Standard tableaux and Klyachko's Theorem on Lie representations

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Outline

- Tensor representations
- Lie representations
- Klyachko's Theorem
- A combinatorial proof

Tensor representations

Schur (1901, 1923): The T_n are semisimple and the irreducible components are parameterised by partitions of n

$$T_n \cong \bigoplus_{\lambda \vdash n} t_\lambda [\lambda]$$

$[\lambda]$ - irreducible GL(V) module corresponding to λ
t_λ - multiplicity

Tensor representations

Example:

$$T_4 \cong [4] \oplus 3\,[3,1] \oplus 2\,[2^2] \oplus 3\,[2,1^2] \oplus [1^4]$$

[4]:

1	2	3	4

3 [3, 1]:

1	2	3
4		

1	2	4
3		

1	3	4
2		

2 [2²]:

1	2
3	4

1	3
2	4

3 [2, 1²]:

1	2
3	
4	

1	3
2	
4	

1	4
2	
3	

[1⁴]:

1
2
3
4

Lie representations

Decomposition into irreducibles			Missing
L_1	$\cong$	$[1]$	$-$
L_2	$\cong$	$[1^2]$	$[2]$
L_3	$\cong$	$[2,1]$	$[3], [1^3]$
L_4	$\cong$	$[3,1] \oplus [2,1^2]$	$[4], [2^2], [1^4]$
L_5	$\cong$	$[4,1] \oplus [3,2] \oplus [3,1^2] \oplus [2^2,1] \oplus [2,1^3]$	$[5], [1^5]$
L_6	$\cong$	$[5,1] \oplus [4,2] \oplus 2[4,1^2] \oplus [3^2] \oplus 3[3,2,1]$ $\oplus [3,1^3] \oplus 2[2^2,1^2] \oplus [2,1^4]$	$[6], [2^3], [1^6]$

Klyachko's Theorem

Let $n \geq 3$ and let $\lambda \vdash n$ with no more than $\dim(V)$ parts. Then

$$l_\lambda > 0 \Leftrightarrow \lambda \neq (1^n), (n), (2^2), (2^3).$$

In other words, almost every irreducible $GL(V)$ module occurs in the Lie representation.

Standard tableaux, descents and major index

Example: $\lambda = (5, 3, 2, 1) \vdash 11$

1	2	4	8	9
3	5	11		
6	10			
7				

$D(T) = \{2, 4, 5, 6, 9\}$

$\mathrm{maj}(T) = 2 + 4 + 5 + 6 + 9 = 26$

Remarks:

- $D(T) \subseteq \{1, \ldots, n-1\}$
- $k - 1 \leq |D(T)| \leq n - \lambda_1$

Kraśkiewicz-Weyman Theorem

Let i and n be coprime.

$$l_\lambda = \text{number of standard tableaux } T \text{ of shape } \lambda \text{ with } \mathrm{maj}(T) \equiv i \bmod n.$$

- Note that i can be any fixed number which is coprime to n.
- It is natural to try to prove Klyachko's Theorem using the Kraśkiewicz-Weyman Theorem

Theorem

Let $n \geq 3$, $\lambda \vdash n$.
$\exists$ a standard tableau of shape λ with major index coprime to n

$$\iff \lambda \neq (1^n), (n), (2^2), (2^3)$$

1	2
3	4

2

1	3
2	4

4

1	2
3	4
5	6

6

1	2
3	5
4	6

10

1	3
2	4
5	6

8

1	3
2	5
4	6

9

1	4
2	5
3	6

12

Main Idea

- We look at standard tableaux with "small" descent sets.
- Let $\lambda \vdash n$ into k parts.
  We can construct a standard tableau of shape λ with at most k descents which has major index coprime to n.

Strategy:

- Two part partitions.
- Rectangles.
- Non-rectangular partitions into more than two parts.

Two part partitions

$n = 2m + 1$, $\lambda = (n - s, s)$:

1	...	s	...	m	m+s+1	...	2m+1
m+1	...	m+s					

$n = 2m$, $\lambda = (n - s, s)$, $1 < s < m$:

1	2	...	s	...	m-1	m+1	m+2	m+s+2	...	2m
m	m+3	...	m+s+1							

1	3	...	2m
2			

1	2	3	...	m-1	m+2
m	m+1	m+3	...	2m-1	2m

Rectangles

Let $n = mk$, $\lambda = (m^k) \vdash n \qquad 0 \leq i \leq k-2 \qquad 1 \leq s \leq m-1.$

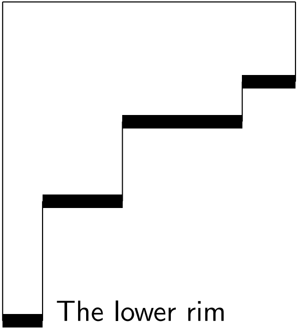
$T =$

<table>
<tr><td>1</td><td colspan="5">...</td><td>m</td></tr>
<tr><td>⋮</td><td colspan="5"></td><td>⋮</td></tr>
<tr><td>(i-1)m+1</td><td colspan="5">...</td><td>im</td></tr>
<tr><td>im+1</td><td>im+2</td><td>...</td><td>im+s</td><td>im+s+2</td><td>...</td><td>(i+1)m+1</td></tr>
<tr><td>im+s+1</td><td>(i+1)m+2</td><td colspan="4">...</td><td>(i+2)m</td></tr>
<tr><td>⋮</td><td colspan="5"></td><td>⋮</td></tr>
<tr><td>(k-2)m+1</td><td colspan="5">...</td><td>(k-1)m</td></tr>
<tr><td>(k-1)m+1</td><td colspan="5">...</td><td>km</td></tr>
</table>

- $\mathrm{maj}(T) = \frac{mk(k-1)}{2} + im + s + 1$
- Show that one of these is coprime to n (technical)

The rest

- Let λ be a non-rectangular partition of n into $k > 2$ parts.
- Write $n = mk + r$ where $0 \leq r < k < n$
- Let $m_1 + \cdots + m_k = n$, $m_i \in \{m, m+1\}$.
- Set $\lambda^{(k)} = \lambda$



Can remove m_i boxes from the lower rim of $\lambda^{(i)}$ to obtain a Young diagram $\lambda^{(i-1)}$ which has $i-1$ rows.

The rest

- For every choice $m_1, \ldots, m_k$ we can construct a standard tableau T of shape λ with descent set

$$D(T) = \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_{k-1}\}$$

- Put the entries

$$m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m_i$$

  from left to right in $\lambda^{(i)} \setminus \lambda^{(i-1)}$
- It can be shown that one of these descent sets gives major index which is coprime to n.