr{N}_k \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^k(S^{\mathsf{op}}) \arrow[l, shift right, "M_k"']
\end{tikzcd}$$

is an isomorphism, so that the functor $M_k : \mathsf{Proj}^k(S^{\mathsf{op}}) \to \mathscr{N}_k$ is full and faithful. We start by showing the fidelity of M_{k+1}.

Claim: The functor M_{k+1} is faithful.

Proof of claim: Suppose $A_1, A_2 \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ and we have a map $\alpha : A_1 \to A_2$. We want to show that if $M_{k+1}(\alpha) = 0$ then $\alpha = 0$. So assume that $M_{k+1}(\alpha) = 0$. By elementary homological algebra, see for instance [28], α lifts to a map of projective resolutions of A_1 and A_2, so we obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, "\kappa"'] & P_1 \arrow[r] \arrow[d, "\pi"'] & A_1 \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & P_2 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

with $K_1, K_2 \in \mathsf{Proj}^k(S^{\mathsf{op}})$ and $P_1, P_2 \in \mathsf{Proj}(S^{\mathsf{op}})$. By construction we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[rr] & & (M_{k+1}(A_1),-) \arrow[rr] & & (M_k(P_1),-) \arrow[rr] & & (M_k(K_1),-) \arrow[rr] & & \cdots \\
0 \arrow[rr] & & (A_1,(C,-)) \arrow[ur, "\sim"'] \arrow[rr] & & (P_1,(C,-)) \arrow[ur, "\sim"'] \arrow[rr] & & (K_1,(C,-)) \arrow[ur, "\sim"'] \arrow[rr] & & \cdots & \\
 & 0 \arrow[rr] & & (M_{k+1}(A_2),-) \arrow[uu] \arrow[rr] & & (M_k(P_2),-) \arrow[uu] \arrow[rr] & & (M_k(K_2),-) \arrow[uu] \arrow[rr] & & \cdots \\
0 \arrow[rr] & & (A_2,(C,-)) \arrow[uu] \arrow[ur, "\sim"'] \arrow[rr] & & (P_2,(C,-)) \arrow[uu] \arrow[ur, "\sim"'] \arrow[rr] & & (K_2,(C,-)) \arrow[uu] \arrow[ur, "\sim"'] \arrow[rr] & & \cdots &
\end{tikzcd}$$

where the arrows on the back face of the cuboid above are given by the obvious compositions and where the blanks are assumed to take values in $\mathscr{M}_{k+1}$. The back face then restricts to the following commuative diagram.

$$\begin{tikzcd}
0 \arrow[r] & (M_{k+1}(A_1),-) \arrow[r] & (M_k(P_1),-) \arrow[r] & (M_k(K_1),-) \arrow[r] & {} \\
0 \arrow[r] & (M_{k+1}(A_2),-) \arrow[r] \arrow[u] & (M_k(P_2),-) \arrow[r] \arrow[u] & (M_k(K_2),-) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

Note that this links the functoriality of M_{k+1} with the functoriality of M_k.

Using this link in the functoriality of the functors M_{k+1} and M_k, we then get the commutative diagram below.

$$\begin{tikzcd}
M_k(K_1) \arrow[r] \arrow[d, "M_k(\kappa)"'] & M_k(P_1) \arrow[r] \arrow[d, "M_k(\pi)"] \arrow[dl, dashed, "M_k(\rho)"'] \arrow[dr, dashed, "0"] & M_{k+1}(A_1) \arrow[r] \arrow[d, "M_{k+1}(\alpha)"] & {} \\
M_k(K_2) \arrow[r] & M_k(P_2) \arrow[r] & M_{k+1}(A_2) \arrow[r] & {}
\end{tikzcd}$$

Since $M_{k+1}(\alpha) : M_{k+1}(A_1) \to M_{k+1}(A_2)$ is zero, it follows that $M_k(\pi) : M_k(P_1) \to M_k(P_2)$ factors through $M_k(K_2)$ as shown. Since, by hypothesis (3) of 6.2.6, M_k is full and faithful, the factoring map has the form $M_k(\rho)$ for some $\rho : P_1 \to K_2$. Hence we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, "\kappa"'] & P_1 \arrow[r] \arrow[d, "\pi"] \arrow[dl, dashed, "\rho"'] & A_1 \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & P_2 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

whence, we have $\alpha = 0$ and M_{k+1} is a faithful functor, proving the claim.

Claim: The functor M_{k+1} is full.

Proof of claim: Let $A_1, A_2 \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$ and suppose we have a map $a : M_{k+1}(B_1) \to M_{k+1}(B_2)$. By construction we have two distinguished triangles, as shown below.

$$\begin{tikzcd}
M_k(K_1) \arrow[r] \arrow[d, dashed, "M_k(\kappa)"'] & M_k(P_1) \arrow[r] \arrow[d, dashed, "M_k(\pi)"'] & M_{k+1}(A_1) \arrow[r] \arrow[d, "a"'] & \Sigma M_k(K_1) \arrow[d, dashed, "\Sigma M_k(\kappa)"'] \\
M_k(K_2) \arrow[r] & M_k(P_2) \arrow[r] & M_{k+1}(A_2) \arrow[r] & \Sigma M_k(K_2)
\end{tikzcd} \tag{6.6}$$

Note that the composite $M_k(P_1) \to M_{k+1}(A_1) \to M_{k+1}(A_2) \to \Sigma M_k(K_2)$ is zero by hypotheses (4) and (6) of Hypotheses 6.2.6. Hypothesis (4) says that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma M_k(K_2)) = 0$, while hypothesis (6) says that $M_k(P_1) \in \mathsf{Add}(C)$ so that $\mathsf{Hom}_{\mathscr{T}}(M_k(P_1), \Sigma M_k(K_2)) = 0$, forcing the composite to be zero. Hence the broken arrow $M_k(\pi) : M_k(P_1) \to M_k(P_2)$

exists making the middle square commutative. It follows that a map $M_k(\kappa) : M_k(K_1) \to M_k(K_2)$ exists, making the whole diagram commutative. The maps $M_k(\pi)$ and $M_k(\kappa)$ have this form because M_k is full by hypothesis (3) of 6.2.6. This then gives a commutative (because M_k is faithful) diagram,

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, "\kappa"'] & P_1 \arrow[r] \arrow[d, "\pi"'] & A_1 \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & P_2 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact. Applying the functors M_k and M_{k+1} we get, by construction, the following commutative diagram of distinguished triangles.

$$\begin{tikzcd}
M_k(K_1) \arrow[r] \arrow[d, "M_k(\kappa)"'] & M_k(P_1) \arrow[r] \arrow[d, "M_k(\pi)"'] & M_{k+1}(A_1) \arrow[r] \arrow[d, "M_{k+1}(\alpha)"'] & \Sigma M_k(K_1) \arrow[d, "\Sigma M_k(\kappa)"'] \\
M_k(K_2) \arrow[r] & M_k(P_2) \arrow[r] & M_{k+1}(A_2) \arrow[r] & \Sigma M_k(K_2)
\end{tikzcd} \tag{6.7}$$

We need to show that $M_{k+1}(\alpha) = a$. By the commutativity of diagrams (6.6) and (6.7), we can consider the following commutative diagram:

$$\begin{tikzcd}
M_k(K_1) \arrow[r] & M_k(P_1) \arrow[r] \arrow[d] \arrow[dr, dashed, "0"] & M_{k+1}(A_1) \arrow[r] \arrow[d, "M_{k+1}(\alpha)-a"] & \Sigma M_k(K_1) \arrow[dl, dashed, bend left, "\exists"] \\
& M_k(P_2) \arrow[r] & M_{k+1}(A_2) &
\end{tikzcd} \tag{6.8}$$

The composite straight broken arrow is zero by the commutativity of diagrams (6.6) and (6.7), so the curved broken arrow $\Sigma M_k(K_1) \to M_{k+1}(A_2)$ exists making the right hand triangle commute. However, we already know any map $\Sigma M_k(K_1) \to M_{k+1}(A_2)$ has to be zero. Hence $M_{k+1}(\alpha) - a = 0$, that is $M_{k+1}(\alpha) = a$. Thus, the functor $M_{k+1} : \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \to \mathscr{N}_{k+1}$ is full, proving the second claim.

