THE BOUSSINESQ AND ANELASTIC LIQUID APPROXIMATIONS FOR CONVECTION IN THE EARTH'S CORE

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Abstract

Convection in the Earth's core is usually studied in the Boussinesq approximation in which the compressibility of the liquid is ignored. The density of the Earth's core varies from ICB to CMB by approximately 20%. The question of whether we need to take this variation into account in core convection and dynamo models is examined. We show that it is in the thermodynamic equations that differences between compressible and Boussinesq models become most apparent. The heat flux conducted down the adiabat is much smaller near the inner core boundary than it is near the core-mantle boundary. In consequence, the heat flux carried by convection is much larger nearer the inner core boundary than it is near the core-mantle boundary. This effect will have an important influence on dynamo models. Boussinesq models also assume implicitly that the rate of working of the gravitational and buoyancy forces, as well as the Ohmic and viscous dissipation, are small compared to the heat flux through the core. These terms are not negligible in the Earth's core heat budget, and neglecting them makes it difficult to get a thermodynamically consistent picture of core convection. We show that the usual anelastic equations simplify considerably if the anelastic liquid approximation, valid if $\alpha T << 1$, where α is the coefficient of expansion and T a typical core temperature, is used. The resulting set of equations are not significantly more difficult to solve numerically than the usual Boussinesq equations. The relationship of our anelastic liquid equations to the Boussinesq equations is also examined.

Keywords: Geodynamo models; Core convection; Boussinesq approximation

1. INTRODUCTION

Thermal and compositional convection are important energy sources for the generation of magnetic field in planets and stars. Because the dynamo equations are difficult to solve, they are often studied in the Boussinesq approximation (BA) where compressibility is neglected. Is the BA adequate for describing the convection in the cores of the planets, particularly for the Earth's core? The density of the core varies by only about 20% from the inner core boundary (ICB) to the core-mantle boundary, so that one might believe that the effect of compressibility would be fairly small. Any errors from using the BA would be negligible compared to the much larger uncertainties arising from our lack of precise knowledge of many physical parameters such as the thermal conductivity and viscosity in the core. We believe that this argument is not valid, (i) partly because convection in the core differs in important respects from a classical Rayleigh-Bénard (RB) problem, and (ii) partly because in the Boussinesq limit a number of issues that are important for the geodynamo are suppressed.

As an example of issue (i) in classical RB convection, the Nusselt number, the ratio of total convected plus conducted flux to the conducted flux, increases as some positive power of the Rayleigh number, generally believed to be somewhat greater than one third. Therefore at very large Rayleigh number the Nusselt number is also very large. In the Earth's core the Rayleigh number is very large because the dimensions of the core are big and the diffusivities small. The convected flux is however only of the same order as the conducted flux, and the Nusselt number is only $O(1)$. It is clear from this that if the BA is used to describe convection in the core the relationship between the quantities calculated in a BA calculation need some interpretation before they can be related to the actual physical variables in the core. In section 7 below we derive a set of Boussinesq equations from the full anelastic equations in a consistent way by letting the dissipation number tend to zero. We can then relate quantities used in the BA approximation, such as the temperature perturbation, to the actual temperature in the core. This is less straightforward than might be imagined.

As an example of issue (ii), an important question for the geodynamo is what fraction of the total heat flux passing through the core is converted into magnetic energy, to then emerge as ohmic dissipation? It is known from entropy production arguments, Hewitt *et al.* (1975), Gubbins, (1977), Braginsky and Roberts, (1995) (subsequently referred to as BR95), Roberts, Jones and Calderwood (2003) (subsequently referred to as RJC03), that this depends on the temperature difference between the ICB and the Core-Mantle Boundary, CMB. There is a "Carnot efficiency" factor, so that this fraction cannot exceed $(T_{icb} - T_{cmb})/T_{cmb}$ (Backus, 1975), which is believed to be around 30% for the Earth's core. However, when the BA is used in a compressible fluid, it is justified by making the thin layer approximation (see e.g. Spiegel, 1971) that the temperature difference between the boundaries is small compared to the absolute temperature on the boundaries. In this limit, the Carnot efficiency factor goes to zero, so the ratio of ohmic dissipation to heat flux through the system also goes to zero in the Boussinesq limit. We need a more sophisticated approach than the simple BA to answer some key questions for geodynamo models.

The equations we derive below are related to those used in mantle convection studies (see e.g. Schubert *et al.* 2001), although conditions in the mantle and the core differ in some important respects. The equations we derive below are not fundamentally any more demanding numerically than those of the traditional BA.

Although the general principles governing models of the core are agreed (see e.g. BR95, RJC03, Lister and Buffett (1995), Labrosse (2003)), there is still much uncertainty about the values of many key physical parameters. However, it is generally agreed that most, if not all, of the outer core is close to adiabatically stratified and the adiabatic temperature gradient at the CMB is at least double that at the ICB. Since the surface area of the CMB is about eight times larger than the surface area of the ICB, the heat flux conducted down the adiabat increases by at least a factor 16 as we pass from the ICB to the CMB. This heat must be supplied by the convection. A substantial amount of heat is released at the ICB due to the growth of the inner core, so there needs to be a large convective heat flux in the lower part of the core. However, as we move towards the CMB, conduction down the adiabat takes up an ever increasing fraction of the input heating, so the convective part of the heat flux correspondingly falls as we move outward. It is possible that the convective part becomes zero, or even negative (Loper, 1978), in which case turbulence being driven by compositional effects might maintain the near adiabatic stratification by pumping heat flux down from the CMB into the core. Clearly this reduction in the convective heat flux is an important effect, and it appears as a cooling term in the convective temperature equation which we call the heat flux deficit. Most of the published BA models neglect this term altogether, often assuming a constant input heat flux through the ICB which emerges unscathed to pass through the CMB, which is not a very realistic assumption. As we shall see below in section 7, it is possible to include an 'equivalent heat sink' term into our Boussinesq limit equations which takes the heat flux deficit into account. The heat flux deficit is offset somewhat by the slow cooling of the core. From the point of view of the convection, this can be thought of as equivalent to a heat source in a steady state temperature model, so this secular cooling provides a heating term in the convective temperature equation.

