Compressible convection in the deep atmospheres of giant planets

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Abstract

Fast rotating giant planets such as Jupiter and Saturn possess alternate prograde and retrograde zonal winds which are stable over long periods of time. We consider a compressible model of convection in a spherical shell with rapid rotation, using the anelastic approximation, to explore the parameter range for which such zonal flows can be produced.

We consider models with a large variation in density across the layer. Our models are based only on the molecular H/He region above the metallic hydrogen transition at about 2Mbar, and we do not include the hydromagnetic effects which may be important if the electrical conductivity is significant. We find that the convective velocities are significantly higher in the low density regions of the shell, but the zonal flow is almost independent of the z-coordinate parallel to the rotation axis. We analyse how this behaviour is consistent with the Proudman-Taylor theorem.

We find that deep prograde zonal flow near the equator is a very robust feature of our models. Prograde and retrograde jets alternating in latitude can occur inside the tangent cylinder in compressible as well as Boussinesq models, particularly at lower Prandtl numbers. However, the zonal jets inside the tangent cylinder are suppressed if a no-slip condition is imposed at the inner boundary. This suggests that deep high latitude jets may be suppressed if there is significant magnetic dissipation.

Our compressible calculations include the viscous dissipation in the entropy equation, and we find this is comparable to, and in some cases exceeds, the total heat flux emerging from the surface. For numerical reasons, these simulations cannot reach the extremely low Ekman number found in giant planets, and they necessarily also have a much larger heat flux than planets. We therefore discuss how our results might scale down to give solutions with lower dissipation and lower heat flux.

Keywords: Atmospheres, Dynamics; Jupiter, Interior; Saturn, Interior.

1 Introduction

The issue of whether the fast zonal flows seen at the surface of giant planets extend thousands of kilometres below the surface, or whether they are restricted to a comparatively shallow weather layer, is still unresolved. Models based on the weather layer are capable of reproducing the
banded structure of alternating easterly and westerly jets (e.g. Cho and Polvani, 1996; Williams, 2003) but so are the deep convection models (Busse, 1976; Heimpel et al. 2005). These deep convection models are currently based on Boussinesq models which do not take into account the rapid increase in density with depth below the surface. In this paper we explore deep convection convection models in a rotating spherical shell using the anelastic approximation (Gilman and Glatzmaier 1981), which allows for a substantial density contrast between the inner and outer boundary. This is a very demanding computational problem, and the numerical methods used in these calculations will be described elsewhere.

The belts of Jupiter and Saturn are associated with prograde and retrograde zonal flows. In both cases there is a broad prograde equatorial belt, reaching 150 m s\(^{-1}\) in the case of Jupiter and 300 m s\(^{-1}\) in the case of Saturn, see e.g. the review of Vasavada and Showman (2005). At higher latitudes the amplitude of the zonal flows falls to a few tens of metres per second, but the alternating flow pattern persists right up to the polar regions (Porco et al. 2003). To discover how deep the zonal flows go into the atmosphere, the Galileo mission parachuted a probe into Jupiter’s atmosphere (Atkinson et al. 1997). It revealed the winds are not confined to the surface, but they persist into the interior. The visible surface of Jupiter extends down to a pressure of about 1 bar. The Galileo probe showed that the wind speed increases to around 170 m s\(^{-1}\) at the 20 bar level, suggesting that the zonal flow persists into the deep interior, driven by the internal heat flux. It is however still possible that solar heating, which does not penetrate below the visible surface, may drive flows well below the surface (Lian and Showman, 2008). A further point is that the Galileo probe entered at latitude 7°N in the equatorial prograde region. It is conceivable, as we see below, that the equatorial prograde flow penetrates considerably deeper than the high latitude banded zonal flows.

According to models of giant planet interiors (e.g. Guillot 1999, 2005), below 15 Mm the pressure inside Jupiter is so high that the material goes into a metallic hydrogen phase. In Saturn, which is smaller than Jupiter, the metallic core only occupies the innermost 50% by radius. It is currently believed that thermal convection is occurring in Jupiter and Saturn everywhere below the one bar level, though the determination of the equation of state, on which this belief rests, is a difficult area of high pressure physics currently under rapid development. It is also believed that the metallic hydrogen region is electrically conducting, rising to a conductivity of around 10\(^5\) S m\(^{-1}\) deduced from shock experiments (Nellis, 2000). The experiments also suggest that the electrical conductivity drops off steadily as the metallicity falls off with increasing radius. It is possible that even quite low values of the electrical conductivity could sustain currents and hence magnetic fields that affect the dynamics of the deep interior of the giant planets (Liu et al. 2008). Even weak magnetic fields in electrically conducting stellar interiors are likely to give rise to Lorentz forces that lead to near uniform rotation rates (magnetic locking) in their deep interiors, and this is believed to be why the radiative interior of the Sun has low differential rotation (Tomczyk et al. 1995, Gough and McIntyre, 1999). Liu et al (2008) also point out that fast deep zonal flows could lead to more ohmic dissipation than is consistent with the surface heat flux coming out of the planet. Further evidence that deep zonal flows are unlikely to penetrate into the magnetic interior comes from the very well-defined internal rotation rate of Jupiter’s magnetic field. This suggests a secular variation of the order of only 10\(^{-3}\) m s\(^{-1}\) (Russell et al.
consistent with estimates of the typical convective velocity required to sustain the dynamo (Starchenko and Jones, 2002). However, as Glatzmaier (2008) points out, these arguments ignore possible alignments of the flow and the magnetic field, which could mean that velocities much larger than $10^{-3}$ m s$^{-1}$ could be present, but would not show up in the magnetic rotation data and would also reduce the ohmic dissipation.

In this paper we do not consider magnetic fields, though we have explored the effect of a no-slip boundary to very crudely model the effect of a magnetically locked metallic hydrogen region at a depth of 15 Mm in Jupiter. While there are difficulties in allowing the fast zonal flows to penetrate into the electrically conducting regions of giant planets, the deep convection hypothesis does have a number of points in its favour. Eastward equatorial jets are a robust feature of deep convection models, whereas in shallow layer models they require special hypotheses about the structure of the atmosphere. Saturn has a much broader equatorial jet than Jupiter, naturally explained by the deeper level at which metallicity occurs. A smaller radius ratio always leads to a broader equatorial jet in deep spherical convection models. Finally, there is a natural explanation for the faster equatorial flow on Saturn, that the effects of interior dissipation are restricted to deeper regions in Saturn, and so have less effect than on Jupiter.

The fundamental mechanisms by which highly turbulent rotating convection can organise itself into banded zonal flows have received much attention recently. Jones et al. (2003) and Rotvig and Jones (2006) considered the simplified Boussinesq two-dimensional Busse annulus model, which allows very low Ekman number to be reached. Alternating jet flows with the width of each band scaling with the Rhines length (Rhines, 1975) were found. Christensen (2001, 2002) considered Boussinesq convection in a rapidly rotating spherical shell, finding strong zonal flows with stress-free boundaries. Heimpel et al. (2005) looked again at the Boussinesq problem, but went to lower Prandtl numbers, higher Rayleigh numbers and larger radius ratio. Higher Rayleigh number and larger radius ratio add to the computational load considerably, but nevertheless they found clear evidence of banded structure, with the jet-width again scaling with the Rhines length. Whereas far fewer bands were observed in their calculations than are observed in Jupiter, it is very likely that this is due to (inevitable) computational limitations, in particular insufficiently small Ekman number. Their calculations did however use stress-free boundary conditions, and also used hyperviscosity. In all these models, the zonal flows are driven by Reynolds stresses created by convection in a rotating frame, and the form of this convection is strongly affected by compressibility (Glatzmaier and Gilman, 1981a; Jones et al. 2009). The rotation leads to a systematic spiralling of the convection columns (Zhang 1992), which leads to non-zero Reynolds stresses (see e.g. Plaut et al. 2008).