Since the functor M_{k+1} is full and faithful and it appears in an adjunction

$$\begin{tikzcd}
\mathscr{N}_{k+1} \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Proj}^{k+1}(S^{\mathsf{op}}) \arrow[l, shift right, "M_{k+1}"']
\end{tikzcd}$$

it follows that the unit η of this adjunction is an isomorphism, thus proving hypothesis (3) for $k+1$. □

Hence we have shown that if Hypotheses 6.2.6 are true for k then they are true for $k+1$. We know, by Lemma 6.2.8 that they are true for $k = 0$. It is clear that the induction terminates at $k = n+1$, thus setting $M = M_{n+1}$ and $\mathscr{M} = \mathscr{M}_{n+1}$ gives the existence of a functor

$$M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{M}$$

which is left adjoint to the functor $\mathsf{Hom}_{\mathscr{T}}(C, -) : \mathscr{M} \to \mathsf{Mod}(S^{\mathsf{op}})$, completing the proof of Theorem 6.2.5.

It is useful for the next section to highlight two important aspects of the proof of Theorem 6.2.5. The first is a general remark about a specific distinguished triangle which is obtained in the proof of Theorem 6.2.5; the second is relates to a useful construction which occurs in the proof of Proposition 6.2.16.

Remark 6.2.17. Given a short exact sequence, $0 \to K \to P \to A \to 0$, in $\mathsf{Mod}(S^{\mathsf{op}})$ with $K \in \mathsf{Proj}^k(S^{\mathsf{op}})$, $P \in \mathsf{Proj}(S^{\mathsf{op}})$ and $A \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$, the proof of Theorem 6.2.5 gives a distinguished triangle $MK \to MP \to MA \to \Sigma MK$.

Remark 6.2.18. In the proof of Proposition 6.2.16 we observed the following construction: if we are given a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "i_1"] \arrow[d, "\kappa"'] & P_1 \arrow[r, "p_1"] \arrow[d, "\pi"'] & A_1 \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & K_2 \arrow[r, "i_2"'] & P_2 \arrow[r, "p_2"'] & A_2 \arrow[r] & 0,
\end{tikzcd}$$

in which the rows are short exact sequences, $A_1, A_2 \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$, $K_1, K_2 \in \mathsf{Proj}^k(S^{\mathsf{op}})$ and $P_1, P_2 \in \mathsf{Proj}(S^{\mathsf{op}})$, we get a commutative diagram,

$$\begin{tikzcd}
MK_1 \arrow[r, "Mi_1"] \arrow[d, "M\kappa"'] & MP_1 \arrow[r, "Mp_1"] \arrow[d, "M\pi"'] & MA_1 \arrow[r] \arrow[d, "M\alpha"'] & \Sigma MK_1 \arrow[d, "\Sigma M\kappa"'] \\
MK_2 \arrow[r, "Mi_2"'] & MP_2 \arrow[r, "Mp_2"'] & MA_2 \arrow[r] & \Sigma MK_2,
\end{tikzcd}$$

in which the rows are distinguished triangles in $\mathscr{T}$.

We shall now regard the functor $M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{M}$ obtained in Theorem 6.2.5 as taking values in $\mathscr{T}$ via composition with the inclusion functor $i : \mathscr{M} \hookrightarrow \mathscr{T}$ (c.f. Theorem 6.1.3).

Corollary 6.2.19. *Viewing the functor obtained in Theorem 6.2.5 as taking values in $\mathscr{T}$, the functor $M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{T}$ is a full embedding.*

6.3 Some consequences

In this section we consider the behaviour of the generalised Moore spectra obtained in the previous section. In particular, we show that the functor M is "well behaved" in a natural way. But firstly, we revisit Jørgensen's construction.

6.3.1 Jørgensen's construction revisited

In this brief section we return to Jørgensen's theorem (stated in this chapter as Theorem 6.1.3). We first note the following specialisation of Theorem 6.2.5.

Proposition 6.3.1. *Let $\mathscr{T}$ be an R-linear triangulated category with set indexed coproducts. Suppose C is a compact object of $\mathscr{T}$ satisfying the following assumptions:*

(1) Its endomorphism algebra $S^{\mathsf{op}} = \mathsf{End}_{\mathscr{T}}(C)^{\mathsf{op}}$ has global dimension 1; and,

(2) We have $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma C) = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1} C) = 0$.

Let $\mathscr{M} = \mathscr{M}_1$ (see Definition 6.2.4). Then, the functor

$$\mathsf{Hom}_{\mathscr{T}}(C, -) : \mathscr{M} \to \mathsf{Mod}(S^{\mathsf{op}})$$

has a left adjoint

$$M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{M}.$$

Moreover, the functor M is a full embedding of the module category $\mathsf{Mod}(S^{\mathsf{op}})$ into the full subcategory $\mathscr{M}$ of $\mathscr{T}$.

Proposition 6.3.1 can be viewed as a generalisation of Theorem 6.1.3. The restriction on the global dimension in Proposition 6.3.1 corresponds to the assumption that the ring R is a principal ideal domain in Setup 6.1.1. The assumption that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma C) = 0$ in Proposition 6.3.1 replaces the assumption that $\mathsf{Hom}_{\mathscr{T}}(C, C)$ is a flat R-module in Setup 6.1.1. In addition, if the unit of the adjunction obtained in Theorem 6.1.3 is an isomorphism then we have

$$R \cong \mathsf{Hom}_{\mathscr{T}}(C, M'(R)) \cong \mathsf{Hom}_{\mathscr{T}}(C, C) = S,$$

where M' denotes the functor obtained in Theorem 6.1.3. Hence, in this case, $\mathsf{Mod}(R^{\mathsf{op}}) \simeq \mathsf{Mod}(S^{\mathsf{op}})$ and the functors M, obtained in Proposition 6.3.1, and M', obtained in Theorem 6.1.3, coincide (up to natural isomorphism).

Remark 6.3.2. Note that the full subcategory $\mathscr{N}_1$, defined in Hypotheses 6.2.6, of the auxilliary category $\mathscr{M}_1$ defined in Definition 6.2.4 coincides with the auxilliary category $\mathscr{M}$ defined in Definition 6.1.2.

6.3.2 The functor M is well behaved

We now show that the functor $M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{T}$ constructed in Theorem 6.2.5 is well behaved with respect to short exact sequences in $\mathsf{Mod}(S^{\mathsf{op}})$ and distinguished triangles in $\mathscr{T}$. The hard work carried out in section 6.2 provides the setting for a functorial proof of Theorem 6.3.3 which differs in character entirely with the corresponding result which it generalises ([35, Theorem 4.11]). However, a proof in the same character as [35, Theorem 4.11] does exist; we give this alternative in section 6.3.3.

Theorem 6.3.3. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Let C be an object of $\mathscr{T}$ satisfying the assumptions of Setup 6.2.3. Let $M : \mathsf{Mod}(S^{\mathsf{op}}) \to \mathscr{T}$ be the functor obtained in Theorem 6.2.5. If $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a} A'' \longrightarrow 0$ is a short exact sequence in $\mathsf{Mod}(S^{\mathsf{op}})$, then there is a distinguished triangle:*

$$MA' \xrightarrow{Ma'} MA \xrightarrow{Ma} MA'' \longrightarrow \Sigma MA'$$

in $\mathscr{T}$.