In the usual BA with uniform heat flux at the ICB and no heat sources within the core, convection starts when the Rayleigh number, Ra, no matter how it is defined, exceeds its critical value Ra_c which is independent of the parameters governing the adiabatic temperature. Since the Rayleigh number in the core is certainly much greater than Ra_c, convection is bound to occur throughout the core in such a BA model. On the other hand, convection in the actual core will exist only if the heat flux through the core exceeds the value of the adiabatic heat flux on the ICB. This is a more delicate issue, as the heat flux conducted down the adiabat is of the same order of magnitude as the total heat flux to be carried. The onset of convection in the core therefore depends crucially on the thermal conductivity of outer core material. If the conductivity is large enough to carry all the heat flux at the ICB down the adiabat, there will be no convection however large the Rayleigh number is.

Another difference is connected with the region in which convection occurs. It is possible that the total heat flux at the ICB is larger than the adiabatic heat flux near the ICB, but the total heat flux near the CMB is less than the adiabatic heat flux there. In this case, there will be some point at which the adiabatic heat flux equals the total heat flux, and above this there will be no convection except for a small amount of penetrative convection. In the absence of compositional convection, the basic state temperature gradient will no longer be adiabatic, but will be determined by Fourier's heat conduction law. This gradient will be less than the adiabatic temperature gradient, so the total temperature difference between the ICB and CMB will be reduced. The BA model with a constant heat flux across the core cannot have this property; convection must occur throughout the core.

RJC03 state that "... in order to produce a vigorous reversing dynamo resembling the geodynamo, ohmic dissipation of the order of 2×10^{12} W is required". Dissipation of this order is comparable with the whole heat flux, $\sim 7\times10^{12}$ W, from the Earth's core on CMB and so it should be taken into account in the heat transport equation. However the BA heat transport equation does not include this dissipation. Unlike the heat flux deficit problem, which can be taken into account using a modified Boussinesq formulation, this dissipation cannot be incorporated into the BA heat equation in a consistent way. Its possible effect on geodynamo simulations is hard to predict, though the ohmic dissipation will locally have its strongest influence on the heat balance where the magnetic field is largest, which may well be key regions for dynamo action.

Some care is needed in formulating the compressible equations, as neither the Ohmic nor the viscous dissipation is an external heat source since their energies are taken from the convection. They take their energies from the heat flux and therefore their dissipation must not change the heat flux. However, if we simply include them in the heat transport equation without including compensating terms they will (erroneously) enhance the heat flux. To resolve this difficulty we must consider the buoyancy force, which also does internal work. This work takes its energy from the heat flux. So the law of energy conservation requires this term to be included in the heat transport equation working as a cooling term and we will refer to it as the Archimedean cooling. The Archimedean cooling, together with the energy released by the compositional convection, integrated over the whole liquid core compensates the dissipation of the magnetic and the kinetic energies. Averaged over a long time period this work equals the sum of the dissipations, but in general there is no local balance, so the dissipation may be located in different places (and at different times) from where the buoyancy work is released.

2. ANELASTIC LIQUID CONVECTION EQUATIONS

General anelastic theory

We start by considering the reference state. The equations governing the pressure $\bar{P}$, temperature $\bar{T}$, and density $\bar{\rho}$ for our adiabatic reference state are

$$\frac{d\bar{P}}{dr} = -g\bar{\rho}, \ \frac{d}{\bar{T}}\frac{d\bar{T}}{dr} = -D, \ \frac{d}{\bar{\rho}}\frac{d\bar{\rho}}{dr} = -\frac{D}{\gamma}, \quad \bar{\xi} = \text{const.}, \ \bar{S} = \text{const.}, \ D = \frac{g\alpha d}{c_p}. \qquad (2.1a-f)$$

Here g is the gravitational acceleration, $\alpha = -\rho^{-1}(\partial\rho/\partial T)_{p,\xi}$ is the coefficient of thermal expansion, γ is the Grüneisen parameter and c_p is the specific heat at constant pressure. The difference between the outer core radius r_{cmb} and the inner core radius r_{icb} is $d = r_{cmb} - r_{icb}$. The composition of the core is assumed to consist of iron and a light component. In general the mass fraction of the lighter component is Ξ and is $\bar{\xi}$ in the reference state. The reference state is assumed well-mixed (hence $\bar{\xi}$ is independent of position) and isentropic with specific entropy $\bar{S}$. Note that α, c_p, γ, g and hence D are all functions of r. They are estimated from a combination of theory and experiment in high pressure physics. Typical values for the Earth's core are given in Table 1, a subscript 0 denoting the value at the point $r_0 = (r_{icb} + r_{cmb})/2$, and see also BR95, RJC03. The first of these equations is the hydrostatic equation, and the second defines the adiabatic temperature gradient. The dimensionless ratio D is called the dissipation number (see e.g. Schubert *et al.* 2001). In the Boussinesq limit $D \to 0$, but here we do not take this limit; D varies considerably across the Earth's core but is typically around 0.3. The Grüneisen parameter γ is hard to estimate accurately. RJC03 used values in the range 1.15 to 1.3 but recent estimates based on *ab initio* quantum calculations are generally around 1.5 (Vočadlo *et al.* 2003), and the density variation in the core reference state is about 20%.