Compressible rotating convection also leads to spiralling convection patterns. The zonal flow builds up until one of two things happens: either (a) the zonal flow gets so large that the differential rotation suppresses convection, leading to the bursting phenomenon (Busse, 2002; Rotvig and Jones, 2006), or (b) the dissipation becomes large enough so that it balances the driving by Reynolds stresses. In the bursting phenomenon, (a), for most of the time the convection is suppressed by the shear from the large zonal flows, but eventually the shear decays allowing a burst of vigorous convection. This rapidly drives up the zonal flow, suppressing convection again until viscosity reduces the zonal flow sufficiently for another convective burst to occur. Scenario
(a) was found in these simulations, and indeed may be occurring in Saturn, where the zonal flow appears to be time-dependent, Sanchez-Lavega et al. (2003), but here we focus on cases where the balance is between Reynolds stresses and viscous dissipation. Since viscous dissipation is fairly weak in the models, and extremely weak in giant planets (even when enhanced by turbulence) there is little to counteract the zonal flows which therefore get much larger than the convection driving them.

We now face a problem which cannot be avoided in numerical simulations of giant planet convection. The lowest possible Ekman number that can be reached in currently affordable simulations is about $10^{-6}$ and even that is very expensive. The Ekman number in giant planets is likely to be at least 5 orders of magnitude smaller than this. In consequence, we must drive our simulations much harder than the giant planets are driven, to compensate for the much higher viscous (and possibly ohmic) dissipation. The dimensional heat flux in a simulation therefore has to be orders of magnitude greater than that coming out of Jupiter or Saturn. Another consequence of the parameter limitation is that while the ratio of zonal flow to convective velocity can be fairly large in a simulation, the ratio cannot reach the very high values found in giant planets. However, as our understanding develops it may become possible to develop scaling laws, analogous to those used in planetary dynamo theory (Christensen and Aubert, 2006, Starchenko and Jones 2002), which might be used to extend simulation results into the actual parameter regime found in giant planets.

The main issues addressed in this paper concern the differences between compressible convection and Boussinesq convection in rotating spherical shells. Linear aspects have been discussed in Glatzmaier and Gilman (1981a), Drew et al. (1995) and recently by Jones et al. (2008). Convective velocities in planets are much lower than the sound speed, and we use the anelastic approximation throughout. An important difference between compressible and Boussinesq rotating convection, emphasised recently by Evonuk (2008) and Evonuk and Glatzmaier (2006), is that vorticity parallel to the rotation axis is stretched only as columns of fluid change length due to changing boundary separation in Boussinesq convection, whereas in anelastic convection it is stretched locally by the volume changes associated with compressibility. In consequence, the slope of the outer boundaries plays an important role in Boussinesq convection. At high Rayleigh numbers, when typically convection columns are short-lived and may not reach the boundaries, we may expect significant differences between Boussinesq and anelastic convection. Furthermore, low densities near the surface require high velocities and large temperature fluctuations to carry the heat flux, while in the deep interior we expect lower velocities and smaller temperature variations. This requirement is in contention with the Proudman-Taylor theorem, which indicates velocities should be independent of $z$, the rotation direction coordinate. We see below how this conflict is resolved. We also study the zonal flows, to see whether they are independent of $z$ in strongly compressible systems. Another feature of compressible convection is that the viscous heating can be of the same order of magnitude as the heat flux through the system, which cannot happen in Boussinesq convection. We therefore consider the effects of viscous heating on the flow.
2 Model for the simulations

We consider a spherical shell lying between \( r = r_i \) and \( r = r_o \), where the unit of length \( d = r_o - r_i \). The radius ratio \( r_i/r_o = \eta \). Details of the model are given in Jones et al. (2009), see also Gilman and Glatzmaier, (1981), here we just outline the essentials. We non-dimensionalise the density, pressure and temperature using their values at the midpoint between the bounding spherical surfaces, \( \rho_0, p_0 \) and \( T_0 \). The basic state of the shell is a polytropic perfect gas, with polytropic index \( n \), so the dimensionless density, pressure and temperature are given by

\[
\rho = \zeta^n, \quad p = \zeta^{n+1}, \quad \text{and} \quad T = \zeta. \tag{2.1a - c}
\]

We assume that the planet is centrally condensed, so gravity falls off as \( 1/r^2 \) in the shell, and then

\[
\zeta = c_0 + \frac{c_1 d}{r}, \quad c_0 = \frac{2\zeta_0 - \eta - 1}{1 - \eta}, \quad c_1 = \frac{(1 + \eta)(1 - \zeta_0)}{(1 - \eta)^2}, \quad \zeta_0 = \frac{\eta + 1}{\eta \exp(N_\rho/n) + 1}. \tag{2.2a - d}
\]

Note we are not using the Lane-Emden polytropic equation for the density here, which includes the self-gravity of the whole planet. Even in the case \( n = 2, \eta = 0.7, N_\rho = 5 \), where the density difference between the two models is the greatest of those considered here, when the densities at the boundaries are chosen to be identical the Lane-Emden model has only a \( \approx 6\% \) higher density at the midpoint by radius than the centrally condensed model. The centrally condensed model has the advantage that simple expressions in terms of \( r \) are available for all the thermodynamic variables. There are \( N_\rho \) density scale heights across the shell, that is \( \rho(r_i)/\rho(r_o) = \exp N_\rho \). The cases \( N_\rho = 5 \) and \( N_\rho = 3.5 \), used below, correspond to density ratios of about 150 and 33 respectively.

The basic state is assumed to be nearly adiabatic, which allows the convection equations to be formulated in terms of the entropy, without the need to solve for the pressure and density at each timestep, which can be eliminated by taking the curl and double curl of the equation of motion. The entropy is given here by

\[
S = c_p \left( \frac{1}{\gamma} \ln \rho - \ln \rho \right). \tag{2.3}
\]

Using the near adiabatic approximation, the dimensionless momentum equation reads (Lantz, 1992, Braginsky and Roberts 1995, Clune et al. 1999, Jones et al. 2008)

\[
\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \boldsymbol{\omega} - 2E^{-1}\hat{z} \times \mathbf{u} - \nabla \left( \frac{\eta'}{\rho} + \frac{1}{2} \mathbf{u}^2 \right) + \mathbf{F}_\nu + \frac{RaS}{r^2} \hat{r} \tag{2.4}
\]

where

\[
\boldsymbol{\omega} = \nabla \times \mathbf{u}, \quad \mathbf{F}_\nu = \frac{1}{\rho} \frac{\partial}{\partial x_j} \rho \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3\rho} \frac{\partial}{\partial x_j} \rho \frac{\partial u_i}{\partial x_j}, \tag{2.5a - b}
\]

where \( \mathbf{u} \) is the velocity, \( \boldsymbol{\omega} \) is the vorticity, \( \Omega \) the angular rotation vector, \( S \) is the entropy. We assume constant eddy viscosity \( \nu \) and constant eddy diffusivity \( \kappa \). This is not the only possible choice, as either \( \rho \nu \) constant (constant dynamical viscosity) or \( \rho \kappa \) constant (constant thermal conductivity), or both, are viable alternatives. Glatzmaier and Gilman (1981b) discussed some of the possible effects these choices can have.
The continuity equation has the anelastic form
\[ \nabla \cdot \rho \mathbf{u} = 0, \] (2.6)
which is valid as long as the motion of the fluid is much slower than the sound speed. This implies that the velocity can be expanded in two potentials \( P \) and \( T \),
\[ \mathbf{u} = \frac{1}{\rho} \nabla \times \nabla \times r P \rho + \frac{1}{\rho} \nabla \times r T \rho. \] (2.7)