Proof: By Theorem 6.2.5 we have an adjoint pair

$$\begin{tikzcd} \mathscr{M} \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{Mod}(S^{\mathsf{op}}), \arrow[l, shift right, "M"'] \end{tikzcd}$$

where $\mathscr{M} = \{X \in \mathscr{T} \mid \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i}X) = 0 \text{ for } i = 1, \ldots, n+1\}$. Suppose we have a short exact sequence

$$0 \longrightarrow B' \xrightarrow{b'} B \xrightarrow{b} B'' \longrightarrow 0. \tag{6.9}$$

We can extend the morphism $Mb' : MB' \to Mb$ to get a distinguished triangle:

$$MB' \xrightarrow{Mb'} MB \xrightarrow{f} Z \xrightarrow{g} \Sigma MB'. \tag{6.10}$$

We first claim that $Z \in \mathscr{M}$. Applying the functor $\mathsf{Hom}_{\mathscr{T}}(C, -)$ to distinguished triangle (6.10) gives the following long exact sequence.

$$\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i}MB) \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i}Z) \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i+1}MB'). \tag{6.11}$$

Since MB and MB' are objects of $\mathscr{M}$, we have $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i}Z) = 0$ for $i = 2, \ldots, n{+}1$. We need to check that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1}Z) = 0$. Looking at the right hand end of (6.11), we have the exact sequence

$$0 \to \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1}Z) \to \mathsf{Hom}_{\mathscr{T}}(C, MB') \hookrightarrow \mathsf{Hom}_{\mathscr{T}}(C, MB),$$

where $\mathsf{Hom}_{\mathscr{T}}(C, MB') \hookrightarrow \mathsf{Hom}_{\mathscr{T}}(C, MB)$ is injective by the fact that the unit of the adjunction in Theorem 6.2.5 is an isomorphism. It follows that $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-1}Z) = 0$ and, thus, that $Z \in \mathscr{M}$, as claimed.

We shall now show that not only is Z an object of $\mathscr{M}$, but that Z is isomorphic to MB'' in $\mathscr{M}$. Let X be an object of $\mathscr{M}$ and apply the functor $\mathsf{Hom}_{\mathcal{S}^{\mathsf{op}}}(-, \mathsf{Hom}_{\mathscr{T}}(C, X))$ to the given short exact sequence, (6.9). This gives a long exact sequence, which forms the top row of a commutative diagram which is given by the adjunction obtained from Theorem 6.2.5:

$$\begin{tikzcd}
0 \arrow[r] & (B'', (C,X)) \arrow[r] \arrow[d, "\sim"] & (B, (C,X)) \arrow[r] \arrow[d, "\sim"] & (B', (C,X)) \arrow[r] \arrow[d, "\sim"] & {} \\
0 \arrow[r] & (MB'', X) \arrow[r] & (MB, X) \arrow[r] & (MB', X) \arrow[r] & {}
\end{tikzcd} \tag{6.12}$$

where we use $(B, (C,X))$ as a shorthand for $\mathsf{Hom}_{\mathcal{S}^{\mathsf{op}}}(B, \mathsf{Hom}_{\mathscr{T}}(C, X))$ and (MB, X) as a shorthand for $\mathsf{Hom}_{\mathscr{T}}(MB, X)$, and so on. Applying the functor $\mathsf{Hom}_{\mathscr{T}}(-, X)$ to distinguished triangle (6.10) also gives an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & (Z, X) \arrow[r] & (MB, X) \arrow[r] & (MB', X) \arrow[r] & {}
\end{tikzcd}$$

which combined with the exact sequence obtained from the bottom row of diagram (6.12) yields the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & (MB'', X) \arrow[r, "(Mb)^*"] \arrow[d, dashed, "\xi^*"', "\sim"] & (MB, X) \arrow[r, "(Mb')^*"] \arrow[d, equal] & (MB', X) \arrow[r] \arrow[d, equal] & {} \\
0 \arrow[r] & (Z, X) \arrow[r, "f^*"'] & (MB, X) \arrow[r, "(Mb)^*"'] & (MB', X) \arrow[r] & {}
\end{tikzcd} \tag{6.13}$$

Hence there is an isomorphism $\xi^* : \mathsf{Hom}_{\mathscr{T}}(MB'', X) \to \mathsf{Hom}_{\mathscr{T}}(Z, X)$ for all $X \in \mathscr{M}$. It follows that $\xi : MB'' \to Z$ is an isomorphism in $\mathscr{M}$, as desired.

From this we obtain the following diagram:

$$\begin{tikzcd}
MB' \arrow[r, "Mb'"] \arrow[d, equal] & MB \arrow[r, "Mb"] \arrow[d, equal] & MB'' \arrow[r, "g\circ\xi"] \arrow[d, "\xi"', "\sim"] & \Sigma MB' \arrow[d, equal] \\
MB' \arrow[r, "Mb'"'] & MB \arrow[r, "f"'] & Z \arrow[r, "g"'] & \Sigma MB'.
\end{tikzcd}$$

The squares on the left and right of the diagram above are commutative. If we can show the middle square commutes, then we shall have obtained an isomorphism of our candidate triangle with a distinguished triangle in $\mathscr{T}$, thus giving us the desired distinguished triangle. In order to do this we have to remember more information from the construction of the isomorphism obtained in (6.13). Diagram (6.13) gives the equality $(Mb)^* = f^* \circ \xi^* = (\xi \circ f)^*$, and hence $Mb = \xi \circ f$, making the central square commute, thus proving the theorem. □

We now aim to prove that for $A, B \in \mathsf{Mod}(S^{\mathsf{op}})$ there are natural maps

$$\mathsf{Ext}^n_{S^{\mathsf{op}}}(A,B) \xrightarrow{\Delta^n_{A,B}} \mathsf{Hom}_{\mathscr{T}}(MA, \Sigma^n MB).$$

We need to appeal to the definitions of a δ-functor, a universal δ-functor and the fact that the functor $\mathsf{Ext}(-,-)$ is a universal δ-functor. The following definitions are taken from [26].

Definition 6.3.4. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. A *(covariant) δ-functor* from $\mathscr{A}$ to $\mathscr{B}$ is a collection of functors $T = (T^i)_{i\geqslant 0}$ together with a morphism

$$\delta^i : T^i(A'') \to T^{i+1}(A')$$

for each short exact sequence $0 \to A' \to A \to A'' \to 0$ and each $i \geqslant 0$, such that

(1) For each short exact sequence, as above, there is a long exact sequence

$$0 \longrightarrow T^0(A') \longrightarrow T^0(A) \longrightarrow T^0(A'') \xrightarrow{\delta^0} T^1(A') \longrightarrow \cdots$$
$$\cdots \longrightarrow T^i(A') \longrightarrow T^i(A) \longrightarrow T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \longrightarrow \cdots$$

(2) For each morphism of one short exact sequence, as above, into another $0 \to B' \to B \to B'' \to 0$, the δs give a commutative diagram

$$\begin{tikzcd}
T^i(A'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\
T^i(B'') \arrow[r, "\delta^i"] & T^{i+1}(B').
\end{tikzcd}$$

A *contravariant δ-functor* is defined similarly.

Definition 6.3.5. A δ-functor $T = (T^i)_{i\geqslant 0} : \mathscr{A} \to \mathscr{B}$ is said to be a *universal δ-functor* if, given any other δ-functor $U = (U^i)_{i\geqslant 0} : \mathscr{A} \to \mathscr{B}$ and any given morphism of functors $f^0 : T^0 \to U^0$, there exists a unique sequence of morphisms $f^i : T^i \to U^i$ for each $i \geqslant 0$, starting with the given f, which commute with the δ^is for each short exact sequence.

It is a well known fact that $\mathsf{Ext}^n(A, -)$ is a covariant universal δ-functor and $\mathsf{Ext}^n(-, B)$ is a contravariant universal δ-functor. We will need the following lemma.

Lemma 6.3.6. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Let C be an object of $\mathscr{T}$ satisfying the assumptions of Setup 6.2.3. Recall that $S^{\mathsf{op}} = \mathsf{End}(C)^{\mathsf{op}}$. Then, we have the following:*

(i) For $A \in \mathsf{Mod}(S^{\mathsf{op}})$ the functor $(\mathsf{Hom}_{\mathscr{T}}(MA, \Sigma^n M(-)))_{n\geqslant 0}$ is a covariant δ-functor.

(ii) For $B \in \mathsf{Mod}(S^{\mathsf{op}})$ the functor $(\mathsf{Hom}_{\mathscr{T}}(M(-), \Sigma^n MB))_{n\geqslant 0}$ is a contravariant δ-functor.

