MATH310201

This question paper consists of 3
printed pages, each of which is
identified by the reference **MATH3102**

No calculators allowed

Examination for the Module MATH3102
(May–June 2009)

MATHEMATICAL LOGIC 2

Time allowed: **3 hours**

Do not answer more than *FOUR* questions.

All questions carry equal marks.

Throughout this paper it can be assumed that the first order theory PA for arithmetic has the special axioms:

(N1) $(x_1 = x_2 \to (x_1 = x_3 \to x_2 = x_3))$
(N2) $(x_1 = x_2 \to (x_1' = x_2'))$
(N3) $(\overline{0} \neq x_1')$
(N4) $(x_1' = x_2' \to x_1 = x_2)$
(N5) $(x_1 + \overline{0} = x_1)$
(N6) $(x_1 + x_2' = (x_1 + x_2)')$
(N7) $(x_1 \times \overline{0} = \overline{0})$
(N8) $(x_1 \times x_2' = x_1 \times x_2 + x_1)$

and the axiom scheme

(N9) If $\varphi(x_1)$ is a wf of $\mathcal{L}_{\text{PA}}$, then

$$(\varphi(\overline{0}) \to ((\forall x_i)(\varphi(x_i) \to \varphi(x_i')) \to (\forall x_i)\varphi(x_i)))$$

is an axiom of PA.

1. (a) Assuming that $\vdash_{\text{PA}} t = t$ for each term t of $\mathcal{L}_{\text{PA}}$, show that

$$\vdash_{\text{PA}} \ (x_1 = x_2 \to x_2 = x_1).$$

(b) Given that the standard structure $\underset{=}{\mathbb{N}} = \langle \mathbb{N}, 0, {}', +, \times, = \rangle$ is a model of PA, show that PA is ω-*consistent* (that is, for each wf $\varphi(x_1)$ of $\mathcal{L}_{\text{PA}}$ for which $\vdash_{\text{PA}} \varphi(\overline{m})$ for every $m \in \mathbb{N}$, we have that $(\exists x_1)\neg\varphi(x_1)$ is *not* provable in PA).

(c) Show that there is no consistent first order theory $\mathcal{T}$ in the language of PA whose theorems are exactly those wfs of $\mathcal{L}_{\text{PA}}$ which are either logically valid, or whose negations are provable in PA.

[You may assume that every logically valid wf φ of $\mathcal{L}_{\text{PA}}$ is provable in PA.]

2. (a) Say what is meant by the terms *representable relation*, *representable function*.

Show that if $P(\overrightarrow{m})$ and $Q(\overrightarrow{m})$ are representable in PA, then so is $P(\overrightarrow{m}) \wedge Q(\overrightarrow{m})$.

Show that the projection functions U_i^n, $i \leq n$, defined by $U_i^n(m_0, m_1, \ldots m_n) = m_i$, are represented in PA by the respective wfs

$$\varphi(x_0, \ldots, x_{n+1}) =_{\text{defn}} (x_{n+1} = x_i).$$

[You may assume any known theorems of PA.]

(b) Show that + is a primitive recursive function.

Show that if $f(\overrightarrow{n}, m)$ is primitive recursive, then so is the *bounded sum* $h(\overrightarrow{n}, p)$ defined by

$$h(\overrightarrow{n}, p) = \sum_{m \leq p} f(\overrightarrow{n}, m).$$

(c) Let $\mathcal{T}$ be a first order theory, and let gn be a Gödel numbering of $\mathcal{T}$ for which

- $\text{Form}_{\mathcal{T}}(m) \Leftrightarrow_{defn} m$ is the Gödel number of a wf of $\mathcal{T}$,
- $MP(m, n, p) \Leftrightarrow_{\text{defn}}$ $\text{Form}_{\mathcal{T}}(m)$, $\text{Form}_{\mathcal{T}}(n)$, $\text{Form}_{\mathcal{T}}(p)$, and $gn^{-1}(n)$ is $(gn^{-1}(m) \to gn^{-1}(p))$, and
- $\text{Gen}(m, n) \Leftrightarrow_{\text{defn}}$ $\text{Form}_{\mathcal{T}}(m)$ and $\text{Form}_{\mathcal{T}}(n)$, and $gn^{-1}(n)$ is derived from $gn^{-1}(m)$ by an application of the generalisation rule,

are all computable relations.

Show that if $\mathcal{T}$ is computably axiomatisable, then

$$\text{Proof}_{\mathcal{T}}(n) \Leftrightarrow_{\text{defn}} gn^{-1}(n) \text{ is a proof in } \mathcal{T}$$

is a computable relation.

3. (a) Define: A is *many-one reducible* to B $(A \leq_m B)$.

Show that if $S \leq_m S'$ with S' computably enumerable, then S is computably enumerable. [You may assume that a set S is computably enumerable if and only if $m \in S$ is a Σ_1^0-relation]

(b) Show that if S is semi-representable in a computably axiomatisable theory $\mathcal{T}$ then $S \leq_m T_{\mathcal{T}}$ (the set of Gödel numbers of theorems of $\mathcal{T}$), and hence S is computably enumerable.
[You may assume that for any computably axiomatisable $\mathcal{T}$ we have $T_{\mathcal{T}}$ computably enumerable.]

(c) Show that any ω-consistent theory $\mathcal{T}$ is consistent.

Show that if $\mathcal{T}$ is an ω-consistent theory in the language of PA in which every computable relation is representable, then every computably enumerable set is semi-representable in the theory $\mathcal{T}$.

Deduce that for any computably axiomatisable theory $\mathcal{T}$ we have $T_{\mathcal{T}} \leq_m T_{\text{PA}}$.
[You may assume PA to be an ω-consistent theory in which every computable relation is representable.]

4. (a) Let $\mathcal{T}'$ be a finite extension of a first order theory $\mathcal{T}$.

Show that $T_{\mathcal{T}'} \leq_m T_{\mathcal{T}}$.

(b) We say a first order theory $\mathcal{T}$ is *strongly undecidable* if and only if $\mathcal{T}$ is finitely axiomatisable and every theory $\mathcal{T}'$ in the language of $\mathcal{T}$ that is consistent with $\mathcal{T}$ (that is, such that $\mathcal{T} \cup \mathcal{T}'$ is consistent) is undecidable.

Show that if RR is a finitely axiomatisable theory in the language of PA in which every computable relation is representable, then RR is strongly undecidable.

[You may assume *Rosser's Theorem* in the form: If $\mathcal{T}'$ is a consistent axiomatisable first order theory in which every computable function is representable, then $\mathcal{T}'$ is incomplete and undecidable.]

Deduce that there is no algorithm for deciding of any given wf φ in the language of PA whether or not it is logically valid.

5. Write an essay on Gödel's Incompleteness Theorem, covering not more than about two sides.

In it, you should mention any points of particular interest or difficulty in the proof of the theorem, and any consequences you can think of for mathematics in general.

END