We use the anelastic equations about this adiabatic reference state, so we write

$$P = \bar{P} + p, \quad \rho = \bar{\rho} + \rho', \quad T = \bar{T} + \vartheta, \quad \Psi = \bar{\Psi} + \psi, \quad \Xi = \bar{\xi} + \xi, \quad S = \bar{S} + s, \quad (2.2a-f)$$

and all these fluctuations about the reference state are due to the convection. The fluctuations are assumed small compared to their reference state values. Here Ψ is the gravitational potential. The momentum and continuity equations are

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P - \rho\nabla\Psi - 2\rho\mathbf{\Omega}\times\mathbf{u} + \mathbf{j}\times\mathbf{B} + \mathbf{F}^v, \quad \frac{\partial\rho}{\partial t} = -\nabla\cdot(\mathbf{u}\rho), \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \ (2.3a,b)$$

where $\mathbf{F}^v$ is the viscous force, $\mathbf{B}$ the magnetic field and $\mathbf{j}$ the current density. $\mathbf{\Omega}$ is the rotation vector of the mantle frame, supposed constant. The equation of state and the Poisson equations are:

$$\rho = \rho(P, T, \Xi), \qquad \nabla^2\Psi = 4\pi G\rho, \qquad (2.4a,b)$$

G being the gravitational constant, and the heat and mass transport equations are given by

$$\rho T\frac{DS}{Dt} = \nabla\cdot(k\nabla T) + H\rho, \qquad \rho\frac{D\Xi}{Dt} = \nabla\cdot(k^\xi\nabla\Xi). \qquad (2.5a,b)$$

Here $k = \rho c_p\kappa$ and $k^\xi = \rho\kappa^\xi$ are the thermal conductivity and mass diffusion coefficient respectively, and any fluxes of Ξ due to gradients of T or P, the Soret effect, are ignored. κ and κ^ξ are the thermal and mass diffusivities. H denotes the internal heating, which we identify precisely in section 3 below. The electromagnetic equations are (e.g. Moffatt, 1978):

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times\mathbf{E}, \qquad \mathbf{E} = \mathbf{j}/\sigma - \mathbf{u}\times\mathbf{B}, \qquad \mu_0\mathbf{j} = \nabla\times\mathbf{B}, \qquad \nabla\cdot\mathbf{B} = 0. \qquad (2.6a-d)$$

Here $\mathbf{E}$ is the intensity of electrical field. σ and μ_0 are the electrical conductivity and magnetic permeability respectively.

We now insert (2.2) into (2.3a) to obtain

$$\bar{\rho}\frac{D\mathbf{u}}{Dt} = -\nabla p - \bar{\rho}\nabla\psi - \rho'\nabla\bar{\Psi} - 2\bar{\rho}\mathbf{\Omega}\times\mathbf{u} + \mathbf{j}\times\mathbf{B} + \mathbf{F}^v, \quad \nabla^2\psi = 4\pi G\rho'. \qquad (2.7a,b)$$

Here we have used (2.1a) to remove the zero order reference state hydrostatic balance terms, and we ignore some terms which are small because the convection is driven by very small departures from the hydrostatic reference state (2.1). The quantity ρ' can be written in three parts:

$$\rho' = s\Big(\frac{\partial\bar{\rho}}{\partial\bar{S}}\Big)_{p,\xi} + p\Big(\frac{\partial\bar{\rho}}{\partial\bar{P}}\Big)_{S,\xi} + \xi\Big(\frac{\partial\bar{\rho}}{\partial\bar{\xi}}\Big)_{p,S} = -\frac{\bar{\rho}\alpha\bar{T}}{c_p}s - \frac{p}{g\bar{\rho}}\frac{d\bar{\rho}}{dr} - \bar{\rho}\alpha^\xi\xi. \qquad (2.8)$$

Here $\alpha^\xi = -\rho^{-1}(\partial\rho/\partial\xi)_{p,S}$ is the adiabatic compositional expansion coefficient. We can now write the momentum equation in a form close to that of BR95:

$$\bar{\rho}\frac{D\mathbf{u}}{Dt} = -\bar{\rho}\nabla\psi - \bar{\rho}\nabla\frac{p}{\bar{\rho}} + \mathbf{1}_r g\bar{\rho}\Big(\frac{\alpha\bar{T}}{c_p}s + \alpha^\xi\xi\Big) - 2\bar{\rho}\mathbf{\Omega}\times\mathbf{u} + \mathbf{j}\times\mathbf{B} + \mathbf{F}^v. \qquad (2.9)$$

We need to relate the entropy fluctuation s to the temperature fluctuation ϑ, so we note that the change in entropy is given by

$$dS = \left(\frac{\partial S}{\partial T}\right)_{p,\xi} dT + \left(\frac{\partial S}{\partial P}\right)_{T,\xi} dP + \left(\frac{\partial S}{\partial \Xi}\right)_{p,T} d\Xi = \frac{c_p}{T} dT - \frac{\alpha}{\rho} dP + \frac{h^\xi}{T} d\Xi, \tag{2.10}$$

see e.g. BR95, equation (D6). Here h^ξ is the heat of reaction. The entropy fluctuation s is therefore

$$s = \frac{c_p}{\bar{T}}\vartheta - \frac{\alpha}{\bar{\rho}} p + \frac{h^\xi}{\bar{T}}\xi. \tag{2.11}$$

The pressure force, thermal buoyancy force and compositional buoyancy force are all believed to be comparable in the Earth's core convection. Compositional and thermal driving are believed to contribute approximately equally, so the two buoyancy sources must be of the same order of magnitude, and in a convecting system the pressure fluctuations are comparable to buoyancy forces, so

$$p/d \sim g\bar{\rho}\alpha\bar{T}s/c_p \sim g\bar{\rho}\alpha\vartheta \sim g\bar{\rho}\alpha^\xi\xi \sim \bar{\rho}\psi/d, \tag{2.12}$$

using (2.11) to relate the typical entropy fluctuation to the typical temperature fluctuation. The estimate for ψ can also be derived from (2.7b), which gives $\psi/gd \sim \rho'/\bar{\rho}$. The second term of equation (2.8) for the density fluctuation, can now be compared with the first term

$$\frac{p}{g\bar{\rho}}\frac{d\bar{\rho}}{dr} = -\frac{pD}{g\gamma d} \sim -\frac{\bar{\rho}\alpha\vartheta D}{\gamma}, \tag{2.13}$$