Proof: Let $A \in \mathsf{Mod}(S^{\mathsf{op}})$ and consider $U^n(-) = \mathsf{Hom}_{\mathscr{T}}(MA, \Sigma^n M(-))$. We claim that $(U^n)_{n\geqslant 0}$ is a covarant δ-functor. Suppose we have a short exact sequence $0 \to B' \to B \to B'' \to 0$ in $\mathsf{Mod}(S^{\mathsf{op}})$. By Theorem 6.3.3, there is a distinguished triangle $MB' \to MB \to MB'' \to \Sigma MB'$. This distinguished triangle induces a long exact sequence

$$\cdots \to U^i(B') \to U^i(B) \to U^i(B'') \to U^{i+1}(B') \to \cdots$$

and gives a morphism $\delta^i : U^i(B'') \to U^{i+1}(B')$.

Now suppose we have another short exact sequence $0 \to C' \to C \to C'' \to 0$ and a morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0,
\end{tikzcd}$$

which gives a commutative diagram of distinguished triangles:

$$\begin{tikzcd}
MB' \arrow[r] \arrow[d] & MB \arrow[r] \arrow[d] & MB'' \arrow[r] \arrow[d] & \Sigma MB' \arrow[d] \\
MC' \arrow[r] & MC \arrow[r] & MC'' \arrow[r] & \Sigma MC'.
\end{tikzcd}$$

This, in turn, yields a commutative diagram:

$$\begin{tikzcd}
U^i(B'') \arrow[r, "\delta^i"] \arrow[d] & U^{i+1}(B') \arrow[d] \\
U^i(C'') \arrow[r, "\delta^i"'] & U^{i+1}(C').
\end{tikzcd}$$

Hence, $(U^n)_{n\geqslant 0}$ is a covariant δ-functor. This proves assertion (i); assertion (ii) is proved similarly. □

Theorem 6.3.7. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Let C be an object of $\mathscr{T}$ satisfying the assumptions of Setup 6.2.3. Recall that $S^{\mathsf{op}} = \mathsf{End}(C)^{\mathsf{op}}$. Then, we have the following:*

(i) For $A, B \in \mathsf{Mod}(S^{\mathsf{op}})$ there exist maps,

$$\mathsf{Ext}^n_{S^{\mathsf{op}}}(A, B) \xrightarrow{\Delta^n_{A,B}} \mathsf{Hom}_{\mathscr{T}}(MA, \Sigma^n MB),$$

which are natural in A and B.

(ii) Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathsf{Mod}(S^{\mathsf{op}})$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_{S^{\mathsf{op}}}(A'', B) \arrow[r] \arrow[d, "\Delta^0_{A'',B}"] & \mathsf{Hom}_{S^{\mathsf{op}}}(A, B) \arrow[r] \arrow[d, "\Delta^0_{A,B}"] & \mathsf{Hom}_{S^{\mathsf{op}}}(A', B) \arrow[r] \arrow[d, "\Delta^0_{A',B}"] & {} \\
0 \arrow[r] & (MA'', MB) \arrow[r] & (MA, MB) \arrow[r] & (MA', MB) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{Ext}^n_{S^{\mathsf{op}}}(A'', B) \arrow[r] \arrow[d, "\Delta^n_{A'',B}"] & \mathsf{Ext}^n_{S^{\mathsf{op}}}(A, B) \arrow[r] \arrow[d, "\Delta^n_{A,B}"] & \mathsf{Ext}^n_{S^{\mathsf{op}}}(A', B) \arrow[r] \arrow[d, "\Delta^n_{A',B}"] & \cdots \\
\cdots \arrow[r] & (MA'', \Sigma^n MB) \arrow[r] & (MA, \Sigma^n MB) \arrow[r] & (MA', \Sigma^n MB) \arrow[r] & \cdots .
\end{tikzcd}$$

(iii) Given a short exact sequence $0 \to B' \to B \to B'' \to 0$ in $\mathsf{Mod}(S^{\mathsf{op}})$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{Hom}_{S^{\mathsf{op}}}(A, B') \arrow[r] \arrow[d, "\Delta^0_{A,B'}"] & \mathsf{Hom}_{S^{\mathsf{op}}}(A, B) \arrow[r] \arrow[d, "\Delta^0_{A,B}"] & \mathsf{Hom}_{S^{\mathsf{op}}}(A, B'') \arrow[r] \arrow[d, "\Delta^0_{A,B''}"] & {} \\
0 \arrow[r] & (MA, MB') \arrow[r] & (MA, MB) \arrow[r] & (MA, MB'') \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{Ext}^n_{S^{\mathsf{op}}}(A, B') \arrow[r] \arrow[d, "\Delta^n_{A,B}"] & \mathsf{Ext}^n_{S^{\mathsf{op}}}(A, B) \arrow[r] \arrow[d, "\Delta^n_{A,B}"] & \mathsf{Ext}^n_{S^{\mathsf{op}}}(A, B'') \arrow[r] \arrow[d, "\Delta^n_{A,B''}"] & \cdots \\
\cdots \arrow[r] & (MA, \Sigma^n MB') \arrow[r] & (MA, \Sigma^n MB) \arrow[r] & (MA, \Sigma^n MB'') \arrow[r] & \cdots .
\end{tikzcd}$$

Note that the use of $(X, \Sigma^n Y)$ *on the bottom row in statements (ii) and (iii) is shorthand for* $\mathsf{Hom}_{\mathscr{T}}(X, \Sigma^n Y)$.

Proof: The functors $(\mathsf{Ext}^n_{S^{\mathsf{op}}}(A, -))_{n \geqslant 0}$ and $(\mathsf{Ext}^n_{S^{\mathsf{op}}}(-, B))_{n \geqslant 0}$ are universal δ-functors. We also know that the functor $(\mathsf{Hom}_{\mathscr{T}}(MA, \Sigma^n M(-)))_{n \geqslant 0}$ is a covariant δ-functor and the functor $(\mathsf{Hom}_{\mathscr{T}}(-, \Sigma^n MB))_{n \geqslant 0}$ is a contravariant δ-functor by Lemma 6.3.6; the theorem now follows. □

6.3.3 An alternative proof of Theorem 6.3.3

In this section we give another proof of Theorem 6.3.3. The reason for including this proof is that the character is so different from the proof of Theorem 6.3.3 above: it is more in the spirit of Jørgensen's original construction, therefore, it may be possible that by using the techniques employed in this section a proof of Theorem 6.2.5 may be obtained in a similar spirit to the version which it generalises in [35].

Recall the conditions of Setup 6.2.3. Suppose we have a short exact sequence $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a} A'' \longrightarrow 0$ in $\mathsf{Mod}(S^{\mathsf{op}})$. The second proof of Theorem 6.3.3 is an induction on the projective dimensions of the S^{op}-modules A', A and A''.

Suppose we have a short exact sequence $0 \longrightarrow P' \xrightarrow{\pi'} P \xrightarrow{\pi} P'' \longrightarrow 0$ of projective S^{op}-modules. Such an exact sequence is split, hence there are splitting maps:

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, shift left, "\pi'"] & P \arrow[l, shift left, "p"] \arrow[r, shift left, "\pi"] & P \arrow[l, shift left, "p"] \arrow[r] & 0
\end{tikzcd}$$

such that $\mathsf{id}_P = p''\pi + \pi' p$. Applying the functor M, constructed in Theorem 6.2.5, to the diagram above yields:

$$\begin{tikzcd}
MP' \arrow[r, shift left, "M\pi'"] & MP \arrow[l, shift left, "Mp"] \arrow[r, shift left, "M\pi"] & MP'' \arrow[l, shift left, "Mp''"]
\end{tikzcd}.$$

This diagram is then a candidate triangle (see Chapter 3). It is well known that such a diagram is isomorphic to a distinguished triangle:

$$MP' \to MP' \coprod MP'' \to MP'' \to \Sigma MP'.$$

Hence, $MP' \xrightarrow{M\pi'} MP \xrightarrow{M\pi} MP'' \xrightarrow{0} \Sigma MP'$ is a distinguished triangle, proving the assertion for projectives.