The entropy equation is (Braginsky and Roberts 1995)
\[ \Pr \frac{\partial S}{\partial t} = -\Pr \mathbf{u} \cdot \nabla S + \frac{1}{\rho T} \nabla \cdot \rho T \nabla S + \frac{D_i}{T} Q_v, \] (2.8)
where \( \Pr = \nu/\kappa \) is the (constant) Prandtl number and the dissipation parameter \( D_i \) (see e.g. Anufriev et al. 2005)
\[ D_i = \frac{\eta(1 + \eta) Pr (\exp(Np/n) - 1)}{(1 - \eta)^2 Ra (1 + \eta \exp(Np/n))}. \] (2.9)
The \( Q_v \) term is the rate of heating by viscous dissipation.
\[ Q_v = 2 \left[ e_{ij} e_{ij} - \frac{1}{3} (\nabla \cdot \mathbf{u})^2 \right], \quad e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \] (2.10a,b)
This is important in this problem; it is usually comparable to the heat flux passing through the layer, and may even exceed it. However, it is only an internal heat source, so in our model the total heat flux entering the inner surface from below must equal the total heat flux leaving the outer surface when averaged over time. This provides a useful check of the numerical scheme: when spectral convergence is inadequate, these quantities are no longer in balance. The eddy diffusion of entropy in this equation comes from averaging over a small turbulent length scale
\[ \overline{\mathbf{u}_T S_T} = -\kappa \nabla S \] (2.11)
in accord with Braginsky and Roberts (1995). As usual, this procedure is only a crude model for the small scale turbulent diffusion, which ignores possible anisotropy. Here we assume that on the very smallest scales of our computation, the turbulence is approximately isotropic. Note that it is entropy, not temperature, that is diffused, because eddies transport the entropy almost adiabatically until the the fluid elements break up and release their entropy to the surroundings. We apply constant entropy boundary conditions \( S = \Delta S \) at \( r = r_i \) and \( S = 0 \) at \( r = r_0 \). We neglect any radiative part of the conductivity, assuming it to be small compared to the turbulent diffusivity. In a steady state, i.e. averaged over time, the dimensionless heat flux through a spherical surface of radius \( r \) is
\[ F = -\int_{S_i} \rho T \frac{dS}{dr} \, ds \]
\[ = -\int_{S_r} \rho T \frac{dS}{dr} \, ds + \int_{S_r} Pr \rho T u_r S \, dS - \int_{V_r} Pr \rho S u \cdot \nabla T \, dv - \int_{V_r} \rho D_i Q_v \, dv \] (2.12)
where the surface integrals are over the inner boundary, \( S_i \), and the spherical surface of radius \( r \), \( S_r \), and the volume integrals are over the volume between these two surfaces. Of the four terms on the right-hand-side, the first is the conducted flux across \( S_r \) and the second is the
convected heat flux across $S_r$. The third and fourth terms, volume integrals, are the work done by buoyancy and the viscous dissipation. These cancel out when integrated over the whole shell, but the integrals up to level $r$ may not exactly cancel.

We have nondimensionalised our equations using the length scale $d$, timescale $d^2/\nu$, where $\nu$ is the constant kinematic viscosity, mass $\rho_0d^3$, unit of entropy $Pr\Delta S$. The six dimensionless parameters that govern anelastic compressible convection are

\[
Ra = \frac{GMd\Delta S}{\nu\kappa c_p}, \quad Pr = \frac{\nu}{\kappa}, \quad E = \frac{\nu}{\Omega d^2}
\]

\[
N_\rho = \ln \left( \frac{\rho(r_i)}{\rho(r_o)} \right), \quad n, \quad \eta = \frac{r_i}{r_o}
\]

(2.13a – f)

$M$ being the total mass of the planet, $Ra$ being the Rayleigh number and $E$ the Ekman number.

### 2.1 Boundary conditions

The dimensionless entropy $S$ satisfies the boundary conditions

\[
S = Pr^{-1} \quad \text{on} \quad r = r_i, \quad S = 0 \quad \text{on} \quad r = r_o.
\]

(2.14a, b)

The velocity boundary conditions are that there is no penetration through the bounding surfaces of the shell, and that the velocity is either no-slip or stress-free there. In terms of the toroidal and poloidal potentials (2.7), the toroidal velocity scalar satisfies on $r = r_i$ or $r = r_o$

\[
\frac{\partial T}{\partial r} - \frac{T}{r} = 0 \quad \text{stress-free}, \quad T = 0 \quad \text{no-slip},
\]

(2.15a, b)

and the poloidal velocity scalar satisfies

\[
\frac{\partial^2 P}{\partial r^2} + \frac{1}{\rho \frac{d\rho}{dr}} \frac{\partial P}{\partial r} = 0 \quad \text{stress-free}, \quad \frac{\partial P}{\partial r} = 0 \quad \text{no-slip}.
\]

(2.16a, b)

The no-penetration condition at the boundaries, $u_r = 0$, becomes

\[
P = 0.
\]

(2.17)

### 2.2 Dimensional values of the physical quantities

The numerical simulations here are performed in dimensionless units, so they can accommodate a range of physical models. To be specific, we give here a possible set of dimensional values approximately corresponding to Jupiter values. We take a shell of radius ratio $\eta = 0.8$ with $n = 2$ and $N_\rho = 5$. Then $\zeta_0 = 0.1675$, $c_0 = -7.325$ and $c_1 = 37.462$. Since Jupiter’s radius is $7 \times 10^7$ m, the unit of length $d = 1.4 \times 10^7$ m. The density and temperature near the transition region to metallic hydrogen are around $1.1 \times 10^3$ kg m$^{-3}$ and $6.8 \times 10^3$ K respectively (Lodders and Fegley, 1998), so the units of density and temperature in our model, $\rho_0$ and $T_0$, are $264$ kg m$^{-3}$ and $3332$ K. Taking the gas constant as $3.5 \times 10^3$ J kg$^{-1}$ K$^{-1}$, the pressure at the outer boundary
is $1.4 \times 10^5$ Pa or 140 bar, so we have to cut the atmosphere off above this level. The temperature there is about 560 K.

The rotation rate of Jupiter is $1.76 \times 10^{-4}$ rad s$^{-1}$, and the typical value of $E$ we can achieve is $10^{-5}$, so our viscosity $\nu \sim 3.4 \times 10^5$ m$^2$ s$^{-1}$ and with $Pr = 0.1$, $\kappa$ is ten times larger. Our unit of velocity is then $v/d \approx 2.5 \times 10^{-2}$ m s$^{-1}$. A typical dimensionless zonal flow of about 5000 corresponds to 125 m s$^{-1}$, about right.