so the pressure fluctuation contribution to the density perturbation in (2.8) is of the same order as the other two terms, so our estimate for ρ' is

$$\rho' \sim \bar{\rho}\alpha\vartheta. \tag{2.14}$$

In the anelastic approximation the term $\partial\rho'/\partial t$ is ignored in (2.3b). If t_{conv} is the timescale of convection, the velocity $\mathbf{u}$ is $O(d/t_{conv})$, so the term $\partial\rho'/\partial t$ is of order $\alpha\vartheta \sim 10^{-8}$ smaller than $\nabla\cdot\rho\mathbf{u}$. This approximation is also equivalent to assuming the typical time for sound waves to cross the core is very short compared to t_{conv}. So we have

$$\nabla\cdot\bar{\rho}\mathbf{u} = \bar{\rho}\left[\nabla\cdot\mathbf{u} - \frac{u_r}{d}\frac{D}{\gamma}\right] = 0. \tag{2.15}$$

Equations (2.9) and (2.15), together with (2.4-2.6), constitute the anelastic equations for the Earth's core, as given in BR95. The key physical assumptions that underpin these equations are (i) the core is close to adiabatically stratified and uniformly mixed, so that the temperature, pressure and density fluctuations are all small compared to their adiabatic values, and (ii) the $\partial\rho'/\partial t$ term in the continuity equation is negligible. If there is a strongly stably stratified layer in the outer core, assumption (i) would be invalid, and these equations would need modification. It would still be true that the thermodynamic fluctuations would be small compared to the reference state, but if the reference state is not adiabatic it is for example no longer possible to write the pressure fluctuation as a gradient of $p/\bar{\rho}$ in (2.9).

The anelastic liquid approximation

The entropy fluctuation equation (2.11) can be greatly simplified if we make the anelastic liquid assumption $\alpha\bar{T} << 1$. This was used by Jarvis and McKenzie (1980) for mantle convection models, and has been used extensively by subsequent workers in that field (see e.g. Schubert *et al.* 2001), but appears not to have been adopted in core convection studies. Since $\alpha \sim 10^{-5}$ K^{-1} and $\bar{T} \sim 5,000$ K, this assumption is reasonable for the core. It is important to note that this approximation is not valid for gases, and so would be inappropriate for a solar dynamo model, where $\alpha\bar{T}$ would be $O(1)$. We need some thermodynamic relations, see e.g. BR95, equations (D18, D22, D24 and E6),

$$\alpha_i^\xi - \alpha^\xi = \frac{\alpha h^\xi}{c_p}, \quad \alpha^\xi c_v = \alpha_i^\xi c_p, \quad c_p - c_v = \gamma\alpha\bar{T}c_v \quad \text{giving} \quad \alpha_i^\xi = -\frac{h^\xi}{\gamma\bar{T}c_p}. \tag{2.16a-d}$$

Here $\alpha_i^\xi = -\rho^{-1}(\partial\rho/\partial\xi)_{p,T}$ is the isothermal compositional expansion coefficient and c_v the specific heat at constant volume. We can now estimate the second and third terms of the right-hand-side of (2.11) using (2.12),

$$\frac{\alpha p}{\bar{\rho}} \sim g\alpha^2 d\vartheta = D(\alpha\bar{T})\Big(\frac{c_p\vartheta}{\bar{T}}\Big), \quad \frac{h^\xi\xi}{\bar{T}} \sim \frac{h^\xi\alpha\vartheta}{\alpha_i^\xi\bar{T}} = \frac{h^\xi\alpha}{\alpha_i^\xi c_p}\Big(\frac{c_p\vartheta}{\bar{T}}\Big) = -\gamma(\alpha\bar{T})\Big(\frac{c_p\vartheta}{\bar{T}}\Big), \tag{2.17a,b}$$

so even if D is $O(1)$ the pressure fluctuation and composition fluctuation terms in (2.11) are negligible because $\alpha\bar{T} << 1$. For an anelastic liquid, therefore,

$$s = \frac{c_p}{\bar{T}}\vartheta. \tag{2.18}$$

On substituting (2.2) and (2.18) into the heat transport equation (2.5a), we obtain

$$\bar{\rho}\bar{T}\frac{Ds}{Dt} = \bar{\rho}c_p\frac{D\vartheta}{Dt} - \frac{\bar{\rho}c_p\vartheta}{\bar{T}}(\mathbf{u}\cdot\nabla)\bar{T} + \bar{\rho}\vartheta(\mathbf{u}\cdot\nabla)c_p = \nabla\cdot k\nabla\vartheta + \nabla\cdot k\nabla\bar{T} - \bar{\rho}\bar{T}\dot{\bar{S}} + \bar{\rho}H. \tag{2.19}$$

The outer core is gradually losing heat energy and becoming less dense as the inner core grows, though overall the core may well be increasing in density. This happens on a billion year time scale, so we can assume the inner core and outer core radii do not change on dynamo timescales (a thousand year timescale). However, the cooling is an important energy source for the dynamo, and so in (2.19) $d\bar{S}/dt = \dot{\bar{S}}$ and $\dot{\bar{\xi}}$ cannot be neglected, but variations in time of the reference state density and temperature (when multiplied by the convecting quantities) can be neglected. The term $\bar{\rho}\bar{T}\dot{\bar{S}}$ can be thought of as equivalent to a heat source, so we write $H' = H - \bar{T}\dot{\bar{S}}$. We have retained the term $\nabla\cdot k\nabla\vartheta$, which seems inconsistent, since it is apparently small compared to $\nabla\cdot k\nabla\bar{T}$. This is discussed below when we consider turbulent diffusion, but for now we merely remark that because this term involves second derivatives, it could be important in thin boundary layers. Using (2.1b) and (2.1e), (2.19) can be written as a temperature equation

$$\frac{\partial\vartheta}{\partial t} + \mathbf{u}\cdot\nabla\vartheta = \frac{1}{\bar{\rho}c_p}\nabla\cdot\bar{\rho}c_p\kappa\nabla\vartheta + \frac{1}{\bar{\rho}c_p}\nabla\cdot\bar{\rho}c_p\kappa\nabla\bar{T} + \frac{H'}{c_p} - \frac{g\alpha}{c_p}u_r\vartheta - \frac{u_r\vartheta}{c_p}\frac{dc_p}{dr}. \tag{2.20}$$