Let $k \geq 1$ and, by induction, suppose that any short exact sequence $0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a} A'' \longrightarrow 0$ in $\mathsf{Proj}^k(S^{\mathsf{op}})$ corresponds to a distinguished triangle $MA' \xrightarrow{a'} MA \xrightarrow{a''}$

$MA'' \longrightarrow \Sigma MA'$. Now suppose we have a short exact sequence $0 \longrightarrow B' \xrightarrow{b'} B \xrightarrow{b} B'' \longrightarrow 0$ with $B', B, B'' \in \mathsf{Proj}^{k+1}(S^{\mathsf{op}})$. This short exact sequence lifts to the diagram below by elementary homological algebra,

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K' \arrow[r] & P' \arrow[r] & B' \arrow[r] \arrow[d, "b'"] & 0 \\
 & & & B \arrow[d, "b"] & \\
0 \arrow[r] & K'' \arrow[r] & P'' \arrow[r] & B'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

with $P', P'' \in \mathsf{Proj}(S^{\mathsf{op}})$ and $K', K'' \in \mathsf{Proj}^{k}(S^{\mathsf{op}})$. By the Horseshoe Lemma, [57, Horseshoe Lemma 2.2.8], we can complete this diagram to the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K' \arrow[r, "f'"] \arrow[d, "\kappa'"] & P' \arrow[r, "g'"] \arrow[d, "\pi'"] & B' \arrow[r] \arrow[d, "b'"] & 0 \\
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\kappa"] & P \arrow[r, "g"] \arrow[d, "\pi"] & B \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & K'' \arrow[r, "f''"] \arrow[d] & P'' \arrow[r, "g''"] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $P = P' \coprod P''$ and $K \in \mathsf{Proj}^{k}(S^{\mathsf{op}})$. By induction, the short exact sequence in the left hand column gives $MK' \xrightarrow{M\kappa'} MK \xrightarrow{M\kappa} MK'' \longrightarrow \Sigma MK'$ and the short exact sequence of projectives sitting in the central column gives a split distinguished triangle $MP' \xrightarrow{M\pi'} MP \xrightarrow{M\pi} MP'' \longrightarrow \Sigma MP'$. By Remark 6.2.17 the short exact sequences forming the rows of the diagram above yield distinguished triangles. These distinguished triangles give us the commutative diagram below:

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] \arrow[d, "M\kappa'"'] & MP' \arrow[r, "Mg'"] \arrow[d, "M\pi'\ (*)"] & MB' \arrow[r] \arrow[d, "Mb'\ \ (**)"] & \Sigma MK' \arrow[d, "\Sigma M\kappa'"] \\
MK \arrow[r, "Mf"] \arrow[d, "M\kappa"'] & MP \arrow[r, "Mg"] \arrow[d, "M\pi\ \ (\dagger)"] & MB \arrow[r] \arrow[d, "Mb\ \ \ (\ddagger)"] & \Sigma MK \arrow[d, "\Sigma M\kappa"] \\
MK'' \arrow[r, "Mf''"] & MP'' \arrow[r, "Mg''"] & MB'' \arrow[r] & \Sigma MK''.
\end{tikzcd}$$

Claim: Mb' is the unique map making squares $(*)$ and $(**)$ commute, and Mb is the unique map making the squares (†) and (‡) commute.

Proof of claim: We show that Mb' is unique. Suppose there was another map $\beta : MB' \to MB$ making $(*)$ and $(**)$ commute, then the composition $(Mb' - \beta) \circ Mg' = 0$, thus we have a commutative diagram:

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] & MP' \arrow[r, "Mg'"] \arrow[dr, "0"'] & MB' \arrow[r] \arrow[d, "Mb'-\beta"] & \Sigma MK'. \arrow[dl, dashed, bend left, "\exists"] \\
 & & MB &
\end{tikzcd}$$

The broken arrow, however, by the proof of Proposition 6.2.16 (see diagram (6.8) on page 130) is known to be zero, and it follows that $Mb' = \beta$, as claimed. The uniqueness of Mb is similar.

This gives us the following commutative diagram:

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] \arrow[d, "M\kappa'"'] & MP' \arrow[r, "Mg'"] \arrow[d, "M\pi'"] & MB' \arrow[r, "h'"] \arrow[d, "Mb'"] & \Sigma MK' \arrow[d, "\Sigma M\kappa'"] \\
MK \arrow[r, "Mf"] \arrow[d, "M\kappa"'] & MP \arrow[r, "Mg"] \arrow[d, "M\pi"] & MB \arrow[r, "h"] \arrow[d, "Mb"] & \Sigma MK \arrow[d, "\Sigma M\kappa"] \\
MK'' \arrow[r, "Mf''"] \arrow[d] & MP'' \arrow[r, "Mg''"] \arrow[d] & MB'' \arrow[r, "h''"] \arrow[d, dashed] & \Sigma MK'' \arrow[d] \\
\Sigma MK' \arrow[r, "\Sigma Mf'"] & \Sigma MP' \arrow[r, "\Sigma Mg'"] & \Sigma MB' \arrow[r, "\Sigma h'"] & \Sigma^2 MK'.
\end{tikzcd} \tag{6.14}$$

The two left hand columns are distinguished triangles, with that in the second column split. The rows are distinguished triangles, and the broken arrow $MB'' \to \Sigma MB'$ exists by virtue of the axioms of triangulated categories. Therefore, all rows and columns of diagram (6.14) are distinguished triangles with the exception of the candidate triangle

$MB' \xrightarrow{Mb'} MB \xrightarrow{Mb} MB'' \longrightarrow \Sigma MB'$. We claim that this is also a distinguished triangle; this is proved by appealing to result of Neeman.

Let C', C and C'' be the mapping cones of $Mf' : MK' \to MP'$, $Mf : MK \to MP$ and $Mf'' : MK'' \to MP''$, respectively. By a lemma of Neeman [45, Lemma 1.7], the mapping cones form a distinguished triangle $C' \xrightarrow{c'} C \xrightarrow{c} C'' \xrightarrow{c''} \Sigma C'$ which occurs in the following commutative diagram.

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] \arrow[d, "M\kappa'"'] & MP' \arrow[r, "\varphi'"] \arrow[d, "M\pi'"] & C' \arrow[r, "\theta'"] \arrow[d, "c'"] & \Sigma MK' \arrow[d, "\Sigma M\kappa'"] \\
MK \arrow[r, "Mf"] \arrow[d, "M\kappa"'] & MP \arrow[r, "\varphi"] \arrow[d, "M\pi"] & C \arrow[r, "\theta"] \arrow[d, "c"] & \Sigma MK \arrow[d, "\Sigma M\kappa"] \\
MK'' \arrow[r, "Mf''"] \arrow[d] & MP'' \arrow[r, "\varphi''"] \arrow[d] & C'' \arrow[r, "\theta''"] \arrow[d, "c''"] & \Sigma MK'' \arrow[d] \\
\Sigma MK' \arrow[r, "\Sigma Mf'"] & \Sigma MP' \arrow[r, "\Sigma\varphi'"] & \Sigma C' \arrow[r, "\Sigma\theta'"] & \Sigma^2 MK'
\end{tikzcd} \tag{6.15}$$

We shall show that the candidate triangle $MB' \xrightarrow{Mb'} MB \xrightarrow{Mb} MB'' \longrightarrow \Sigma MB'$ is isomorphic to the distinguished triangle $C' \xrightarrow{c'} C \xrightarrow{c} C'' \xrightarrow{c''} \Sigma C'$ obtained from Neeman's lemma. In order to show this, note that we have a commutative diagram

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] \arrow[d, equal] & MP' \arrow[r, "Mg'"] \arrow[d, equal] & MB' \arrow[r, "h'"] \arrow[d, dashed, "\gamma'"] & \Sigma MK' \arrow[d, equal] \\
MK' \arrow[r, "Mf'"'] & MP' \arrow[r, "\varphi'"'] & C' \arrow[r, "\theta'"'] & \Sigma MK''
\end{tikzcd} \tag{6.16}$$

in which the rows are distinguished triangles and γ' is an isomorphism. Similar diagrams give the existence of isomorphisms $\gamma : MB \to C$ and $\gamma'' : MB'' \to C''$.