Although this is encouraging, the problems arise when we consider the heat flux in this model. For a Rayleigh number of around $GMd\Delta S/\nu\kappa c_p = 10^9$, with $G = 6.67 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$ and $M = 1.9 \times 10^{27}$ kg and $c_p \sim 10^4$ J kg$^{-1}$ K$^{-1}$ our unit of heat flux is $\nu_0 T_0 \Delta s/d \sim 1.4 \times 10^5$ W m$^{-2}$, corresponding to a unit of surface luminosity of about $4 \times 10^{20}$ W. Actually, our simulations typically give a surface heat flux of several hundred dimensionless units, so our models are producing about $10^{23}$ W, about a factor $10^8$ too large! How is this happening? For numerical reasons, we cannot reduce $E$ below $10^{-5}$, but to be in the right parameter regime we need to reduce both diffusivities by a factor of about $10^5$. This would give $E \sim 10^{-10}$ leading to a critical $Ra$ a factor $\sim 5 \times 10^6$ larger than at $E \sim 10^{-5}$. If we keep $Ra$ at about 20 times critical, $\Delta S$ would drop by a factor $5 \times 10^{-4}$, and the unit of heat flux by a factor $5 \times 10^{-9}$, bringing the heat flux in line with Jupiter’s observed heat flux. Of course, Ekman numbers this small would lead to very small dissipation length scales (as there probably are on Jupiter) and these would be impossible to resolve numerically. However, it is plausible that reducing the diffusion coefficients and the heat flux would not change the large scale flows dramatically, but would merely extend the turbulent subrange to smaller length scales.

### 3 General properties of compressible convection

#### 3.1 Scaling arguments

Convection in rapidly rotating systems usually takes a columnar form, unless the Prandtl number is very low, when a highly oscillatory form may be preferred. In our compressible calculations we also find a columnar structure (see also Jones et al. 2009), in which the columns have a much shorter width $\ell$ than their height $d$ parallel to the rotation axis. Near the surface, where the scale height can be short, columns with length significantly smaller than $d$, the distance across the shell, are found, but we ignore this in our initial scaling argument. Comparing the order of magnitude of the terms in the vorticity equation suggests the three-term balance between inertia, planetary vorticity stretching and buoyancy, (see e.g. Ingersoll and Pollard, 1982; Jones, 2007)

$$u_\ast^2 d^2 / \ell^2 \sim 2E^{-1}u_\ast \sim \frac{Ra S^* d}{r^2 \ell}$$ (3.1)

viscosity being assumed negligible at high $Ra$ and small $E$ (see e.g. Gillet and Jones, 2006). From (2.12), the convective heat flux is of order

$$F \sim 4\pi r^2 \rho T u_\ast S^*,$$ (3.2)

assuming that the work done by buoyancy and the viscous heating approximately balance. Estimate (3.2) also assumes that the velocity and entropy are strongly correlated, i.e. hot fluid
rises. In rapidly rotating systems, the correlation maybe less strong, leading to somewhat larger velocities than this simple argument suggests. Solving (3.1) and (3.2) gives the scalings

\[
\frac{\ell}{d} \sim \frac{Ra^{1/5} F^{1/5} E^{3/5}}{2\pi^{1/5} r^{4/5} (\rho T)^{1/5}}, \quad u_\ast \sim \frac{Ra^{2/5} F^{2/5} E^{1/5}}{2\pi^{2/5} r^{8/5} (\rho T)^{2/5}}, \quad S^* \sim \frac{Ra^{-2/5} F^{3/5} E^{-1/5}}{2\pi^{3/5} r^{2/5} (\rho T)^{3/5}}.
\]

These arguments are quite crude, but the results are broadly consistent with our simulations. In particular, we see that the expected convective velocity increases significantly as the low density region near the outer surface is approached. Note that for the fairly thin shells considered, the variation in \( r \) is less significant than the variation in \( \zeta \). The physical reason why the convective velocities generally increase near the outer boundary is that the heat flux there has to be carried by much lower density material, so the product of velocity and entropy perturbations must increase there. The dynamical argument suggests there is both a larger velocity and a larger entropy fluctuation there. The larger velocity predicted near the surface by this argument conflicts to some extent with the Proudman-Taylor theorem which we now examine.

### 3.2 Compressible Proudman-Taylor theorem

Taking the curl of the equation of motion (2.3) gives

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla (\mathbf{u} \cdot \nabla) - 2 E^{-1} \frac{\partial \mathbf{u}}{\partial z} = \nabla \times \left( \frac{RaS}{r^2} \mathbf{r} \right) + \nabla \times \mathbf{F}.
\]

The \( \phi \) component of this is the thermal wind equation

\[
\frac{\partial u_\phi}{\partial t} + \omega (\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla) \omega_\phi - s (\nabla \cdot \mathbf{u}) u_\phi - 2 E^{-1} \frac{\partial u_\phi}{\partial z} = -\frac{Ra}{r^2} \frac{\partial S}{\partial \theta} + (\nabla \times \mathbf{F})_\phi,
\]

where \( s = r \sin \theta \). Since \( E \) is very small, we expect the dominant term in this equation to be \( 2 E^{-1} \partial u_\phi / \partial z \), and so to obtain balance with the other terms, \( u_\phi \) must be nearly independent of \( z \), consistent with the Proudman-Taylor theorem. In section 5 we discuss this result in the light of the numerical simulations, which do indeed give large \( z \)-independent zonal flows.

Given that large zonal flows appear in simulations, which we expect to be \( z \)-independent from the argument above, how are they maintained? We must look at the \( \phi \) component of the momentum equation, averaged over \( z \) and \( \phi \), that is the surface integral over a coaxial cylinder of radius \( s \),

\[
\int_S \rho \frac{\partial u_\phi}{\partial t} s \phi dz = -\int_S \left\{ \frac{\partial}{\partial z} (\rho u_z u_\phi) + \frac{1}{s^2} \frac{\partial}{\partial s} (s^2 \rho u_s u_\phi) \right\} s \phi dz + \int_S \rho F_\phi s \phi \phi dz.
\]

Because we included the \( \rho \) factor in this integral, the Coriolis term gives zero, because there is no net mass flux across the cylinder (apply the divergence theorem to (2.5) integrated over the volume inside the cylinder). The first term on the right-hand-side is the forcing due to the Reynolds stresses. Note that the \( z \)-derivative term can be integrated, but leaves a non-zero surface integral. The \( s \)-derivative term is non-zero because the spiralling of the convection in a rotating system gives a non-zero average for \( u_s u_\phi \). In a steady state, this Reynolds stress must be balanced against the viscous forces. Since these forces may be small in a strongly driven system, particularly if there are stress-free boundaries, a large zonal flow may result.
Next we consider the $z$-component of the vorticity equation (3.4). This is the equation we used in our scaling arguments,

\[
\frac{\partial \omega_z}{\partial t} + \omega_z (\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla) \omega_z - (\omega \cdot \nabla) u_z + 2E^{-1}(\nabla \cdot \mathbf{u}) - 2E^{-1} \frac{\partial u_z}{\partial z} = \frac{Ra}{r^3} \frac{\partial S}{\partial \phi} + (\nabla \times \mathbf{F})_z.
\] (3.7)

At first sight this seems to have a similar structure to the $\phi$ vorticity equation, but there are important differences. As emphasized by Evonuk and Glatzmaier (2008), see also Glatzmaier et al. (2009), the $2E^{-1}(\nabla \cdot \mathbf{u})$ term is non-zero in compressible flow, so in regions where there are strong $\rho$-gradients (near the surface) $u_z$ will not be $z$-independent. Furthermore, the convection has a small scale structure in the $s$ and $\phi$ directions (the length scale $\ell$), so that although the global Rossby number is small, the inertial terms are still significant on this small length scale. So although the $z$-length scale of the columns is longer than the horizontal scale, the convection columns typically do not reach right across the shell.

