STRONG SPATIAL RESONANCE AND TRAVELLING WAVES IN BENARD CONVECTION

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We study the normal form for the onset of convection in a fluid layer when conditions are such that two modes whose horizontal wavenumbers are in the ratio 1:2 bifurcate simultaneously. It is shown that when the boundary conditions on the convection layer are such that there is no symmetry between the top and bottom of the layer, then the normal form possesses additional quadratic terms, and these equations have been shown to admit solutions in the form of travelling waves, modulated waves and standing waves. We give a discussion of the stability properties of these waves and describe a further form of nonsteady motion, namely an attracting homoclinic trajectory. The generality of the circumstances leading to this resonant behaviour suggests that such complex time-dependence may play a role in convective disorder just above threshold. The results may also explain phenomena observed in laboratory experiments on cylindrical convection by Azouni, and Azouni and Normand.

Busse and Or [1] have recently noted the possibility of mixed mode type solutions in two-dimensional weakly nonlinear Benard convection. Their solutions generalize the ideas of Knobloch and Guckenheimer [2] and Segel [3] who investigated the interaction of modes of horizontal wave numbers in the ratio l:$l+1$. These papers assumed the Boussinesq approximation and supposed that the boundary conditions at the top and bottom of the layer were the same. In this circumstance there is a symmetry between the upper and lower half of the layer, and this ensures that the normal form equations for the evolution of the amplitudes contain no quadratic terms, and take the form

$$\dot{A}_i = \mu_i A_i - a_i A_i |A_i|^2 - b_i A_i |A_{i+1}|^2 + \mathrm{O}(5) , \qquad (1)$$

where $i=1, 2$; $A_3 \equiv A_1$. To the order studied these equations couple only the *amplitudes* of the order parameters, but not the phases. We now change the boundary conditions to destroy the symmetry (for example, by making the lower boundary rigid, but the upper one stress-free – this would be easy to achieve in the laboratory), and we also allow departures from the Boussinesq approximation (this has a similar symmetry breaking effect). Then if we consider the dynamics of two bifurcating modes with horizontal wavenumbers k, say and $2k$, (1) has to be supplemented by extra quadratic terms representing the spatially resonant interaction of the two wavenumbers. The new normal form can be written, after appropriate scaling, as

$$\dot{A}_1 = \alpha_1 A_2 \bar{A}_1 + \mu_1 A_1 - a_1 A_1 |A_1|^2 - b_1 A_1 |A_2|^2 ,$$

$$\dot{A}_2 = -\alpha_2 A_1^2 + \mu_2 A_2 - a_2 A_2 |A_2|^2 - b_2 A_2 |A_1|^2 , \qquad (2)$$

where $\alpha_1 \alpha_2 = \pm \alpha^2$, $|\alpha_1| = |\alpha_2| = \alpha$ and A_2 is the amplitude of the mode which has twice the horizontal wavenumber of A_1. Note that if the linear and cubic terms in (2) are ignored, the equations have two invariants, namely

$$|A_1|^2 \pm |A_2|^2 \qquad (3)$$

and

$$A_1^2\bar{A}_2 - \bar{A}_1^2 A_2 \,. \tag{4}$$

Thus if the dynamics is described by eq. (2) with $\alpha_1\alpha_2 > 0$ a positive definite quantity (the "energy") is preserved (ref. [4], p. 130), and the cubic terms are needed to give sensible finite amplitude behaviour when $\mu_1, \mu_2 \neq 0$. When $\alpha_1\alpha_2 < 0$ the dynamics is quite different (and less rich). However it can be shown that while the effect of relaxing the Boussinesq approximation leads to values of $\alpha_1\alpha_2$ that can be of either sign, but are typically negative, the effect of changing the boundary conditions for a Boussinesq fluid will always result in a *positive* $\alpha_1\alpha_2$. This is because of the character of the nonlinear terms, which are of "energy preserving" form in the Boussinesq case. Eqs. (2) have been studied in a container of *finite* horizontal extent when the lack of symmetry is due to surface tension effects [5] (here $\alpha_1\alpha_2 < 0$). They can also be expected to describe the interaction of two axisymmetric vortices in Taylor-Couette flow, though the sign of $\alpha_1\alpha_2$ is not known in this case. In the rest of this note we suppose that $\alpha_1 = \alpha_2 = \alpha$ (this can be achieved without loss of generality by suitable scaling provided $\alpha_1\alpha_2 > 0$).