Consider the commutative diagrams (6.14) and (6.15). One sees that commutative diagrams of the form (6.16) provide a link between diagrams (6.14) and (6.15). We therefore obtain a three dimensional commutative diagram with (6.14) placed above (6.15) with the maps between the two layers provided diagram (6.16) and its analogues. In order to avoid any ambiguity in the reader's visualisation of the diagram below, we shall refer to diagram (6.14) as the *upper face* of the diagram below and (6.15) as the *lower face*.

$$\begin{tikzcd}
MK' \arrow[rr] \arrow[dd] \arrow[dr, equal] & & MP' \arrow[rr] \arrow[dd] \arrow[dr, equal] & & MB' \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \Sigma MK' \arrow[dd] \arrow[dr, equal] & \\
& MK' \arrow[rr] \arrow[dd] & & MP' \arrow[rr] \arrow[dd] & & C' \arrow[rr] \arrow[dd] & & \Sigma MK' \arrow[dd] \\
MK \arrow[rr] \arrow[dd] \arrow[dr, equal] & & MP \arrow[rr] \arrow[dd] \arrow[dr, equal] & & MB \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \Sigma MK \arrow[dd] \arrow[dr, equal] & \\
& MK \arrow[rr] \arrow[dd] & & MP \arrow[rr] \arrow[dd] & & C \arrow[rr] \arrow[dd] & & \Sigma MK \arrow[dd] \\
MK'' \arrow[rr] \arrow[dd] \arrow[dr, equal] & & MP'' \arrow[rr] \arrow[dd] \arrow[dr, equal] & & MB'' \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \Sigma MK'' \arrow[dd] \arrow[dr, equal] & \\
& MK'' \arrow[rr] \arrow[dd] & & MP'' \arrow[rr] \arrow[dd] & & C'' \arrow[rr] \arrow[dd] & & \Sigma MK'' \arrow[dd] \\
\Sigma MK' \arrow[rr] \arrow[dr, equal] & & \Sigma MP' \arrow[rr] \arrow[dr, equal] & & \Sigma MB' \arrow[rr] \arrow[dr, dashed] & & \Sigma^2 MK' \arrow[dr, equal] & \\
& \Sigma MK' \arrow[rr] & & \Sigma MP' \arrow[rr] & & \Sigma C' \arrow[rr] & & \Sigma^2 MK'
\end{tikzcd} \tag{6.17}$$

The maps between the third column of the upper face and the third column of the lower face appear in the commutative diagram shown below.

$$\begin{tikzcd}
MB' \arrow[r, "Mb'"] \arrow[d, "\gamma'"'] & MB \arrow[r, "Mb"] \arrow[d, "\gamma"'] & MB'' \arrow[r] \arrow[d, "\gamma''"'] & \Sigma MB' \arrow[d, "\Sigma\gamma'"'] \\
C' \arrow[r, "c'"'] & C \arrow[r, "c"'] & C'' \arrow[r, "c''"'] & \Sigma C'.
\end{tikzcd} \tag{6.18}$$

Note that diagram (6.18) is the only part of diagram (6.17) which is not known to be commutative immediately. If (6.18) is commutative then, since, γ', γ and γ'' are isomorphisms, it would follow that the candidate triangle $MB' \xrightarrow{Mb'} MB \xrightarrow{Mb} MB'' \longrightarrow \Sigma MB'$ is isomorphic to a distinguished triangle, and thus, is itself a distinguished triangle.

Claim: Diagram (6.18) is commutative.

Proof of claim: Consider the following subdiagram of (6.17):

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] \arrow[d, "M\kappa'"'] & MP' \arrow[r, "Mg'"] \arrow[d, "M\pi'"'] & MB' \arrow[r, "h'"] \arrow[d] & \Sigma MK' \arrow[d, "\Sigma M\kappa'"'] \\
MK \arrow[r, "Mf"'] & MP \arrow[r, "\varphi"'] & C \arrow[r, "\theta"'] & \Sigma MK.
\end{tikzcd}$$

This is just the diagonal from the first row of the upper face to the second row of the lower face. Note that there are two ways of going from MB' to C, namely

$$MB' \xrightarrow{Mb'} MB \xrightarrow{\gamma} C \text{ and } MB' \xrightarrow{\gamma'} C' \xrightarrow{c'} C.$$

We shall show that these two compositions are in fact the same, thus showing the commutativity of the right hand square of the diagram (6.18).

By the commutativity of the rest of the diagram, $(\gamma \circ Mb' - c' \circ \gamma') \circ Mg' = 0$. Writing $\delta = \gamma \circ Mb' - c' \circ \gamma'$, we have the commutative diagram shown below.

$$\begin{tikzcd}
MK' \arrow[r, "Mf'"] & MP' \arrow[r, "Mg'"] \arrow[dr, "0"'] & MB' \arrow[r, "h'"] \arrow[d, "\delta"] & \Sigma MK'. \arrow[dl, dashed, "\exists"] \\
& & C &
\end{tikzcd}$$

However, $C \cong MB \in \mathscr{M}$, so again by the proof of Theorem 6.2.5, the broken arrow $\Sigma MK' \to C$ must be zero. It follows that $\delta = 0$ and that $\gamma \circ Mb' = c' \circ \gamma'$. This shows the commutativity of the right hand square of (6.18). The commutativity of the remaining two squares is proved by a similar argument, completing the proof of the claim.

It follows that $MB' \xrightarrow{Mb'} MB \xrightarrow{Mb} MB'' \longrightarrow \Sigma MB'$ is a distinguished triangle. Therefore, Theorem 6.3.3 follows by induction.

6.4 An example from u-cluster categories

In this section we shall consider a special case of Theorem 6.2.5 when the endomorphism ring of the compact object C of the triangulated category $\mathscr{T}$ is right coherent. In particular, this allows us to specialise Theorem 6.2.5 to the category $\mathsf{mod}(S^{\mathsf{op}})$, the full subcategory of $\mathsf{Mod}(S^{\mathsf{op}})$ consisting of finitely presented S^{op}-modules. We shall then apply this specialisation to the case of a path algebra of a quiver which has no oriented cycles. Such a path algebra is well known to be hereditary, and as such coherent, see [42] for example. When applying this specialisation of the main theorem to this case, the full embedding of the theorem recovers the canonical embedding of the module category into its u-cluster category, where $u \geqslant 2$ is an integer.

6.4.1 A version of the main result for finitely presented modules

We first specialise the main result to the case for finitely presented S^{op}-modules. For the basic facts on coherent rings and modules we refer to [42].

Definition 6.4.1. A ring R is said to be *right coherent* if every finitely generated right ideal of R is also finitely presented as a right R-module.

A finitely generated right R-module A is said to be *coherent* if every finitely generated submodule of A is finitely presented.

Recall that a ring R is right coherent if and only if any finitely presented right R-module is coherent. It follows that, if R is right coherent then every finitely generated projective right R-module is coherent. It is well known that the kernel of a homomorphism of finitely generated projective R-modules is also finitely generated, see [29, Lemma 2.11]. Hence, the kernel of a homomorphism of finitely generated right R-modules is finitely presented. The following is now an easy lemma.

Lemma 6.4.2. *Suppose R is a right coherent ring. A finitely presented right R-module A with finite projective dimension k has a projective resolution of length k consisting of finitely generated projective right R-modules.*

In light of Lemma 6.4.2 and the usual finite version of Proposition 6.2.1, we now obtain the following version of Theorem 6.2.5.

Theorem 6.4.3. *Let $\mathscr{T}$ be a triangulated category with set indexed coproducts. Suppose C is an object of $\mathscr{T}$ satisfying the following assumptions:*

(1) Its endomorphism algebra $S^{\mathsf{op}} = \mathsf{End}_{\mathscr{T}}(C)^{\mathsf{op}}$ has finite global dimension n;

(2) We have $\mathsf{Hom}_{\mathscr{T}}(C, \Sigma^i C) = \mathsf{Hom}_{\mathscr{T}}(C, \Sigma^{-i} C) = 0$ for $i = 1, \dots, n+1$; and,

(3) The endomorphism algebra S is right coherent.