Heimpel and Aurnou (2007) showed that in their Boussinesq model, the width of the zonal flow jets was approximately given by the Rhines length based on the zonal flow speed,

\[ l_R \sim \left( \frac{u_\phi}{\beta} \right)^{1/2}. \] (3.8)

In a compressible flow it is not so clear what $\beta$ should be. In a Boussinesq flow with convection columns stretching between the boundaries, the vortex stretching balance (ignoring the nonlinear terms, buoyancy and viscosity) is

\[
\frac{\partial \omega_z}{\partial t} + \beta u_s = 0, \quad \text{where} \quad \beta = -2E^{-1} \frac{1}{h} \frac{dh}{ds},
\] (3.9)

leading to westward propagating Rossby waves inside the tangent cylinder where $dh/ds > 0$, and eastward propagating waves outside the tangent cylinder. Indeed Heimpel and Aurnou (2007) point out that

\[
\frac{1}{h} \frac{dh}{ds} = \frac{1}{r_o} \left( \frac{\tan \theta}{(\sigma^2 - \sin^2 \theta)^{1/2}} \right), \quad s < s_c; \quad \frac{1}{h} \frac{dh}{ds} = -\frac{1}{r_o \cos^2 \theta}, \quad s > s_c,
\] (3.10)

$s_c$ being the radius of the tangent cylinder. In the compressible case we have an additional term from the non-zero divergence,

\[
\frac{\partial \omega_z}{\partial t} + 2E^{-1} \nabla \cdot \mathbf{u} - 2E^{-1} \frac{\partial u_z}{\partial z} = 0.
\] (3.11)

We can still integrate the third term $-2E^{-1} \frac{\partial u_z}{\partial z}$ over $z$ leading to the $\beta u_s$ form, but whereas in the Boussinesq case $u_s$ is reasonably constant along the convection columns this is typically not the case in the compressible case. Quite rapid changes in $u_s$ can occur near the outer boundary, where the boundary conditions are applied. The divergence term $2E^{-1} \nabla \cdot \mathbf{u} = -2E^{-1} (1/\rho) \mathbf{u} \cdot \nabla \rho$. If $u_r$ is positively correlated with $u_s$, $\beta$ will be increased by compressibility, since $d\rho/dr < 0$. However, in compressible convection, as in Boussinesq convection (Aurnou et al. 2008), $u_z$ can be larger than $u_s$ so it is possible that $u_r$ can have the opposite sign to $u_s$, so the argument is rather uncertain, and it is not easy to apply formula (3.8) in compressible flow.
<table>
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<tr>
<th>Run</th>
<th>$E$</th>
<th>$N_\rho$</th>
<th>$\eta$</th>
<th>$Ra$</th>
<th>$Pr$</th>
<th>$N_r$</th>
<th>$N_l = N_m$</th>
<th>Velocity unit</th>
<th>Boundary conditions</th>
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<td>0.7</td>
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<td>256</td>
<td>0.739ms$^{-1}$</td>
<td>SF</td>
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<td></td>
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<tr>
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<td>384</td>
<td>0.0493ms$^{-1}$</td>
<td>SF</td>
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<td>384</td>
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<td>0.85</td>
<td>$6 \times 10^9$</td>
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<td>64</td>
<td>384</td>
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</tr>
<tr>
<td>F</td>
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<td>0.85</td>
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<td>384</td>
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</table>

Table 1: Runs performed. NS means a no-slip boundary, SF stress-free. TC refers to the tangent cylinder. Velocity unit is $E(1-\eta)\Omega_J R_J$.  

### 4 Fully three-dimensional simulations

The velocity is decomposed into toroidal and poloidal parts according to (2.7). The two equations for these quantities are given by $\mathbf{f} \cdot \nabla \times$ the equation of motion (2.4) and $\mathbf{f} \cdot \nabla \times \nabla \times (2.4)$. The entropy equation (2.8) then gives us three equations for the $T$, $P$ and $S$. Note that the expansion (2.7) automatically ensures that the divergence condition (2.6) is satisfied exactly. The numerical scheme is pseudo-spectral and uses spectral decomposition with up to degree $N_l$ Legendre polynomials and up to order $N_m$ Fourier modes with respect to the horizontal directions $\theta$ and $\phi$. We use high-order finite differences with a non-uniform mesh in the radial direction. The runs performed are listed in table 1. The velocities in the figures are given in dimensionless units, based on $v/d = E(1-\eta)\Omega r_o$. Since the rotation rates and radii are known reasonably accurately for the planets, these form the most natural basis for dimensional comparisons. In table 1, we give the unit of velocity based on the rotation rate and radius of Jupiter, so that for example for run E the peak zonal flow of $\sim 2 \times 10^4$ dimensionless units corresponds to $0.00739 \times 2 \times 10^4 = 148$ms$^{-1}$, while in run F the peak zonal flow of $\sim 10^4$ corresponds to 110ms$^{-1}$.

In figure 1, the results for near Boussinesq convection, run A, are compared with those from the strongly compressible run B. The Ekman number is set at a moderate value of $2 \times 10^{-4}$, and the Prandtl number is set to 1. The polytropic index $n=2$ in all runs here, consistent with the value chosen in Jones et al. (2008). The radius ratio $\eta = 0.7$, between the expected values for Saturn and Jupiter. For these moderate values $N_r = 64$ radial grid points (spaced at zeros of the Chebyshev polynomials) and a triangular truncation with $N_l = N_m = 256$ spherical harmonics, $0 \leq m \leq 256$ with $m \leq l \leq 256$ was found to be adequate for these parameters for moderate $N_\rho$.

In the near Boussinesq case, $N_\rho = 0.1$ the amplitudes of the highest harmonics near the cutoff
in the truncation are typically nine orders of magnitude less than for the largest components, so these are well resolved. For the strongly compressible case the ratio is only four orders of magnitude, and the spectrum drops off quite slowly near $l \approx 256$, so this is not quite so well resolved. However, increasing the resolution to $N_r = 96$, $N_l = 384$, i.e. including all $0 \leq m \leq 384$ with $m \leq l \leq 384$ did not significantly affect the figures shown, though the energy in the highest harmonics near the cutoff are typically a factor 30 lower than in the $N_l = 256$ truncation. The higher resolution was actually used in figures 1d-f.

### 4.1 Boussinesq and compressible convection compared

In figure 1a, we show an equatorial section of $u_r$ for the near Boussinesq case, which can be compared with the same quantity for the strongly compressible case figure 1d. It is immediately apparent that the convection is very different in the two cases. As expected from the scaling argument, in the strongly compressible case the convective velocity is much larger near the outer boundary where the density is lower. In contrast, in the Boussinesq case the convective velocity is somewhat larger near the inner boundary. It is also clear that the dominant azimuthal wavenumber is much larger in the compressible case. This was found in the linear results of Jones et al. (2008), and this effect persists into the nonlinear regime. Another feature of interest in the near Boussinesq run, figure 1a, is the longitudinal asymmetry, there being a significant contribution from the $m = 1$ mode. This was also seen in Busse (2002), who attributes it to an influence of the zonal flow on the convection. In figures 1b and 1e we show the meridional cross-section of the zonal flow for the two cases. Note that in both cases the zonal flow is remarkably independent of $z$. In neither case is there significant zonal flow at high latitudes outside the tangent cylinder. Figures 1c and 1f show the surface zonal flow pattern as a function of latitude. In the compressible case, small scale weak jets do occur at high latitudes, but these are transient, not permanent, features of the flow. Note also the strong eastward equatorial jet in both cases. This is a very robust feature of the flow, seen in all simulations we have run over a wide range of parameters. Since there are no internal heat sources, the heat flux in at the bottom of the shell equals the heat flux out at the top on time-average, and this was checked in the code. In the $N_r = 5$ case, the Nusselt number is just under 2, so the convecting heat flux and the conduction heat flux are comparable. In the near Boussinesq case, the Nusselt number varies more with time, around an average value of about 1.5.

