Solutions 5

1. By using the series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots ,$$

find a series for $\cosh x$. Hence find the sum of the series

$$\frac{2}{0!} + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \cdots .$$

Solution:

As

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

for all x, we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!}$$

by substituting $-x$ for x. Thus

$$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + 2\frac{x^6}{6!} + \cdots \qquad (*)$$

As

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

by definition, it follows that

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}.$$

Set $x = 1$ in the previous equation (*) to obtain

$$e + e^{-1} = \frac{2}{0!} + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \cdots .$$

2. Express

$$f(x) = \frac{e^x - 1}{x} - 1 - \frac{x}{2} \qquad (x \neq 0),$$

as a power series. Differentiate term by term and hence evaluate

$$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \cdots .$$

Solution:

$$f(x) = \frac{e^x - 1}{x} - 1 - \frac{x}{2} = \frac{1}{x}\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1 - \frac{x}{2} = \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots \qquad (x \neq 0),$$

so $f(x)$ converges absolutely for every $x \neq 0$. Differentiating term by term gives

$$f'(x) = \frac{xe^x - (e^x - 1)}{x^2} - \frac{1}{2} = \frac{2x}{3!} + \frac{3x^2}{4!} + \frac{4x^3}{5!} + \cdots \qquad (x \neq 0),$$

putting $x = 1$, we find

$$\frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \cdots = \frac{1}{2}.$$

3. Find the terms of the Maclaurin series for $\frac{1}{(1+x)^2}$, as far as the term in x^3. Give an expression for the remainder $R_4(x)$.

Solution:

Set $f(x) = (1+x)^{-2}$. We have

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + R_4(x),$$

where

$$\begin{aligned}
f^{(0)}(x) &= \frac{1}{(1+x)^2} \quad \text{so} \quad f^{(0)}(0) = 1 \,, \\
f^{(1)}(x) &= -2(1+x)^{-3} \quad \text{so} \quad f^{(1)}(0) = -2! \,, \\
f^{(2)}(x) &= (-2)(-3)(1+x)^{-4} \quad \text{so} \quad f^{(2)}(0) = 3! \,, \\
f^{(3)}(x) &= (-2)(-3)(-4)(1+x)^{-5} \quad \text{so} \quad f^{(3)}(0) = -4! \,, \\
f^{(4)}(x) &= (-2)(-3)(-4)(-5)(1+x)^{-6} = 5!(1+x)^{-6} \,.
\end{aligned}$$

Thus,

$$f(x) = 1 - 2x + 3x^2 - 4x^3 + R_4(x),$$

where

$$R_4(x) = \frac{f^{(4)}(\xi)}{4!}x^4 = \frac{5}{(1+\xi)^6}x^4,$$

for some ξ between 0 and x.

4. Find the terms of the Taylor series with centre 9 for $\sqrt{x}$ as far as the term in $(x-9)^2$, giving an expression for the remainder $R_3(x)$. Use these terms to obtain an approximation for the value of $\sqrt{10}$ and to estimate the error in your approximation.

Solution:

Let $f(x) = \sqrt{x}$. Then

$$f(x) = f(9) + \frac{f^{(1)}(9)}{1!}(x-9) + \frac{f^{(2)}(9)}{2!}(x-9)^2 + R_3(x),$$

where

$$\begin{aligned}
f^{(0)}(x) &= f(x) = \sqrt{x} \quad \text{so} \quad f^{(0)}(9) = 3 \,, \\
f^{(1)}(x) &= \frac{1}{2}x^{-1/2} \quad \text{so} \quad f^{(1)}(9) = \frac{1}{2}\frac{1}{3} = \frac{1}{6} \,, \\
f^{(2)}(x) &= \frac{1}{2}\left(\frac{-1}{2}\right)x^{-3/2} \quad \text{so} \quad f^{(2)}(9) = \frac{-1}{4}\frac{1}{27} = \frac{-1}{108} \,, \\
f^{(3)}(x) &= \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2} = \frac{3}{8}\frac{1}{x^{5/2}} \,.
\end{aligned}$$

Thus,

$$f(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + R_3(x),$$

where

$$R_3(x) = \frac{f^{(3)}(\xi)}{3!}(x-9)^3,$$

with ξ between 9 and x. Now

$$f^{(3)}(\xi) = \frac{3}{8}\frac{1}{\xi^{5/2}} \Rightarrow \frac{f^{(3)}(\xi)}{3!} = \frac{1}{16 \cdot \xi^{5/2}}.$$

In order to compute the value of $\sqrt{10}$, we take $x = 10$ in the previous polynomial.

$$\sqrt{10} = f(x=10) = 3 + \frac{1}{6}(10-9) - \frac{1}{216}(10-9)^2 + R_3(x=10)$$

$$\Rightarrow \sqrt{10} = f(x=10) = 3 + \frac{1}{6} - \frac{1}{216} + R_3(x=10)\ .$$

To estimate the error notice that the function

$$h(u) = \frac{1}{16 \cdot u^{5/2}}$$

assumes its maximum between $a = 9$ and $x = 10$, when $u = 9$ (as it is a decreasing function). Thus

$$|R_3(x=10)| \leq \frac{1}{16 \cdot 9^{5/2}}(10-9)^3 = \frac{1}{16 \cdot 9^{5/2}}\ .$$

Please evaluate this last number with the calculator to obtain a bound for the error.