We also have the equation for the composition obtained from inserting (2.2) into (2.5b),

$$\frac{\partial\xi}{\partial t} + \mathbf{u}\cdot\nabla\xi = \frac{1}{\bar{\rho}}\nabla\cdot\bar{\rho}\kappa^\xi\nabla\xi - \dot{\bar{\xi}}. \tag{2.21}$$

Finally, we replace the entropy perturbation in the equation of motion (2.9) using (2.18) to get

$$\bar{\rho}\frac{D\mathbf{u}}{Dt} = -\bar{\rho}\nabla\psi - \bar{\rho}\nabla\frac{p}{\bar{\rho}} + \mathbf{1}_r g\bar{\rho}(\alpha\vartheta + \alpha^\xi\xi) - 2\bar{\rho}\mathbf{\Omega}\times\mathbf{u} + \mathbf{j}\times\mathbf{B} + \mathbf{F}^v. \tag{2.22}$$

Equations (2.15), (2.20), (2.21) and (2.22) are the basic system for an anelastic compressible liquid. The essential difference between the anelastic liquid and the full anelastic system of BR95 and Glatzmaier and Roberts (1996) is that in the liquid the entropy perturbation s given by (2.11) is dominated by the temperature perturbation, and the other terms are ignored. This allows s to be eliminated from the system. How useful this is depends somewhat on the numerical method adopted to solve the equations. In some methods, the pressure perturbation is found explicitly, for example by taking the divergence of the equation of motion. There is then not much advantage in exploiting $\alpha T << 1$, because if p, s and ξ are computed explicitly, ϑ and ρ' can be found trivially from (2.11) and (2.8). If, however, p is eliminated by dividing (2.9) by $\bar{\rho}$ and taking the curl, s and ξ will be the only thermodynamic variables computed. If the heat transport equation and boundary conditions can be formulated entirely in terms of s and ξ rather than ϑ, this is not a problem, but otherwise it will be necessary to do extra computing to find p and hence the other thermodynamic variables. If the anelastic liquid approximation is adopted, the heat and composition equations only involve ϑ and ξ, and if as is likely the boundary conditions can be formulated in terms of these variables, computing the remaining thermodynamic variables is unnecessary.

The anelastic equations are significantly different from the Boussinesq approximation. Apart from the obvious difference that the fluid is no longer incompressible, (2.15), the main differences are in the temperature equation. The second term on the right-hand-side of equation (2.20), $(1/\bar{\rho}c_p)\nabla\cdot\bar{\rho}c_p\kappa\nabla\bar{T}$, which we call the heat flux deficit, arises from the divergence of the heat flux conducted down the adiabat. This must be balanced by the convection and the other heat source terms. This heat flux deficit is an important source term for convection in the Earth's core; as we see in section 7 below, it is possible to adopt a modified BA which takes this term into account. The third term, H'/c_p, includes any radioactive heating, any secular cooling of the core, and as we see in the next section, the ohmic heating and the heat produced by viscous dissipation. The fourth term, $(g\alpha/c_p)u_r\vartheta$, has been called the adiabatic cooling term (see e.g. Schubert *et al.* 2001) as it is independent of the diffusive processes. It is linked to the work done by the buoyancy forces (see below) so Archimedean cooling might be a better name for this term than the rather bland 'adiabatic cooling'. The fifth and final term on the right-hand-side of (2.20) will be small, because c_p only varies by about 2% across the outer core (Stacey (1992), RJC03). It is important to identify these source terms in the temperature equation correctly, because numerical models indicate that there are significant differences between models with uniform heating (e.g. Grote and Busse, 2001) and prescribed heat flux at the ICB (e.g. Christensen *et al.* 1999), see also Busse *et al.* (2003).

The diffusion term $(1/\bar{\rho}c_p)\nabla\cdot k\nabla\vartheta$ in (2.20) plays a rather different role from the corresponding ohmic and viscous diffusion terms. The magnetic and kinetic energy of the fluid is dissipated by these diffusion processes whereas thermal diffusion merely disperses the temperature without affecting the overall energy.

Turbulent diffusion

As noted above, the value of k is very small in (2.20), so that molecular diffusion of ϑ cannot be simulated numerically directly, and the same applies to the diffusion terms in (2.21) and (2.22). It is therefore necessary to resort to subgrid scale modelling by introducing turbulent diffusivities. For a compressible treatment of this problem see e.g. Speziale et al. (1988) and Xie & Toomre, (1993). We illustrate with the temperature equation (2.20) which we can write in the form

$$\bar{\rho}c_p\frac{\partial\vartheta}{\partial t} + \nabla\cdot(\bar{\rho}c_p\mathbf{u}\vartheta) = \nabla\cdot\bar{\rho}c_p\kappa\nabla\vartheta + \nabla\cdot\bar{\rho}c_p\kappa\nabla\bar{T} - \bar{\rho}g\alpha u_r\vartheta + \bar{\rho}H'. \tag{2.23}$$

The superadiabatic temperature and the flow velocity can be divided into two parts: the resolved large-scale values and the unresolved turbulent values which have zero mean:

$$\mathbf{u} = \breve{\mathbf{u}} + \mathbf{u}^T, \qquad \vartheta = \breve{\vartheta} + \vartheta^T, \qquad \langle\vartheta^T\rangle = 0, \qquad \langle\mathbf{u}^T\rangle = 0, \tag{2.24}$$

where $\langle\rangle$ denotes averaging over the small grid-size scale. Now the averaged (2.23) can be written:

$$\bar{\rho}c_p\frac{\partial\breve{\vartheta}}{\partial t} + \nabla\cdot(\bar{\rho}c_p\breve{\mathbf{u}}\,\breve{\vartheta}) = \nabla\cdot\bar{\rho}c_p\Big(\kappa\nabla\breve{\vartheta} - \langle\mathbf{u}^T\vartheta^T\rangle\Big) - \bar{\rho}g\alpha(\breve{u}_r\breve{\vartheta} + \langle u_r^T\vartheta^T\rangle) + \nabla\cdot\bar{\rho}c_p\kappa\nabla\bar{T} + \bar{\rho}H'. \tag{2.25}$$

