

MATH1050

MOCK EXAM MATH1050

December 2008

Calculus and mathematical analysis

Time allowed: **2 hours**

Attempt all ten questions from Section A and no more than three questions from Section B.

Section A is worth 40% and Section B 60% of the available marks.

You must show your working in the answers to **all** questions.

SECTION A

A1. Show by mathematical induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1), \quad \text{for } n \geq 1.$$

A2. Define the function $\sinh(x)$ and make use of the definition to show that $\sinh'(x) = \cosh(x)$.

A3. Find $f'(x)$ for the function $f(x) = (2x - \tanh x)^2$.

A4. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, provide the definition of their composition $g \circ f$.

A5. Compute the following integral

$$\int_0^1 dx x^5 e^x.$$

A6. Let $w = (1 - 7i)^7$. Find the modulus and complex conjugate of w .

A7. Define the function $\cosh(x)$ and make use of the definition to show that $\cosh(iz) = \cos(z)$.

A8. For the sequence $\{a_n = \frac{1}{n^7}\}$, ($n \geq 1$) with limit $L = 0$ and for $\varepsilon = \frac{1}{10^7}$, find an integer N corresponding to the given ε , satisfying

$$n > N \Rightarrow |a_n - L| < \varepsilon.$$

A9. Determine, with reasons, whether the following infinite series converges or diverges:

$$\sum_{n=1}^{+\infty} \frac{n^9}{n!}.$$

A10. Given $f(x) = \cos(x)$, compute its Taylor series about $a = \frac{\pi}{4}$.

SECTION B

B1. (a) Provide an example of a function that is continuous and differentiable.

(b) Complete the following statement. **A function $f : A \rightarrow B$ is continuous at $a \in A$ if ...**

(c) If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a . Make use of the Chain Rule to provide the value of $(f \circ g)'(a)$.

- (d) Let $f(x) = \cos(x)$ for $x \in [0, \pi]$ and consider its inverse $f^{-1}(x) = \arccos(x)$ for $x \in [-1, 1]$.
Make use of the chain rule to show that

$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}.$$

- (e) Find the values of a, b, c for which the function $f(x)$ is continuous and differentiable at every $x \in \mathbb{R}$.

$$f(x) = \begin{cases} ax + b, & x < -2, \\ x^2, & -2 \leq x \leq 0, \\ cx, & x > 0. \end{cases}$$

- B2.** (a) Let $a, b \in \mathbb{R}$ such that $a < b$. Define a **partition** of the interval $[a, b]$.
 (b) Suppose f is bounded on $[a, b]$ and P is a partition of $[a, b]$. Define the **lower sum** $L(f, P)$ and the **upper sum** $U(f, P)$.
 (c) Find the area between $f(x) = \sin x$ and the x -axis on $[-2\pi, 2\pi]$. Sketch a graph of the function and shade the region.
 (d) Find $\int dx e^x \sin(7x)$.
 (e) Find $\int_a^b dx \sin x e^x$.
- B3.** (a) Provide the definition of **complex number**. Given $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$, two complex numbers, define their product $z_1 \cdot z_2$, in terms of a_1, b_1, a_2, b_2 .
 (b) Find the real and imaginary parts, the modulus, the principal argument and the complex conjugate of $z = (-7, 7)$.
 (c) Find $\cos 9\theta$ in terms of $\sin \theta$ and $\cos \theta$.
 (d) Find all values of $(-i)^{\frac{1}{6}}$.
 (e) Given that $2 - 3i$ is a root of

$$p(z) = z^4 - 6z^3 + 26z^2 - 46z + 65,$$

find the other three roots of $p(z)$.

- B4.** (a) Find an integer N corresponding to the given ε , satisfying

$$n > N \Rightarrow |a_n - L| < \varepsilon, \quad \text{for} \quad a_n = \frac{1}{n+5}, \quad L = 0, \quad \varepsilon = \frac{1}{10^5}.$$

- (b) Provide an example of a bounded sequence and an example of an unbounded sequence.
 (c) Determine, using tests for convergence, whether the following infinite series

$$\sum_{n=1}^{+\infty} \frac{(2n)!}{(3n)!(2 \cdot 4 \cdot 6 \cdots (2n))},$$

converges or diverges. **Say which test** you use. [You do NOT have to find out what the sum is!]

- (d) Find the Taylor series expansion of $f(x) = \sin x$ at $a = -\frac{\pi}{4}$.
 (e) Find the Maclaurin series expansion of $g(x) = \log(1+x)$.