Imposition of lateral boundary conditions on the problem severely restricts the phase relationships between A_1 and A_2. If, however, we suppose an effectively infinite layer, or a cylindrical container, the relative phases of the two modes are governed by eqs. (2), and interesting dynamics can occur. To see this, we write

$$A_1 = \rho\, e^{i\theta} \,, \quad A_2 = \sigma\, e^{i\phi} \,, \quad \chi = 2\theta - \phi \,,$$

so that (2) becomes

$$\dot{\rho} = \alpha\rho\sigma \cos\chi + \mu_1\rho - a_1\rho^3 - b_1\rho\sigma^2 \,,$$

$$\dot{\sigma} = -\alpha\rho^2 \cos\chi + \mu_2\sigma - a_2\sigma^3 - b_2\sigma\rho^2 \,,$$

$$\dot{\chi} = \alpha(\rho^2/\sigma - 2\sigma) \sin\chi \,. \tag{5}$$

The reduction to third order is a consequence of the translational invariance. The μ_i, a_i, b_i are real due to a further symmetry (under reflection in a vertical plane) which holds for all the problems mentioned above. Eqs. (5) are a special case of the normal form for the Hopf bifurcation with 1:2 resonance [6]. A partial description of the dynamics is given in a recent paper by Dangelmayr [7]. (Note that even when the boundary conditons prevent the appearance of quadratic terms, the phases of A_1 and A_2 are still coupled by 5th order terms. Armbruster [8] has given an account of the dynamics in that case.) He treats the general $m{:}n$ resonance problem (including $m{:}n = 1{:}2$), and notes the existence of travelling waves and modulated waves as well as standing waves. However for the 1:2 case he only analyses behaviour in the immediate neighbourhood of the bifurcation point. We are able to extend the analysis to larger amplitudes, and are also able to give a complete analysis of the modulated waves close to the origin, making use of the integral invariant (4). A detailed discussion of (5) will follow in a latter paper: in this letter we summarize the principal results. Since $\dot{\theta}$ and $\dot{\phi}$ are proportional to sin ϕ, truly steady solutions for A_1, A_2 must have $\cos\chi = \pm 1$. So steady solutions of (5), if they exist, satisfy

$$\rho = 0 \,, \quad \sigma^2 = \mu_2 a_2^{-1} \quad \text{(single mode) [P]} \,,$$

or

$$\rho \neq 0 \,, \quad \pm\alpha\sigma + \mu_1 - a_1\rho^2 - b_1\sigma^2 = 0$$

$$\text{(mixed modes)} \,, \tag{6}$$

$$\mp\alpha\rho^2 + \mu_2\sigma - a_2\sigma^3 - b_2\sigma\rho^2 = 0 \quad [M_+ ; M_-] \,.$$

However there is another branch of steady solutions to (5) with $\cos\chi \neq \pm 1$; which satisfy

$$\rho^2 = 2\sigma^2 = \frac{2(2\mu_1 + \mu_2)}{4a_1 + 2(b_1 + b_2) + a_2} \,,$$

$$\cos\chi = \frac{1}{\alpha\sigma} \frac{\mu_2(2a_1 + b_1) - \mu_1(2b_2 + a_2)}{4a_1 + 2(b_1 + b_2) + a_2} \,. \tag{7}$$

These solutions represent *travelling waves* (in either direction) since $\dot{\theta}$ and $\dot{\phi}$ are non-zero. They exist in the part of the (μ_1, μ_2) plane given by $|\cos\chi| \leqslant 1$, or

$$[(2a_1 + b_1)\mu_2 - (2b_2 + a_2)\mu_1]^2$$

$$\leqslant \alpha^2(2\mu_1 + \mu_2)[4a_1 + 2(b_1 + b_2) + a_2] \,. \tag{8}$$

When equality holds in (8) the solution branch ends at a pitchfork bifurcation on one or other of the mixed modes given by (6).