Let $\mathscr{M} = \mathscr{M}_{n+1}$, then there exists a full embedding $M : \mathsf{mod}(S^{\mathsf{op}}) \to \mathscr{M}$.

Recall that the category $\mathsf{mod}(S^{\mathsf{op}})$ of finitely presented right S-modules is an abelian category if and only if S is right coherent. Thus, given Lemma 6.4.2, it follows that the proofs of Theorems 6.3.3 and 6.3.7 can be used to prove versions of these theorems for finitely presented right S-modules.

Remark 6.4.4. Note that in Theorem 6.4.3 we do not obtain that M is left adjoint to $\mathsf{Hom}_{\mathscr{T}}(C,-)$ because it is not clear that $\mathsf{Hom}_{\mathscr{T}}(C,-)$ applied to $\mathscr{M}$ necessarily takes values in $\mathsf{mod}(S^{\mathsf{op}})$. The construction of Theorem 6.2.5 applies to Theorem 6.4.3 by virtue of Lemma 6.4.2 and the fact that one can still show at each stage of the construction that there is an adjunction

$$\begin{tikzcd}
\mathscr{N}_k \arrow[r, shift right, "\mathsf{Hom}_{\mathscr{T}}(C,-)"'] & \mathsf{proj}^k(S^{\mathsf{op}}), \arrow[l, shift right, "M_k"']
\end{tikzcd}$$

where $\mathscr{N}_k = \{X \in \mathscr{T} \mid X \cong M_k(A) \text{ for some } A \in \mathsf{proj}^k(S^{\mathsf{op}})\}$ and where $\mathsf{proj}^k(S^{\mathsf{op}})$ denotes the full subcategory of $\mathsf{mod}(S^{\mathsf{op}})$ consisting of finitely presented S^{op}-modules of projective dimension at most k.

6.4.2 u-cluster categories

Cluster categories were introduced by Buan, Marsh, Reineke, Reiten and Todorov in [15]. They were also introduced indepentently for the type A case in [16]. We give a brief sketch of the definition of the u-cluster category following the exposition given in [30]. The u-cluster category was first introduced by Bernhard Keller in [40, Section 8.4]. Let k be an algebraically closed field and H be an hereditary k-algebra. For an integer $u \geqslant 1$, the *u-cluster category* $\mathscr{C}$ is defined by $\mathscr{D}^f(H^{\mathsf{op}})/\tau^{-1}\Sigma^u$, where τ is the AR translation of $D^f(H^{\mathsf{op}})$ (see [2] or [4], for example) and Σ is its suspension. Here $\mathscr{D}^f(H^{\mathsf{op}})$ is shorthand for $\mathscr{D}^f(\mathsf{mod}(H^{\mathsf{op}}))$.

By Keller, [40, Section 4, Theorem], the canonical projection functor $\pi : \mathscr{D}^f(H^{\mathsf{op}}) \to \mathscr{C}$ is triangulated. Hence by composition with the inclusion functor we obtain a full embedding

$$\begin{tikzcd}
\mathsf{mod}(H^{\mathsf{op}}) \arrow[r, hook, "\iota"] \arrow[dr, hook] & \mathscr{D}^f(H^{\mathsf{op}}) \arrow[d, "\pi"] \\
 & \mathscr{C},
\end{tikzcd} \tag{6.19}$$

for $u \geqslant 2$.

Now let $H = kQ$ be the path algebra of a quiver Q with no loops or oriented cycles. Then H is an hereditary algebra, hence coherent, and $H \in \mathscr{D}^f(H^{\mathsf{op}})/\tau^{-1}\Sigma^u = \mathscr{C}$ is maximal u-orthogonal (see [30]). In particular, we have:

- $\mathsf{Hom}_{\mathscr{C}}(H, \Sigma^i H) = 0$ for $i = 1, \ldots, u$;

- $\mathsf{Hom}_{\mathscr{C}}(H, \Sigma^{-i}H) = 0$ for $i = 1, \ldots, u$.

In addition, we have that $\mathsf{Hom}_{\mathscr{C}}(H, H) \cong H$, therefore the endomorphism algebra has global dimension 1, and for $u \geqslant 2$, H satisfies the hypotheses of Theorem 6.4.3. Therefore there exists a full embedding $M : \mathsf{mod}(H^{\mathsf{op}}) \to \mathscr{M}$ where $\mathscr{M} = \mathscr{M}_2$ in Definition 6.2.4. Hence, composing with the inclusion functor, we have a full embedding

$$\begin{tikzcd}
\mathsf{mod}(H^{\mathsf{op}}) \arrow[r, hook, "M"] \arrow[dr, hook] & \mathscr{M} \arrow[d, hook] \\
& \mathscr{C}.
\end{tikzcd} \tag{6.20}$$

We claim that this embedding coincides with that of (6.19) from [40].

It is clear that embeddings (6.19) and (6.20) are canonically equivalent on $\mathsf{add}(H)$. Thus they are equivalent on $\mathsf{proj}(H^{\mathsf{op}})$, so we only need to extend the equivalence to projective dimension 1 since H is hereditary. Let A and B be H^{op}-modules of projective dimension 1 and suppose we have a module homomorphism $a : A \to B$. Take projective resolutions of A and B:

$$0 \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \longrightarrow 0,$$
$$0 \longrightarrow Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} B \longrightarrow 0.$$

By elementary homological algebra, see [28], we can lift the homomorphism $a : A \to B$ to a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "f_1"] \arrow[d, dashed, "p_1"'] & P_0 \arrow[r, "f_0"] \arrow[d, dashed, "p_0"'] & A \arrow[r] \arrow[d, "a"'] & 0 \\
0 \arrow[r] & Q_1 \arrow[r, "g_1"'] & Q_0 \arrow[r, "g_0"'] & B \arrow[r] & 0.
\end{tikzcd} \tag{6.21}$$

Since there is a natural isomorphism $\tau : M|_{\mathsf{proj}(H^{\mathsf{op}})} \to \pi \circ \iota|_{\mathsf{proj}(H^{\mathsf{op}})}$ there are commutative diagrams:

$$\begin{tikzcd}
MP_1 \arrow[r, "Mf_1"] \arrow[d, "\tau_{P_1}"', "\sim"] & MP_0 \arrow[d, "\tau_{P_0}"', "\sim"] \\
\pi \circ \iota(P_1) \arrow[r, "\pi\circ\iota(f_1)"'] & \pi \circ \iota(P_0)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
MQ_1 \arrow[r, "Mg_1"] \arrow[d, "\tau_{Q_1}"', "\sim"] & MQ_0 \arrow[d, "\tau_{Q_0}"', "\sim"] \\
\pi \circ \iota(Q_1) \arrow[r, "\pi\circ\iota(g_1)"'] & \pi \circ \iota(Q_0).
\end{tikzcd} \tag{6.22}$$

In addition, applying the functors M and $\pi \circ \iota$ to diagram (6.21) yields the following commutative diagrams, respectively:

$$\begin{tikzcd}
MP_1 \arrow[r, "Mf_1"] \arrow[d, "Mp_1"'] & MP_0 \arrow[r, "Mf_0"] \arrow[d, "Mp_0"'] & MA \arrow[r, "h"] \arrow[d, "Ma"'] & \Sigma MP_1 \arrow[d, "\Sigma Mp_1"'] \\
MQ_1 \arrow[r, "Mg_1"'] & MQ_0 \arrow[r, "Mg_0"'] & MB \arrow[r, "j"'] & \Sigma MQ_1,
\end{tikzcd} \tag{6.23}$$

and

$$\begin{tikzcd}
\pi\circ\iota(P_1) \arrow[r, "\pi\circ\iota(f_1)"] \arrow[d, "\pi\circ\iota(p_1)"'] & \pi\circ\iota(P_0) \arrow[r, "\pi\circ\iota(f_0)"] \arrow[d, "\pi\circ\iota(p_0)"'] & \pi\circ\iota(A) \arrow[r, "\theta"] \arrow[d, "\pi\circ\iota(a)"'] & \Sigma(\pi\circ\iota(P_1)) \arrow[d, "\Sigma(\pi\circ\iota(p_1))"'] \\
\pi\circ\iota(Q_1) \arrow[r, "\pi\circ\iota(g_1)"'] & \pi\circ\iota(Q_0) \arrow[r, "\pi\circ\iota(g_0)"'] & \pi\circ\iota(B) \arrow[r, "\varphi"'] & \Sigma(\pi\circ\iota(Q_1)).
\end{tikzcd} \tag{6.24}$$

Note that diagram (6.24) comes by virtue of Keller's theorem that the canonical projection functor is triangulated [40, Section 4, Theorem].

