MATH5102M01

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH5102M**

No calculators allowed

Examination for the Module MATH5102M

(May–June 2009)

ADVANCED LOGIC

Time allowed: **3 hours**

Do not answer more than *FOUR* questions.

All questions carry equal marks.

Throughout this paper it can be assumed that the first order theory PA for arithmetic has the special axioms:

(N1) $(x_1 = x_2 \rightarrow (x_1 = x_3 \rightarrow x_2 = x_3))$

(N2) $(x_1 = x_2 \rightarrow (x_1' = x_2'))$

(N3) $(\overline{0} \neq x_1')$

(N4) $(x_1' = x_2' \rightarrow x_1 = x_2)$

(N5) $(x_1 + \overline{0} = x_1)$

(N6) $(x_1 + x_2' = (x_1 + x_2)')$

(N7) $(x_1 \times \overline{0} = \overline{0})$

(N8) $(x_1 \times x_2' = x_1 \times x_2 + x_1)$

and the axiom scheme

(N9) If $\varphi(x_1)$ is a wf of $\mathcal{L}_{\text{PA}}$, then

$$(\varphi(\overline{0}) \rightarrow ((\forall x_i)(\varphi(x_i) \rightarrow \varphi(x_i')) \rightarrow (\forall x_i)\varphi(x_i)))$$

is an axiom of PA.

1. (a) Show that if t_1, t_2 and t_3 are any terms of $\mathcal{L}_{\text{PA}}$, then

$$(t_1 = t_2 \rightarrow (t_1 = t_3 \rightarrow t_2 = t_3))$$

and

$$(t_1 + \overline{0} = t_1)$$

are theorems of PA.

(b) Show that if m and n are natural numbers and $\overline{m}$ and $\overline{n}$ are the corresponding numerals, then

$$\vdash_{\text{PA}} (\overline{m} = \overline{m})$$

and

$$\vdash_{\text{PA}} (\overline{m} = \overline{n} \rightarrow \overline{n} = \overline{m}).$$

(c) If $\mathbf{M} = \langle M, 0,', +, \times =\rangle$ is a model of PA, and $c \in M$, we say that c is a *non-standard element* of $\mathbf{M}$ if and only if c is not the interpretation of any numeral $\overline{m}$.

Show that if $\mathbf{M}$ is a model of PA, and c is a non-standard element of $\mathbf{M}$, then c' (the successor of c in $\mathbf{M}$) is also a non-standard element of $\mathbf{M}$.

Deduce that any non-standard model of PA has infinitely many non-standard elements. [You may assume that $(x_1 \neq x_1')$ is a theorem of PA.]

2. (a) Say what is meant by the terms *representable relation, representable function*.

Show that if $P(\overrightarrow{m})$ and $Q(\overrightarrow{m})$ are representable in PA, then so is $P(\overrightarrow{m}) \wedge Q(\overrightarrow{m})$.

Show that the projection functions U_i^n, $i \leq n$, defined by $U_i^n(m_0, m_1, \ldots m_n) = m_i$, are represented in PA by the respective wfs

$$\varphi(x_0, \ldots, x_{n+1}) =_{\text{defn}} (x_{n+1} = x_i).$$

[You may assume any known theorems of PA.]

(b) Show that + is a primitive recursive function.

Show that if $f(\overrightarrow{n}, m)$ is primitive recursive, then so is the *bounded sum* $h(\overrightarrow{n}, p)$ defined by

$$h(\overrightarrow{n}, p) = \sum_{m \leq p} f(\overrightarrow{n}, m).$$

(c) Let $\mathcal{T}$ be a first order theory, and let gn be a Gödel numbering of $\mathcal{T}$ for which

- $\text{Form}_{\mathcal{T}}(m) \Leftrightarrow_{defn} m$ is the Gödel number of a wf of $\mathcal{T}$,
- $MP(m, n, p) \Leftrightarrow_{\text{defn}} \text{Form}_{\mathcal{T}}(m)$, $\text{Form}_{\mathcal{T}}(n)$, $\text{Form}_{\mathcal{T}}(p)$, and $gn^{-1}(n)$ is $(gn^{-1}(m) \rightarrow gn^{-1}(p))$, and
- $\text{Gen}(m, n) \Leftrightarrow_{\text{defn}} \text{Form}_{\mathcal{T}}(m)$ and $\text{Form}_{\mathcal{T}}(n)$, and $gn^{-1}(n)$ is derived from $gn^{-1}(m)$ by an application of the generalisation rule,

are all computable relations.

Show that if $\mathcal{T}$ is computably axiomatisable, then

$$\text{Proof}_{\mathcal{T}}(n) \Leftrightarrow_{\text{defn}} gn^{-1}(n) \text{ is a proof in } \mathcal{T}$$

is a computable relation.

3. (a) Define the term *computably enumerable (c.e.) set.*

Show that S is c.e. if and only if there is a computable relation R such that

$$m \in S \Leftrightarrow \exists n R(m, n).$$

(b) Show that if S is computable then S is c.e., and that S is computable if and only if both S and the complement of S are c.e.

(c) Show that if A and B are c.e. than so is $A \cup B$.

(d) Define: A is many-one reducible to B ($A \leq_m B$).

Show that if A is many-one reducible to a c.e. set B, then A is c.e.

(e) Let $\{W_e\}_{e \geq 0}$ be a standard listing of all the c.e. sets. A set S is said to be *productive* if there is a computable function f such that if $W_e \subseteq S$ then $f(e) \in S - W_e$.

Show that if S is productive then S is not c.e., and hence that both S and $\overline{S}$ are infinite.

Show that if $S = \{x | x \notin W_x\}$ then S is productive.

4. Let $\mathcal{T}$ be a computably axiomatisable first order theory.

(a) Using the result of question 2 part (c), or otherwise, show that the set $T_{\mathcal{T}}$ of Gödel numbers of theorems of $\mathcal{T}$ is computably enumerable.

(b) Let $S = \{m \,|\, \text{Form}_{\mathcal{T}}(m) \& \vdash_{\mathcal{T}} \neg gn^{-1}(m)\}$, where gn and $\text{Form}_{\mathcal{T}}$ are as in question 2, part (c).

Show that $S \leq_m T_{\mathcal{T}}$, and hence that S is c.e.

(c) If $\mathcal{T}$ is complete, show that

$$\overline{T}_{\mathcal{T}} \; (= \mathbb{N} - T_{\mathcal{T}}) \; = S \cup \{m \,|\, \neg\text{Form}_{\mathcal{T}}(m)\},$$

and hence that $\mathcal{T}$ is decidable.

(d) If every c.e. set W is semi-representable in $\mathcal{T}$ (that is, for each such W there is a wf $\varphi(x_0)$ for which $m \in W \Leftrightarrow \vdash_{\mathcal{T}} \varphi(\overline{m})$, each $m \in \mathbb{N}$), show that $W \leq_m T_{\mathcal{T}}$, each c.e. W.

Using part (e) of question 3, or otherwise, show that there is a non-c.e. set $S \leq_m \overline{T}_{\mathcal{T}}$, and hence that $\overline{T}_{\mathcal{T}}$ is not c.e.

Deduce that $\mathcal{T}$ is not decidable.

[You may assume for question 4 any result whose proof is asked for in question 3 above.]

5. Write an essay on Gödel’s Incompleteness Theorem, covering not more than about three sides.

In it, you should mention any points of particular interest or difficulty in the proof of the theorem, and any consequences you can think of for mathematics, and our understanding of the world in general.

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