As usual, we describe the term $-\langle \mathbf{u}^T \vartheta^T \rangle$ as a turbulent diffusion which can be written as

$$-\langle \mathbf{u}^T \vartheta^T \rangle = \kappa_T \nabla \breve{\vartheta} \tag{2.26}$$

where the turbulent diffusivity is assumed to be much larger than the molecular one: $\kappa_T \gg \kappa$. We also rewrite the term $g\alpha\bar{\rho}\langle u_r^T \vartheta^T \rangle$ by analogy with the previous term in the form

$$\bar{\rho} g \alpha \langle u_r^T \vartheta^T \rangle = \bar{\rho} c_p \frac{D}{d} \langle u_r^T \vartheta^T \rangle = -\bar{\rho} c_p D \frac{\kappa_T}{d} \frac{\partial \breve{\vartheta}}{\partial r}. \tag{2.27}$$

Then dividing (2.25) by $\bar{\rho} c_p$ we obtain the equation for the superadiabatic temperature averaged over the turbulent length scale:

$$\frac{\partial \vartheta}{\partial t} + (\mathbf{u} \cdot \nabla)\vartheta = \frac{1}{\bar{\rho} c_p} \nabla \cdot \bar{\rho} c_p (\kappa_T \nabla \vartheta + \kappa \nabla \bar{T}) - u_r \vartheta \Big(\frac{D}{d} + \frac{1}{c_p} \frac{dc_p}{dr} \Big) + \frac{H'}{c_p} + D \frac{\kappa_T}{d} \frac{\partial \vartheta}{\partial r}. \tag{2.28}$$

For simplicity we have omitted the $\breve{}$ over $\mathbf{u}$ and ϑ.

Equation (2.28) differs from (2.20) not only by the presence of the turbulent thermal diffusivity instead of the molecular one, but also by one additional term (the last on the R.H.S.) which can be treated as turbulent cooling due to the work on unresolved space scales. Similarly (2.21) becomes

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = \frac{1}{\bar{\rho}} \nabla \cdot \bar{\rho} \kappa_T^{\xi} \nabla \xi - \dot{\xi}, \tag{2.29}$$

where κ_T^{ξ} is the turbulent compositional diffusion coefficient, defined by $-\langle \mathbf{u}^T \xi^T \rangle = \kappa_T^{\xi} \nabla \xi$.

The turbulent diffusion of momentum gives rise to a term $\langle \partial_i (\bar{\rho} u_i^T u_j^T) \rangle = \partial_i \sigma_{ij}$, where σ_{ij} is the turbulent stress tensor. In general, assuming this depends linearly on gradients of $\mathbf{u}$, we could have $\sigma_{ij} = \bar{\rho} \nu_{ijmn}^T \partial_m u_n$, taking into account the anisotropy induced by the Coriolis and Lorentz force, see BR95. Obtaining a consistent prescription for the 36 independent components of ν_{ijmn}^T is a formidable task, and here we make the traditional simplification that the turbulent stress tensor has the same form as the molecular stress tensor with no bulk viscosity term,

$$\sigma_{ij} = \bar{\rho} \nu_T (\partial_j u_i + \partial_i u_j - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u}), \tag{2.30}$$

so the viscous force become

$$F_i^v = \partial_j \sigma_{ij} = \partial_j [\bar{\rho} \nu_T (\partial_j u_i + \partial_i u_j) - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u}], \tag{2.31}$$

and this completes the specification of (2.22). In practice, simulations suggest (Rotvig and Jones 2002) that viscosity is only important in the boundary layers adjacent to the ICB and the CMB. Since the radial velocity is small there, $\nabla \cdot \mathbf{u} \approx 0$ and so the usual Boussinesq form of the viscous force,

$$F_i^v \approx \bar{\rho} \nu_T \nabla^2 u_i, \tag{2.32}$$

is unlikely to give significant error.

For energetic consistency, we need an equation for the turbulent part of the velocity. The Reynolds stress term here can be quite complicated in general, but in the Earth's core we do not expect the Reynolds stress of the small-scale flow to be energetically important so we assume the turbulent diffusivity has the same form as the molecular difusivity so that

$$\bar{\rho} \frac{\partial \mathbf{u}^T}{\partial t} + \bar{\rho} \mathbf{u} \cdot \nabla \mathbf{u}^T = -\bar{\rho} \nabla \psi^T - \bar{\rho} \nabla \frac{p^T}{\bar{\rho}} + \mathbf{1}_r \bar{\rho} g (\alpha \vartheta^T + \alpha_i^{\xi} \xi^T) - 2\bar{\rho} \mathbf{\Omega} \times \mathbf{u}^T + \mathbf{F}^{vT}. \tag{2.33}$$

Note that since the molecular magnetic diffusion in the Earth's core is large enough for numerical schemes to handle, we do not require a turbulent form of the induction equation.

