University of Leeds
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MATH 3102/5102 MATHEMATICAL LOGIC 2
Problems 6

1) Let $\mathcal{T}$ be a computably axiomatisable first-order theory.
(i) Show that $S \leq_m T_{\mathcal{T}}$ where

$$S = \{n \mid \mathrm{Form}(n) \ \& \ \vdash_{\mathcal{T}} \neg gn^{-1}(n)\}\,.$$

(ii) Deduce that S is c.e.
(iii) Show that if $\mathcal{T}$ is complete then

$$\overline{T}_{\mathcal{T}} = S \cup \{m \mid \neg\mathrm{Form}(m)\}\,,$$

and hence that $\mathcal{T}$ is decidable.
(iv) Show that Gödel's Incompleteness Theorem for PA can be deduced from the fact that PA is undecidable.

2) Show that there is no computably axiomatisable first-order theory $\mathcal{T}$ whose theorems are exactly the true statements of arithmetic (in the language of PA).

3) Show that if $\mathcal{T}$ is a computably axiomatisable theory, then $T_{\mathcal{T}}$ is semi-representable in PA.
Deduce that $T_{\mathcal{T}} \leq_m T_{\mathrm{PA}}$.

4) Define $S \equiv_m S' \Leftrightarrow_{\mathrm{defn}} S \leq_m S'$ and $S' \leq_m S$.
Show that $\equiv_m$ is an equivalence relation (where the equivalence classes are called *many-one degrees*).
Show that the set of all recursive sets (other than $\emptyset$ and $\mathbb{N}$) forms a many-one degree (called $\mathbf{0}_m$).

5) Let $\mathcal{T}$ be a computably axiomatisable (ω-consistent) theory in which all the recursive functions are representable.
Show that

$$T_{\mathrm{PA}} \leq_m T_{\mathcal{T}} \ \text{ and } \ T_{\mathcal{T}} \leq_m T_{\mathrm{PA}}\,,$$

and hence that T_{PA} and $T_{\mathcal{T}}$ occupy the same many-one degree.
Calling this degree $\mathbf{0}'_m$, show that $\mathbf{0}_m \neq \mathbf{0}'_m$.

6) Show that there is no axiomatisable first-order theory $\mathcal{T}$ whose theorems are the wfs of PA which are *either* logically valid *or* whose negations are provable in PA.

7) Show that if $\mathcal{T}$ is a first-order theory, all of whose theorems in the language of PA are true in the standard model $\mathbb{N}$ of PA, then $\mathcal{T}$ has a finite extension in which all the recursive functions are representable.
Deduce that $\mathcal{T}$ is undecidable.
[**Careful** : There are two cases depending on whether $\mathcal{T}$ is computably axiomatisable or not.]

8) We say a first order theory $\mathcal{T}$ is *strongly undecidable* iff $\mathcal{T}$ is finitely axiomatisable and every theory $\mathcal{T}'$ in the language of $\mathcal{T}$ that is consistent with $\mathcal{T}$ (that is, such that $\mathcal{T} \cup \mathcal{T}'$ is consistent) is undecidable.
Show that RR is strongly undecidable.

HAND IN SOLUTIONS TO **TWO** QUESTIONS.