A closer inspection of the zonal flow patterns in figures 1b and 1e reveal an interesting difference. While there is no systematic long-term zonal flow in the compressible case, there is a small scale transient zonal flow which exists close to the surface but does not penetrate into the deep interior. This indicates that the convection pattern must be different in the compressible and Boussinesq cases, and figures 2a and 2b show this is indeed the case. In the strongly compressible case the convection is fully developed inside the tangent cylinder, whereas in the near Boussinesq case the convection only occurs outside the tangent cylinder. This is despite the fact that the Rayleigh number in the near Boussinesq case is 8.5 times critical, but the fully compressible case is only 5 times critical. Note, however, that Tilgner and Busse (1997) showed that at much higher $Ra$ convection becomes fully developed inside the tangent cylinder even in the Boussinesq case. In all the cases shown in this paper, the individual convective eddies are transient. Generally,
Figure 1: $E = 2 \times 10^{-4}$, $Pr = 1$, $\eta = 0.7$, $n = 2$, $N_r = 64$, $N_l = N_m = 256$. (a-c) Run A. $N_r = 0.1$, almost Boussinesq, $Ra = 10^7 \approx 8.5Ra_{crit}$. (a) Equatorial section of $u_r$, (b) meridional section of the zonally averaged $u_\phi$, (c) surface zonal flow as a function of latitude. (d-f) Run B. $N_r = 5.0$, strongly compressible, $Ra = 6 \times 10^7 \approx 5Ra_{crit}$, $N_r = 96$, $N_l = N_m = 384$. (d) Equatorial section of $u_r$, (e) meridional section of the zonally averaged $u_\phi$, (f) surface zonal flow.
Figure 2: $E = 2 \times 10^{-4}$, $Pr = 1$, $\eta = 0.7$, $n = 2$. (a) Run A. Meridional section of $u_r$, $N_\rho = 0.1$, almost Boussinesq, $Ra = 10^7 \approx 8.5 Ra_{crit}$, $N_r = 64$, $N_l = N_m = 256$. (b) Run B. Meridional section of $u_r$, $N_\rho = 5.0$, strongly compressible, $Ra = 6 \times 10^7 \approx 5 Ra_{crit}$, $N_r = 96$, $N_l = N_m = 384$.

the smaller the eddies are in size, the shorter their lifetime.

4.2 Zonal flow inside the tangent cylinder

Figures 3a-c show strongly compressible convection at lower Ekman number and Prandtl number. The compressibility $N_\rho$ has been reduced here for numerical reasons. The lower Ekman number means that the convection columns are considerably thinner. Even in the nonlinear regime the width of the convection columns scales as approximately $E^{1/3}$, so a factor 10 reduction of Ekman number requires a doubling of the resolution in all directions. Furthermore, the timestep also has to be reduced, so low $E$ is very computationally demanding. The final problem is that very long runs are necessary before the final pattern of the zonal flow is established, at least of order a viscous diffusion time. This is because the zonal flow is driven by the small Reynolds stress balancing the small viscous damping, and we must wait a full diffusion time before this balance is fully established. In order to reduce the computational demand, figure 3 was obtained applying eightfold symmetry in the $\phi$-direction. Heimpel et al (2005) noted that this assumption made little difference to the nature of the solution, because the very small column lengths means there is little interaction over large azimuthal distances. The columns do not influence each other over a long range. The azimuthal modes included in the calculation are then just the multiples of 8. However, imposing such symmetry does affect the flow near the poles, as the possibility of flow passing over the poles is artificially eliminated when the $m = 1$ mode is excluded.

Figure 3c shows the zonal flow as a function of latitude. The dashed line is a snapshot, while the solid line is the time average taken over 0.03 viscous diffusion times, which is about a hundred convective turn-over times. The most notable difference between figures 3c and 1f is that at the
Figure 3: $E = 2 \times 10^{-5}$, $Pr = 0.1$, $\eta = 0.8$, $n = 2$, $N_p = 3.5$, $Ra = 5 \times 10^8 \approx 7.3Ra_{crit}$, $N_r = 96$, $N_i = N_m = 384$, with eightfold azimuthal symmetry imposed. (a-c) are with stress-free boundaries at inner and outer boundary. (a) a meridional section of $u_r$, (b) meridional section of the zonally averaged $u_\phi$, (c) surface zonal flow as a function of latitude. Solid line, time average, dashed line single snapshot. (d-f) as (a-c) but with a no-slip boundary condition at the inner boundary, stress-free outer boundary.
lower Prandtl number and Ekman number there is a substantial zonal flow inside the tangent cylinder, as well as the usual eastward equatorial jet. Two effects are important here. First the lower Prandtl number, 0.1 rather than 1, increases the relative importance of the Reynolds stress term in the equation of motion (Glatzmaier and Gilman, 1982), and also gives more spiralling, even in the nonlinear regime (Zhang, 1992, and compare figures 7a and 7b of Gillet and Jones 2006). Admittedly, the enhanced spiralling at low Prandtl number is usually clearly observable in the convection outside the tangent cylinder. Inside the tangent cylinder the convection is more like rising and falling plumes than the Busse roll form, but nevertheless there is a small but systematic tilt of the convection pattern driving a Reynolds stress. Second, the much lower Ekman number means that the friction counteracting the Reynolds stress is much smaller.

4.3 Effect of no-slip boundaries

To establish that the low dissipation is crucial, we change the boundary condition from a stress-free boundary at the inner shell surface to a no-slip boundary, figures 3d-f. This has the effect of killing off the mean zonal flow outside the tangent cylinder, compare figures 3c with 3f and 3b with 3e. The flow outside the tangent cylinder is not much affected, and the eastward equatorial jet is hardly affected at all. The small scale convection is also not greatly affected by the change of boundary condition even inside the tangent cylinder, compare figures 3a and 3d. The no-slip boundary layer gives rise to Ekman pumping which strongly enhances the zonal flow dissipation compared to the stress-free case. This result may be significant for giant planets, because the effect of the magnetic field in the dynamo region may be to reduce the differential rotation there to a rather low value (Starchenko and Jones 2002, Liu et al. 2008). An almost rigidly rotating metallic hydrogen core may act more like a no-slip boundary than a stress-free boundary. However, the actual nature of a boundary layer in which there is zonal shear and in which the electrically conductivity drops smoothly to zero will be studied in a forthcoming paper. The transient zonal flows noted in figure 1f are still found here, but the long-term average zonal flow, the solid line in figures 3c and 3f, shows that there is very little long term zonal flow with a no-slip bottom boundary condition. A similar result was found by Christensen (2002) in the Boussinesq case when both inner and outer boundaries were no-slip. Our result shows that only one boundary needs to be no-slip, and that compressibility does not prevent the strong coupling between the bottom boundary and the flow in the shell itself.