Let us see how these compressible equations convert into Boussinesq equations when the compressibility parameter $D = g\alpha d/c_p \sim \Delta\bar{T}/\bar{T} \to 0$ and the reference state density and specific heat is assumed constant. An advantage of the anelastic liquid approximation over the full anelastic model is that because the heat transport equation reduces to a temperature equation a fairly direct comparison can be made. In the temperature equation (2.28), the terms describing the Archimedean and the turbulent cooling obviously vanish as $D \to 0$, and the diffusion of temperature reduces to the standard Boussinesq form $\kappa_T\nabla^2\vartheta$. The heat flux deficit and the internal heating form inhomogeneous terms, which are allowed in the Boussinesq equations, though the usually considered cases are uniform internal heating or no internal heating, the energy source then being a flux input at the bottom boundary. Sarson *et al.* 1997 considered a nonuniform heat source, stronger nearer the ICB, to model the spatial inhomogeneity of the heat flux deficit term mentioned in the introduction. The compositional equation is usually not solved for in Boussinesq models, on the grounds that the forcing will be similar to that of the thermal convection. We note that this depends on the boundary conditions for ϑ and ξ being identical. What about the momentum and the continuity equations? BA supposes that the only source of density variations and consequently the buoyancy force are the variations of temperature and composition. The momentum equation (2.22) includes an additional buoyancy, a term proportional to the pressure: $\bar{\rho}\nabla(p/\bar{\rho}) = \nabla p - \mathbf{1}_r p(\bar{\rho}^{-1}d\bar{\rho}/dr)$. Since $(\bar{\rho}^{-1}d\bar{\rho}/dr) = D/\gamma d$, this term vanishes in the limit $D \to 0$, so the momentum equation reduces to the standard Boussinesq form. The continuity equation can be written in the form:

$$\nabla\cdot\mathbf{u}\bar{\rho} = \bar{\rho}\nabla\cdot\mathbf{u} + u_r\frac{d\bar{\rho}}{dr} = \bar{\rho}\Big[\nabla\cdot\mathbf{u} + u_r\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dr}\Big] = \bar{\rho}\Big[\nabla\cdot\mathbf{u} - \frac{u_r}{d}\frac{D}{\gamma}\Big] = 0 \tag{2.34}$$

In the limit $D \to 0$ the terms proportional to D vanish and the compressible approach transforms into the Boussinesq approximation.

3. THE ENERGY EQUATION

We now demonstrate that our anelastic liquid equations are energetically self-consistent, and also define precisely what must be included in the internal heating term H introduced in (2.5a). We define the internal, gravitational and magnetic energy densities

$$\varepsilon^{\vartheta} = \bar{\rho}c_p\vartheta, \qquad \varepsilon^{g} = \bar{\rho}\mu\xi, \qquad \varepsilon^{m} = \frac{\mathbf{B}^2}{2\mu_0}, \tag{3.1}$$

where

$$\mu = -\int_{r_{icb}}^{r} g\alpha^{\xi}\,dr \tag{3.2}$$

is the chemical potential. Note that we can add an arbitrary constant to this potential without affecting any physical quantities. Note also that α^{ξ} in this formula is the adiabatic compositional expansion coefficient $-(1/\rho)(\partial\rho/\partial\xi)_{p,s}$, but from (2.16) $\alpha^{\xi} = \alpha^{\xi}_i(1+\gamma\alpha\bar{T})$ so the difference between this and the isothermal expansion coefficient α^{ξ}_i is of $O(\alpha\bar{T})$ and is therefore negligible in our anelastic liquid approximation. From now on we drop the subscript i from the isothermal coefficient. Multiplying (2.28) by $\bar{\rho}c_p$, the heat energy equation is

$$\frac{\partial\varepsilon^{\vartheta}}{\partial t} = -\nabla\cdot\Big[\mathbf{u}\varepsilon^{\vartheta} - \bar{\rho}c_p\Big(\kappa_T\nabla\vartheta + \mathbf{1}_r\kappa\frac{\partial\bar{T}}{\partial r}\Big)\Big] + \bar{\rho}H' - \bar{\rho}g\alpha\vartheta u_r + \bar{\rho}g\alpha\kappa_T\frac{\partial\vartheta}{\partial r}. \tag{3.3}$$

Multiply (2.29) by $\bar{\rho}\mu$ to obtain

$$\bar{\rho}\mu\frac{\partial\xi}{\partial t} + \nabla\cdot(\bar{\rho}\mu\mathbf{u}\xi) - \bar{\rho}\xi\mathbf{u}\cdot\nabla\mu = \nabla\cdot(\bar{\rho}\mu\kappa^{\xi}_T\nabla\xi) - \bar{\rho}\kappa^{\xi}_T\nabla\mu\cdot\nabla\xi - \bar{\rho}\mu\dot{\bar{\xi}}, \tag{3.4}$$

which using (3.2) can be written

$$\frac{\partial\varepsilon^{g}}{\partial t} = -\nabla\cdot\Big[\mathbf{u}\varepsilon^{g} - \bar{\rho}\mu\kappa^{\xi}_T\nabla\xi\Big] - \bar{\rho}\mu\dot{\bar{\xi}} - \bar{\rho}g\alpha^{\xi}u_r\xi + \bar{\rho}g\alpha^{\xi}\kappa^{\xi}_T\frac{\partial\xi}{\partial r}. \tag{3.5}$$

The magnetic energy balance obtained from the induction equation has the form

$$\frac{\partial \varepsilon^m}{\partial t} = -\nabla\cdot \frac{\mathbf{B}\times\mathbf{E}}{\mu_0} - \mathbf{u}\cdot\mathbf{j}\times\mathbf{B} - Q_j, \qquad Q_j = \eta\mu_0 \mathbf{j}^2, \tag{3.6}$$

Q_j being the ohmic dissipation per unit volume. A minor complication arises in the kinetic energy equation, because the subgrid scale turbulent part of the kinetic energy should be included, so

$$\varepsilon^k = \frac{1}{2}\bar\rho(\mathbf{u}^2 + (\mathbf{u}^T)^2). \tag{3.7}$$

To derive the equation for the kinetic energy we take the scalar product of the averaged momentum equation (2.22), (2.31) with $\mathbf{u}$ and add the averaged scalar product of the subgrid scale momentum equation (2.33) with $\mathbf{u}^T$. As before, we write the average as

$$\langle \bar\rho g \mathbf{1}_r \cdot \mathbf{u}^T(\alpha\vartheta^T + \alpha^\xi \xi^T)\rangle = -\bar\rho g(\alpha\kappa_T \frac{\partial\vartheta}{\partial r} + \alpha^\xi \kappa_T^\xi \frac{\partial\xi}{\partial r}). \tag{3.8}$$