Applying the functor M and $\pi\circ\iota$ to the top row of diagram (6.21) and using the natural isomorphism highlighted in diagrams (6.22) gives the following commutative diagram:

$$\begin{tikzcd}
MP_1 \arrow[r, "Mf_1"] \arrow[d, "\tau_{P_1}"', "\sim"] & MP_0 \arrow[r, "Mf_0"] \arrow[d, "\tau_{P_0}"', "\sim"] & MA \arrow[r, "h"] \arrow[d, dashed, "\sigma_A"', "\sim"] & \Sigma MP_1 \arrow[d, "\Sigma\tau_{P_1}"', "\sim"] \\
\pi\circ\iota(P_1) \arrow[r, "\pi\circ\iota(f_1)"'] & \pi\circ\iota(P_0) \arrow[r, "\pi\circ\iota(f_0)"'] & \pi\circ\iota(A) \arrow[r, "\theta"'] & \Sigma(\pi\circ\iota(P_1)),
\end{tikzcd} \tag{6.25}$$

where σ_A exists by (TR3) of Definition 3.1.1 and is an isomorphism by the Five Lemma for triangulated categories, Corollary 3.1.5.

Likewise, one obtains the following diagram:

$$\begin{tikzcd}
MQ_1 \arrow[r, "Mg_1"] \arrow[d, "\tau_{Q_1}"', "\sim"] & MQ_0 \arrow[r, "Mg_0"] \arrow[d, "\tau_{Q_0}"', "\sim"] & MB \arrow[r, "j"] \arrow[d, dashed, "\sigma_B"', "\sim"] & \Sigma MQ_1 \arrow[d, "\Sigma\tau_{Q_1}"', "\sim"] \\
\pi\circ\iota(Q_1) \arrow[r, "\pi\circ\iota(g_1)"'] & \pi\circ\iota(Q_0) \arrow[r, "\pi\circ\iota(g_0)"'] & \pi\circ\iota(B) \arrow[r, "\varphi"'] & \Sigma(\pi\circ\iota(Q_1)).
\end{tikzcd} \tag{6.26}$$

Combining diagrams (6.23), (6.24), (6.25), and (6.26) gives the following three dimensional diagram:

$$\begin{tikzcd}
& \pi\circ\iota(P_1) \arrow[rr] \arrow[dd] & & \pi\circ\iota(P_0) \arrow[rr] \arrow[dd] & & \pi\circ\iota(A) \arrow[rr] \arrow[dd] & & \Sigma(\pi\circ\iota(P_1)) \arrow[dd] \\
MP_1 \arrow[rr, crossing over] \arrow[ur, "\sim"] \arrow[dd] & & MP_0 \arrow[rr, crossing over] \arrow[ur, "\sim"] & & MA \arrow[rr, crossing over] \arrow[ur, dashed, "\sim"] & & \Sigma MP_1 \arrow[ur, "\sim"] & \\
& \pi\circ\iota(Q_1) \arrow[rr] & & \pi\circ\iota(Q_0) \arrow[rr] & & \pi\circ\iota(B) \arrow[rr] & & \Sigma(\pi\circ\iota(Q_1)) \\
MQ_1 \arrow[rr] \arrow[ur, "\sim"] & & MQ_0 \arrow[rr] \arrow[ur, "\sim"] \arrow[from=uu, crossing over] & & MB \arrow[rr] \arrow[ur, dashed, "\sim"] \arrow[from=uu, crossing over] & & \Sigma MQ_1, \arrow[ur, "\sim"] \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{6.27}$$

where each square and cube is known to commutative except for those involving the broken arrows. We claim that the commutativity of the rest of the diagram forces the whole diagram to commute.

By the commutativity of the rest of the diagram, we have another diagram of distinguished triangles:

$$\begin{tikzcd}
MP_1 \arrow[r, "Mf_1"] \arrow[d, "\gamma_1"'] & MP_0 \arrow[r, "Mf_0"] \arrow[d, "\gamma_0"'] & MA \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma MP_1 \arrow[d, "\Sigma\gamma_1"'] \\
\pi \circ \iota(Q_1) \arrow[r, "\pi\circ\iota(g_1)"'] & \pi \circ \iota(Q_0) \arrow[r, "\pi\circ\iota(g_0)"'] & \pi \circ \iota(B) \arrow[r, "\varphi"'] & \Sigma(\pi \circ \iota(Q_1)),
\end{tikzcd}$$

where we have

$$\begin{aligned}
\gamma_1 &= \pi \circ \iota(p_1) \circ \tau_{P_1} - \tau_{Q_1} \circ Mp_1 \\
\gamma_0 &= \pi \circ \iota(p_0) \circ \tau_{P_0} - \tau_{Q_0} \circ Mp_0 \\
\gamma &= \pi \circ \iota(a) \circ \sigma_A - \sigma_B \circ Ma.
\end{aligned}$$

By the commutativity of diagrams (6.22), we have $\gamma_1 = \gamma_0 = \Sigma\gamma_1 = 0$. Thus we obtain the diagram:

$$\begin{tikzcd}
MP_1 \arrow[r, "Mf_1"] & MP_0 \arrow[r, "Mf_0"] \arrow[dr, "0"'] & MA \arrow[r, "h"] \arrow[d, "\gamma"'] & \Sigma MP_1 \arrow[dl, dashed, "\exists"] \\
 & & \pi \circ \iota(B). &
\end{tikzcd}$$

From diagram (6.26) we have that $\pi \circ \iota(B) \cong MB$, thus $\pi \circ \iota(B) \in \mathscr{M}$. As in the second proof of Theorem 6.3.3, given in section 6.3.3, the broken arrow, $\Sigma MP_1 \to \pi \circ \iota(B)$ must be zero by the proof of Theorem 6.2.5. Hence, it follows that $\gamma = 0$, so that

$$\pi \circ \iota(a) \circ \sigma_A = \sigma_B \circ Ma.$$

This then forces diagram (6.27) to commute, as claimed. In particular, we obtain the following commutative diagram for any module homomorphism $a : A \to B$ in $\mathsf{mod}(H^{\mathsf{op}})$:

$$\begin{tikzcd}
MA \arrow[r, "Ma"] \arrow[d, "\sigma_A"', "\sim"] & MB \arrow[d, "\sigma_B"', "\sim"] \\
\pi \circ \iota(A) \arrow[r, "\pi\circ\iota(a)"'] & \pi \circ \iota(B).
\end{tikzcd}$$

Hence there exists a natural isomorphism $\sigma : M \to \pi \circ \iota$ on $\mathsf{mod}(H^{\mathsf{op}})$. We have, therefore, proved the following theorem.

Theorem 6.4.5. *Let $H = kQ$ be the path algebra of a quiver Q with no loops or oriented cycles and let $\mathscr{C} = \mathscr{D}^f(H)/\tau^{-1}\Sigma^u$ be the u-cluster category as defined in [30] and [40] for $u \geqslant 2$. Then the canonical embedding, $\pi \circ \iota$, obtained in [40, Section 4, Theorem] (see diagram* (6.19)*) and the full embedding obtained in Theorem 6.4.3 (see diagram* (6.20)*) are naturally equivalent.*

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