A limit which is useful for obtaining understanding of (5) is when $(\mu_1, \mu_2) = \epsilon^2(\tilde{\mu}_1, \tilde{\mu}_2)$, $(\rho, \sigma) = \epsilon(\tilde{\rho}, \tilde{\sigma})$ where $\epsilon \ll 1$. It can be shown that the travelling waves (7) are stable provided

$$\tilde{\mu}_1 - \tilde{\mu}_2 > \frac{3(b_1 - a_2)(2\tilde{\mu}_1 + \tilde{\mu}_2)}{4a_1 + 2(b_1 + b_2) + a_2}\,. \tag{9}$$

Values of $\tilde{\mu}_1$, $\tilde{\mu}_2$ may always be found so that (8) and (9) are satisfied simultaneously, so the existence of stable travelling waves is a generic feature of convection problems with strong resonance. Numerical solutions of (5) indicate that stable travelling waves are not restricted to regions of the parameter space where μ_1, μ_2 are small but occur for a wide range of parameter space. Indeed, it is possible to analyse (5) in the limit of *large* ρ, σ (or equivalently small α). In this limit it can be shown that travelling waves are stable *whenever* they exist provided the quantity $(a_1a_2 - b_1b_2)/[4a_1 + 2(b_1 + b_2) + a_2] > 0$. The apparent contradiction between this result and (9) (or its contrapositive) can be understood by examining the stability of the mixed modes. When the mode (M_+ say) with the upper sign is considered, it is found that it loses stability at a Hopf bifurcation. The resulting oscillations are *standing waves*, since only the amplitudes and not the relative phases of A_1 and A_2 change. The curve in (μ_1, μ_2) space on which this bifurcation occurs will in general meet the boundary of the region of travelling waves given by (8). Standard methods of bifurcation theory then require that the other line of Hopf bifurcations (the boundary of the region given by eq. (9)) also intersect this singular point. Fig. 1 gives a sketch of a representative bifurcation diagram. The latter bifurcation gives rise to *modulated waves.* In the limit of small μ_1, μ_2 (as above) these waves can be described at leading order by the integrable system given by (2) with linear and cubic terms set to zero. The equations can be solved in terms of elliptic functions, with the two invariants (3) and (4) as parameters. The slow evolution of these invariants can be determined by averaging methods, and thus the stability and fate of the modulated wave branch can be found. The modulated wave branch can be shown to be *stable* if the coefficients a_i, b_i obey the relation