4.4 The multiple jet regime

The zonal flow pattern seen in figure 3e has a fairly simple columnar structure. There is some evidence of a high-latitude westward jet in the southern hemisphere, but multiple jets were not much in evidence. Following the experience of Heimpel et al. (2005) we increased the radius ratio, from $\eta = 0.8$ to $\eta = 0.85$, increased the Rayleigh number and reduced the Ekman number still further. Some results can be seen in figures 4a-f. The conversion rates from the dimensionless units to dimensional values of the velocities, based on the rotation rate and radius of Jupiter are also given. Figures 4a-c, Run E, are for a near Boussinesq case at 47 times critical Rayleigh number with $E = 4 \times 10^{-6}$, and can therefore be compared to those of Heimpel et al. (2005).
Figure 4: (a-c) $E = 4 \times 10^{-6}$, $Pr = 0.1$, $\eta = 0.85$, $n = 2$ $N_\rho = 0.1$, almost Boussinesq, $Ra = 6 \times 10^9 \approx 47Ra_{crit}$, $N_r = 96$, $N_l = N_m = 384$, with eightfold azimuthal symmetry imposed. (a) Meridional section of $u_r$, (b) meridional section of the zonally averaged $u_\phi$, (c) surface zonal flow as a function of latitude. With the dimensional units given in table 1, the peak zonal flow of $2 \times 10^4$ corresponds to 150 ms$^{-1}$. (d-f) $E = 6 \times 10^{-6}$, $Pr = 0.1$, $\eta = 0.85$, $n = 2$ $N_\rho = 5.0$, strongly compressible, $Ra = 2 \times 10^{10} \approx 23Ra_{crit}$, $N_r = 96$, $N_l = N_m = 384$, with eightfold azimuthal symmetry imposed. (d) Equatorial section of $u_r$, (e) meridional section of the zonally averaged $u_\phi$, (f) surface zonal flow. The solid line is the time-averaged zonal flow, the dashed line is a particular snapshot. With the dimensional units given in table 1, the peak zonal flow of $10^4$ corresponds to 110 ms$^{-1}$. 

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This simulation was run for a total of approximately 0.5 diffusion times, and some hyperdiffusion was used to improve the convergence of the highest harmonics in the truncation. The form of the hyperdiffusion used in this run was to replace the diffusion operator $\nabla^2$ by $\nabla^2 - Hyp\nabla^4 \perp$ where $\nabla^2 \perp$ means the part of the Laplacian involving only the $\theta$ and $\phi$ derivatives. The parameter $Hyp = 0.001$ in this run, to minimise the effect of the hyperdiffusion on the larger scale modes. Multiple jets are apparent, as they were in the Heimpel et al. simulation. The number of jets depends on the initial conditions, at least no long term tendency for the number of jets in either hemisphere to change after an initial transient period was apparent. In the run shown there are more jets in the northern hemisphere than the southern hemisphere, but of course there is no particular significance to this. In other cases where the initial conditions were more symmetric between the hemispheres, a more symmetric final state results, but the asymmetry shown in figures 4a-c are more typical. There is of course a possibility that the solution would symmetrize if run for many diffusion times which we cannot exclude. Two long runs with different initial conditions were run, both for this case run E and the strongly compressible case discussed below, run F. In all cases, a statistically steady state was reached, but with a different final jet configuration, and in consequence a different final kinetic energy. Some tests were carried out for this run on the effect of changing the hyperdiffusion parameter. Lowering the hyperdiffusion significantly below $10^{-3}$ led to numerical instability in run E at the truncation levels stated in table 1 (including more modes was prohibitively expensive). At $Hyp = 2 \times 10^{-3}$, the zonal flow pattern was very similar to that shown in figure 4c, but the amplitude of the zonal flow was approximately 5% greater.

In figures 4d - 4f, run F, similar behaviour is found for a strongly compressible case. To obtain these solutions, the initial condition was from a near Boussinesq run. The state was then integrated forward with hyperdiffusion similar to that used for the Boussinesq case for approximately 0.5 diffusion time. This corresponds to several thousand convective turnover times $1/u_*$, $u_*$ being a typical convective velocity of $5 \times 10^3$ in dimensionless units. By this stage the heat flux, kinetic energy and zonal flow pattern were showing no long term trends. The hyperdiffusion was then switched off, and run further for a few hundred turnover times, and unlike the run E case the solution remained stable, though a smaller timestep was needed. The surface zonal flow, figure 4f, shows a solid line, time-averaged over approximately the last hundred turnover times, and a dashed curve showing a snapshot at one instant of time. The much greater time variability of the zonal flow in the compressible case compared with the near Boussinesq case is evident from comparing figures 4c and 4f. In 4c the average over the last hundred turnover times is not very different from the snapshot, but in 4f there is much more noise due to the relatively large amplitude near surface convection. The pattern of the time-averaged part of the zonal flow was similar both when hyperdiffusion was switched on and when it was switched off. As in the Boussinesq case, the main effect of the hyperdiffusion (apart from allowing a larger time-step and promoting numerical stability) appears to be to increase the amplitude of the zonal flow slightly, while leaving the shape unchanged. When no hyperdiffusion is used, the spectral convergence is poorer than in the the near Boussinesq case. Nevertheless, the large scale features of the zonal flow were stable over a significant fraction of a diffusion time, and also were not sensitive to varying the mode truncation levels, provided of course these are
large enough. It is of interest to note that multiple jets can be found without hyperdiffusion.

5 Discussion

5.1 Proudman-Taylor theorem

In the compressible case, it is important to distinguish between the convective part of the velocity and the zonal flow. The convective velocity and the entropy perturbations increase rapidly as \( r \to r_o \), as predicted by the scaling argument. The convective velocity varies significantly with \( z \) despite the Proudman-Taylor theorem, but the structure of the zonal jets is still columnar. This leaves an opportunity for the explanation of planetary zonal jets by deep interior convection. Indeed, the jets have very long lifetimes, suggesting they reflect conditions in the interior. The breakdown of the Proudman-Taylor theorem for the convective part of the flow is partly due to the non-zero linear divergence term in (3.7), but mainly due to the thinness of the convection rolls. This means that the nonlinear advective terms are important in (3.7), and can balance the vortex stretching term \( 2E^{-1}\partial u_z/\partial z \) even when the \( z \)-length scale is relatively short.

Why does the zonal flow satisfy the Proudman-Taylor when the convective velocity does not? The main reason is that the zonal flow is a large scale flow formed by the Reynolds stresses having a non-zero average over a much larger length scale than that of individual convective eddies. In the Reynolds stress terms of (3.6) the convective parts of \( u_\phi \) and \( u_s \) vary rapidly with distance and with time, but they nevertheless give a fairly consistent long-term contribution to the integral.

When large horizontal scale jets build up by the Reynolds stresses, as happens in our simulations, very little variation of \( z \) is produced by the nonlinear terms in (3.5). Whereas these Reynolds stresses can compete with the very small viscous forces, they would be overwhelmed by the \( 2E^{-1}\partial u_\phi/\partial z \) if there was any significant \( z \)-variation in \( u_\phi \). The physical mechanism is that any non-zero shear of \( u_\phi \) in the \( z \)-direction stretches out the planetary vorticity giving a rapid growth of \( \omega_\phi \). This then redistributes \( u_\phi \) so that it no longer has a \( z \)-variation.

It is possible that small scale convection can give a significant non-zero average for the nonlinear terms in (3.5) near the surface, where they are large, but this does not seem to happen much at depth, where the convective velocities are smaller. Similarly, the thermal wind term proportional to \( \partial S/\partial \theta \) can be significant near the surface, where there are substantial entropy fluctuations, but it is less effective at depth, because of the scaling argument above. At these very low Ekman numbers, viscosity is small, and cannot give rise to a significant \( z \) variation in \( u_\phi \) in (3.5). In summary, when there are large scale jets, we expect the zonal flow to be rather independent of \( z \) and this is borne out in all our simulations. This conclusion is, however, model dependent on the nature of the bottom boundary conditions. If for example, a new model of the equation of state near the transition zone suggested a possible stably stratified layer there, then considerations such as those suggested for the solar differential rotation (e.g. Rempel, 2005) might apply.
5.2 Viscous heating

