University of Leeds
Department of Pure Mathematics

MATH 3102/ 5102 – MATHEMATICAL LOGIC 2

Modifications and Additions to the Problem Sheet Solutions.

Lecturer: Dr. Charles M. Harris.
Office: Maths. 10.21f. **e-mail:** harris.charles@gmail.com.
Web: http://www.amsta.leeds.ac.uk/~charlie.

1 Modifications.

The hand written answers contain some notation that is out of date, and some references to what were formerly hand written notes and which refer to different elements (Theorems etc) of the printed notes found on the Course Notes web page. Accordingly you should be aware of the following points.

1. At some points in the solution sheets you might see for example (N1), $\mathcal{N}$, $T_{\mathcal{N}}$ or $\vdash_{\mathcal{N}}$. This is the old notation for what is now $(PA1)$, $\mathcal{PA}$, $T_{\mathcal{PA}}$, $\vdash_{\mathcal{PA}}$.
2. Question Sheet 5. Question 6(a). "Theorem 8.3" should read (i.e. relative to the printed notes) "Theorem 8.5".
3. Question Sheet 5. Question 6(b) "Theorem 8.6" should read "Theorem 8.12".
4. Question Sheet 5. Question 7. "Corollary 8.7" should read "Corollary 8.14".
5. Question Sheet 5. Question 7. "Theorem 8.6" should read "Theorem 8.12".
6. Question Sheet 6. Question 1(ii). "Theorem 8.3" should read "Theorem 8.5" (I originally had called Theorem 8.5, Corollary 8.5 but this has now been changed.)
7. Question Sheet 6. Question 2. "Corollary 10.2" should read "Corollary 10.5".
8. Question Sheet 6. Question 3. "Corollary 8.3" should read "Theorem 8.5".
9. Question Sheet 6. Question 5. "Corollary 10.3" should read "Corollary 10.6". Note also that in this question, as mentioned above $\mathcal{N}$ and $T_{\mathcal{N}}$ should read $\mathcal{PA}$ and $T_{\mathcal{PA}}$.

2 Additions.

Question Sheet 2, Question 5. The proof below completes the proof that is in the hand written solutions.

As in the written solutions we suppose that S is represented in $\mathcal{PA}$ by $\varphi(x_1)$ and also that $\psi(x_1, x_2) =_{\text{def}} (\varphi(x_1) \wedge x_2 = \overline{1}) \vee (\neg\varphi(x_1) \wedge x_2 = \overline{0})$.

The fact that $C_S(m) = n \Rightarrow \vdash_{\mathcal{PA}} \psi(\overline{m}, \overline{n})$ for any $m, n \in \mathbb{N}$, is proved in the written solutions.

We consider any $m, n \in \mathbb{N}$ and we suppose that $C_S(m) \neq n$. Then by definition,

$$m \in S \text{ and } n \neq 1 \qquad \text{or} \qquad m \notin S \text{ and } n \neq 0 \tag{2.1}$$

and this imples that

$$\Big(\vdash_{\mathcal{PA}} \varphi(\overline{m}) \text{ and } \vdash_{\mathcal{PA}} \neg(\overline{n} = \overline{1}) \Big) \qquad \text{or} \qquad \Big(\vdash_{\mathcal{PA}} \neg\varphi(\overline{m}) \text{ and } \vdash_{\mathcal{PA}} \neg(\overline{n} = \overline{0}) \Big)$$

Thus there are two cases to consider.

Case 1. $\vdash_{\mathcal{PA}} \varphi(\overline{m})$ and $\vdash_{\mathcal{PA}} \neg(\overline{n} = \overline{1})$

• Note that $\neg(\overline{n} = \overline{1}) \to \neg(\varphi(\overline{m}) \wedge \overline{n} = \overline{1})$ is an instance of a tautology of the form $\neg\mathscr{B} \to \neg(\mathscr{A} \wedge \mathscr{B})$. Hence we have both that

$$\vdash_{\mathcal{PA}} \neg(\overline{n} = \overline{1}) \tag{2.2}$$

and that

$$\vdash_{\mathcal{PA}} \neg(\overline{n} = \overline{1}) \to \neg(\varphi(\overline{m}) \wedge \overline{n} = \overline{1}) \tag{2.3}$$

and so using MP relative to (2.2) and (2.3) we get that

$$\vdash_{\mathcal{PA}} \neg(\varphi(\overline{m}) \wedge \overline{n} = \overline{1})\,. \tag{2.4}$$

• Likewise $\varphi(\overline{m}) \to \neg(\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0})$ is an instance of a tautology of the form $\mathscr{A} \to \neg(\neg\mathscr{A} \wedge \mathscr{B})$. Hence we have both that

$$\vdash_{\mathcal{PA}} \varphi(\overline{m}) \tag{2.5}$$

and that

$$\vdash_{\mathcal{PA}} \varphi(\overline{m}) \to \neg(\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0}) \tag{2.6}$$

and so using MP relative to (2.5) and (2.6) we get that

$$\vdash_{\mathcal{PA}} \neg(\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0})\,. \tag{2.7}$$

Thus, if *Case 1.* holds then we know that

$$\vdash_{\mathcal{PA}} \neg(\varphi(\overline{m}) \wedge \overline{n} = \overline{1}) \qquad \text{and} \qquad \vdash_{\mathcal{PA}} \neg(\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0})\,. \tag{2.8}$$

Case 2. $\vdash_{\mathcal{PA}} \neg\varphi(\overline{m})$ and $\vdash_{\mathcal{PA}} \neg(\overline{n} = \overline{0})$

• It is clear that we can apply similar arguments to those used in *Case 1.* to show that (2.8) also holds in this case.

Thus we know that, $C_S(m) \neq n$ implies that (2.8) holds. Moreover,

$$\vdash_{\mathcal{PA}} \neg(\varphi(\overline{m}) \wedge \overline{n} = \overline{1}) \to \Big(\neg(\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0}) \to \neg\big((\varphi(\overline{m}) \wedge \overline{n} = \overline{1}) \vee (\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0}) \big) \Big) \tag{2.9}$$

since the wf in (2.9) is an instance of a tautology of the form $\neg\mathscr{A} \to \big(\neg\mathscr{B} \to \neg(\mathscr{A} \vee \mathscr{B})\big)$. Hence we can apply MP twice to the statements in (2.8) and (2.9) to obtain that

$$\vdash_{\mathcal{PA}} \neg\big((\varphi(\overline{m}) \wedge \overline{n} = \overline{1}) \vee (\neg\varphi(\overline{m}) \wedge \overline{n} = \overline{0}) \big)\,.$$

In other words $C_S(m) \neq n \Rightarrow \vdash_{\mathcal{PA}} \neg\psi(\overline{m}, \overline{n})$. This completes the proof.