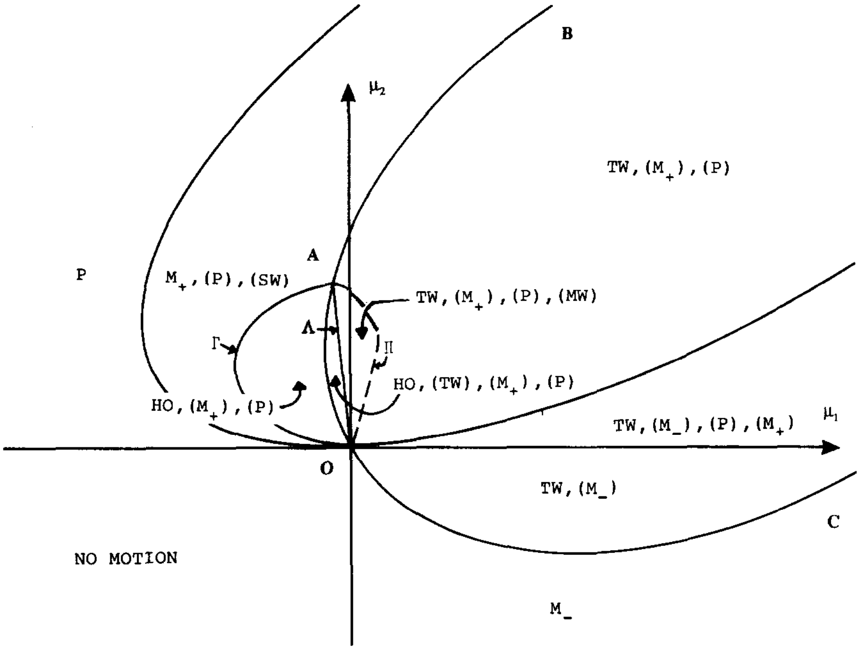

Fig. 1. Bifurcation diagram in the (μ_1, μ_2) plane for eqs. (5). Simple solutions that can occur are marked in each region, with unstable ones bracketed. P single mode, $M_\pm$ mixed modes, TW travelling wave, SW standing wave, MW modulated wave, HO homoclinic orbit. Travelling waves exist inside the parabola COAB, while modulated waves exist between the segments Λ and Π. The line Π is conjectural, although its initial slope is known, since the dynamics becomes fully three dimensional between O and A. The line Γ is the Hopf bifurcation to standing waves. For this example $\alpha = 1$, $a_1 = 1$, $a_2 = 3$, $b_1 = b_2 = 0$, so $a_1a_2 - b_1b_2 > 0$. $a_1 + 2(b_1 + b_2) + a_2 > 0$ and inequality (10) does not hold.

$$3 > \frac{2a_1 + (b_1 + b_2) - 4a_2}{b_1 - a_2} . \tag{10}$$

(If (10) is not satisfied periodic solutions exist but are unstable.) These modes must be interpreted as *quasiperiodic* travelling waves. The branch of modulated waves ends as a homoclinic orbit connecting (at leading order) both the pure mode with $\rho = 0$ and the mixed mode M_+.

No simple analysis can be performed to determine the stability of the standing wave oscillations described above. In many regimes we investigated, the standing waves are unstable. Then the oscillations grow until they lose stability with respect to phase. They then are attracted to a homoclinic orbit (with infinite period). The orbit emerges from the solution

$$\rho = 0 , \quad \sigma^2 = \mu_1 a_2^{-1} , \tag{11}$$

and goes through a short phase in which σ is small before returning to this solution. Asymptotic analysis is possible for this orbit, details of which will appear in a subsequent paper. The analysis indicates that the conditions for the orbit to exist and be attracting are that

$$0 > \mu_1 - \mu_2 b_1 a_2^{-1} > -\alpha(\mu_2 a_2^{-1})^{1/2} . \tag{12}$$

Infinite period orbits have been found in amplitude equations arising in a different context, see refs. [9,10]. A remarkable feature of this homoclinic orbit is that it exists and is attracting over a wide range of parameters, as indicated by (12). The degeneracy that this implies can be resolved only by adding new terms [10] or by the addition of noise [9].

An example of a laboratory experiment where the above analysis is relevant is that of cylindrical convection, Azouni [11] and Azouni and Normand [12]. In these experiments penetrative convection occurs in water near freezing point, which has maximum density near 4°C. Convection is induced by cooling the cylinder from below. Azouni and Normand report that time-dependent behaviour occurs when the aspect ratio is such that the modes with azimuthal dependence $e^{im\phi}$, $m = 1$ and $m = 2$ are excited. Linear theory indicates the principle of exchange of stabilities holds [12], so that the observed time-dependence cannot be explained in terms of linear theory. The observations indicate that two types of time-dependence are found depending on the aspect ratio of the apparatus [4] (ref. [12], pp. 214–219). In type (a) the amplitudes of the modes $m = 1$ and $m = 2$ are fairly constant, so that this behaviour corresponds to the travelling wave solution (7) of eqs. (2). In Azouni and Normand's type (b) time dependence, the amplitudes of the two modes $m = 1$ and $m = 2$ are found to vary, so that this regime corresponds either to the quasi-periodic solutions or to the approach to the homoclinic orbit. Further calculations would be needed to decide which type of behaviour is closest to the experimental phenomenon. Although the analysis of this paper focusses on a simple case of strong resonance, the new dynamical effects brought to light may be of importance in understanding the nature of the irregular behaviour in a real fluid near the instability boundary in Benard convection [13] and in Taylor–Couette flow [14]. The usual modulation theories have no information on the phase relation between the constituent unstable modes; here we see that non-adiabatic effects associated with the 1:2 resonance can profoundly affect the behaviour even when one or other of μ_1, μ_2 is negative. We conjecture that in a fluid layer of large spatial extent such resonant effects may arise as a hysteretic phenomenon dependent on a small but finite initial amplitude of modes with the appropriate resonant wavenumbers.

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