The viscous heating in the entropy equation plays a significant role in the global energy balance. If it is switched off in the entropy equation, the heat flux at surface is substantially less than the input heat flux at the inner boundary, so energy conservation is significantly violated. In the runs for figures 3a-c, the total rate of working of the buoyancy forces, which equals the amount viscous heating, is about 30% of the heat flux passing through the shell. In the highly compressible very low $E$ run of figures 4d-f, this rises to about 60% of the heat flux passing through the shell. There is no reason why the viscous dissipation has to be less than the heat flux passing through the system; in a compressible convecting fluid the constraint is an entropy constraint, not an energy constraint, (e.g Hewitt, McKenzie and Weiss, 1975) and the viscous dissipation is only bounded by

$$F \frac{T_{\text{bottom}} - T_{\text{top}}}{T_{\text{top}}}$$

(5.1),

$F$ being the total heat flux passing through shell and $T_{\text{top}}$ and $T_{\text{bottom}}$ being the temperatures at the respective boundaries. To check this, a case was run with $E = 2 \times 10^{-4}$, $\eta = 0.7$, $Pr = 1$, $N_\rho = 5$, and $Ra = 2 \times 10^8$, i.e. similar to the parameters for figures 1d-f, but with a higher Rayleigh number. This is well-resolved, and the viscous dissipation rate is just under twice the heat flux passing through the system. This is well below the theoretical entropy maximum (5.1), but it nevertheless shows that it is indeed possible for the dissipation to exceed the heat flux through the system. In a Boussinesq fluid, the viscous dissipation rate is small compared to the heat flux through the system.

5.3 Zonal flows

The strong eastward equatorial jet is a very robust feature of all these simulations, as it is in the Boussinesq case too. It is not significantly affected by a bottom no-slip boundary condition. Its latitudinal extent is not affected much by the effect of compressibility, compare figure 4c and 4f, but its extent is affected by the depth at which the impermeable boundary is placed. The radius ratio of 0.85 still gives a wider jet than is observed on Jupiter, the maximum westward flow occurring at latitude $\sim 30^\circ$ rather than the observed $\sim 20^\circ$ (see e.g. Vasavada and Showman, 2005), suggesting that the magnetic influence extends further out than the usual location of the metallic hydrogen transition region, for reasons discussed in Liu et al. 2008.

The nature of the jets inside the tangent cylinder in figure 4 looks similar to that found by Heimpel et al. (2005) for the Boussinesq case, and our compressible results in figure 4f are not so different, given the inevitable differences resulting from different parameter choices and different numerical constraints. However, there are some issues with the width of the jets. As emphasised in section (3.2) it is not clear how to define the Rhines scale in compressible flow, but very crudely we might say that similar zonal flow speeds in figures 4c and 4f are leading to roughly similar jet widths, consistent with jet-widths of approximately $(U/\beta)^{1/2}$. However, the zonal jets inside the tangent cylinder in figure 3c have a longer length scale, indeed in the northern hemisphere a single jet almost fills the whole region, despite having a lower zonal flow and therefore an expected shorter Rhines scale. Clearly the selection of the jet flow length scale is a very complex problem,
and indeed even in the Boussinesq case there are clearly different regimes in which different jet width selection mechanisms are occurring (Rotvig and Jones, 2006). We are still some way off understanding this problem.

5.4 The heat flux problem

In section 2.2 we pointed out that for numerical reasons the heat flux in the simulations is necessarily many orders of magnitude greater than a giant planet heat flux. How serious is this for our models? Our belief is that it does not invalidate the fundamental picture of deep convection. The same problem arises in geodynamo simulations, but nevertheless the models there do a remarkably good job of simulating the geomagnetic field. The reason is the that although the models cannot be run in the correct parameter regime, they can be scaled down into that regime (Starchenko and Jones, 2002, Christensen and Aubert 2006), and the ‘missing’ small scales of motion do not appear to be so important in determining the large scale observable behaviour. However, without further calculations we cannot be certain that these ‘missing’ small scales do not have an impact on the angular momentum transport and hence on the zonal flow distribution.

The inertial theory of rapidly rotating convection sketched in section (3.1) predicts that the convective velocity is independent of the diffusivities, and scales with $F$ to the two-fifths power. If this is correct, and the simulations support it, then if the flux is reduced by ten orders of magnitude, say, the convective velocity is reduced by four orders of magnitude, and the thickness of the convective eddies $\ell$ will be reduced by two orders of magnitude from (3.3a). This will reduce the Reynolds stresses in (3.6) by six orders of magnitude. However, the zonal flow is determined by the balance of Reynolds stresses against friction (3.6), and friction could well be six orders of magnitude too large in the simulations. Indeed, as shown in section (2.2) this is just the order of magnitude reduction in the diffusivity we need to obtain the reduction in the heat flux. Furthermore, this scaling will lead to a regime in which the zonal flow is essentially unchanged from the simulated value, but the convective velocity is much reduced. This would fit much better with the observations, which suggest that vertical motion is a factor at least $10^3$ smaller than the zonal wind speed (Vasavada and Showman, 2005), a much larger ratio than the convective velocity/zonal wind velocity found in our simulations.

We cannot yet say whether our simulations are truly in the asymptotic regime, but analysing the scaling behaviour of large-scale simulations is just about feasible with very large parallel computer clusters, so there is hope that this can be established, and the optimistic view would be that if these simulations were scaled to giant planet values, the structure would become more like the giant planets than the current simulations. For this reason we do not consider the very large heat fluxes in deep convection simulations to be an overwhelming objection.

6 Conclusions

To summarise our main findings:

(i) In both Boussinesq and compressible cases, there is a strong eastward (prograde) jet near
the equator, as on Jupiter and Saturn. This is a very robust feature of these rapidly rotating simulations.

(ii) High latitude eastward and westward jets occur as in giant planets in anelastic compressible simulations when we impose stress-free boundaries, but they are hard to find with a no-slip inner boundary. This suggests that these jets are less robust. Clearly more work is needed on the nature of the transition region between the molecular and metallic regions of the giant planets, especially its dynamics. The issue of how strongly coupled the magnetic interior is to the molecular H/He region is clearly crucial.

(iii) The Proudman-Taylor theorem strongly constrains the zonal flow to independent of $z$, but the convective eddies carrying the heat are small and transient, so small-scale nonlinear vorticity transport can lead to convective eddies with a $z$-extent much less deep than the shell thickness. The convective eddies are however, still strongly anisotropic, with the $z$-length scale much longer than the very narrow eddy width.

(iv) The amount of heat generated by viscous dissipation can exceed the heat flux passing through the planet.

(v) The heat flux in the models exceeds the heat flux in the giant planets by many orders of magnitude. However, simple scaling arguments suggest that large zonal flows will still be present if the flux is reduced to giant planet values, and the ratio of zonal flow speed to convective velocity speed will be greatly increased, consistent with giant planet observations.

(vi) Most observational evidence for deep jets refers to the equatorial current. The Galileo probe went in near the equator and found wind speed increases with depth there. The equatorial jet of Saturn is broader compared to that of Jupiter, which reflects its smaller metallic hydrogen core. The equatorial current is probably deep, extending to at least a significant fraction of the distance to the metallic core interface region. The situation for the higher latitude jets inside the tangent cylinder still remains unclear. As pointed out by Liu et al. 2008, it is hard to see how the jets can survive the ohmic dissipation in the metallic hydrogen transition zone, which may extend over several thousand kilometres. However, the basic picture of deep convection driving the zonal flows still has many attractions, so it may be worth exploring ways out of the difficulty. Our knowledge of the equation of state and the electrical conductivity at the high pressures and comparatively low temperatures is still far from definitive. Changes in the equilibrium model, such a stably stratified layer near the transition region, or a more rapid transition in electrical conductivity reducing the ohmic dissipation, may give possible ways out of this difficulty.

**BIBLIOGRAPHY**


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