We obtain

$$\frac{\partial\varepsilon^k}{\partial t} = -\nabla\cdot\mathbf{I}^k + \bar\rho g u_r(\alpha\vartheta + \alpha^\xi\xi) - \bar\rho g(\alpha\kappa_T\frac{\partial\vartheta}{\partial r} + \alpha^\xi\kappa_T^\xi\frac{\partial\xi}{\partial r}) + \mathbf{u}\cdot\mathbf{j}\times\mathbf{B} - Q_v, \quad Q_v = \sigma_{ij}\partial_j u_i, \tag{3.9}$$

Q_v being the viscous dissipation, where the kinetic energy flux

$$\mathbf{I}^k = \mathbf{u}\varepsilon^k + p\mathbf{u} + \bar\rho\psi\mathbf{u} + \langle \mathbf{u}^T p^T\rangle + \bar\rho\langle \mathbf{u}^T\psi^T\rangle - u_j\sigma_{ij}. \tag{3.10}$$

To obtain the equation for the whole energy we sum the equations (3.3), (3.5), (3.6) and (3.9):

$$\frac{\partial}{\partial t}(\varepsilon^\vartheta + \varepsilon^g + \varepsilon^m + \varepsilon^k) = -\nabla\cdot(\mathbf{I}^\vartheta + \mathbf{I}^g + \mathbf{I}^m + \mathbf{I}^k) - \bar\rho\mu\dot{\bar\xi} - \bar\rho\bar T\dot{\bar S} + \bar\rho H - Q_v - Q_j, \tag{3.11}$$

where the fluxes are given by (3.10) and

$$\mathbf{I}^\vartheta = \mathbf{u}\varepsilon^\vartheta - \bar\rho c_p(\kappa_T\nabla\vartheta + \mathbf{1}_r\kappa\frac{\partial\bar T}{\partial r}), \quad \mathbf{I}^g = \mathbf{u}\varepsilon^g - \bar\rho\mu\kappa_T^\xi\nabla\xi, \quad \mathbf{I}^m = \frac{\mathbf{B}\times\mathbf{E}}{\mu_0}. \tag{3.12a,b,c}$$

Note that we have restored $H = H' + \bar T\dot{\bar S}$ in (3.11). In the absence of any secular change and with no radioactive heating, energy conservation tells us that the only way a small volume of fluid can change its energy is through the flux terms in (3.11). Ohmic heating and viscous heating are internal processes, so they cannot be in the overall energy balance, so if the radioactive heating is H^R

$$\bar\rho H = Q_v + Q_j + \bar\rho H^R. \tag{3.13}$$

This tells us that Q_v and Q_j are included in the heat transport equation (2.19) as part of $\bar\rho H$, and so H' in (2.20) is now fully defined. We also denote the sum of the secular cooling and any radioactive heating as H_{int}, so

$$H_{int} = H^R - \bar T\dot{\bar S}, \quad \bar\rho H' = Q_v + Q_j + \bar\rho H^R - \bar\rho\bar T\dot{\bar S}. \tag{3.14a,b}$$

Equations (3.11) and (3.12) form the energy transport equation which is not the same as the heat transport equation (3.3). In the BA, the magnetic and kinetic energies are small compared to the heat energy, so that then (3.3) can be considered to be the energy equation, but our compressible approximation makes it clear that the heat energy is only a part of the total energy budget. Note that all the work done by the Archimedean terms and the turbulent cooling and turbulent mixing has cancelled out.

4. BOUNDARY CONDITIONS AND GLOBAL ENERGY BALANCE

Boundary conditions

The mechanical boundary conditions usually applied in dynamo calculations are that there is no flow through the ICB and CMB, and there is no slip at these boundaries. The rate of change of position of the ICB is so slow it can be ignored on the dynamo time scale. If the inner core is rotating relative to the mantle (whose rotation defines $\mathbf{\Omega}$), then if the core flow is divided in radial and tangential parts, $\mathbf{u} = u_r \mathbf{1}_r + \mathbf{u}_t$,

$$u_r = 0, \quad \mathbf{u}_t(\mathbf{r}) = \mathbf{\Omega}_{IC} \times \mathbf{r}, \quad \text{at} \quad r = r_{icb}, \quad \text{and} \quad \mathbf{u} = 0, \quad \text{at} \quad r = r_{cmb}, \tag{4.1}$$

where $\mathbf{\Omega}_{IC}$ is the rotation rate of the inner core relative to the mantle. It has been argued (Kuang and Bloxham, 1997) that since the Ekman layer is so thin (less than 100 metres even with a strongly turbulent boundary layer) it should be ignored, and a stress-free condition applied instead.

The electromagnetic boundary conditions at the ICB are that $\mathbf{B}$ is continuous across the ICB and the continuity of tangential $\mathbf{E}$ implies that $\eta \mathbf{j}_t$ is also continuous. Denoting the jump in a quantity across the boundary by square brackets [ ],

$$[\mathbf{B}] = 0, \quad [j_r] = 0, \quad [\eta \mathbf{j}_t] = 0, \quad \text{at} \quad r = r_{icb} \tag{4.2a}$$

where r denotes the radial component and t the tangential component. Since the electrical conductivity of the mantle is much less than that of the core, there is negligible current in the mantle, and the core field has to match to a potential field at the CMB. This implies

$$[\mathbf{B}] = 0, \quad [j_r] = 0. \tag{4.2b}$$

This assumption does, of course, rule out any electromagnetic coupling between core and mantle, which might be important for length of day models.

We also need a condition on the temperature and the composition at both the ICB and CMB. We denote the average over a spherical surface S_r at radius r by $<>_r$, so, for example, the averaged radial component of the part of the heat flux due to convection is

$$I^{conv}(r) = \frac{1}{S_r} \int_{S_r} \left(\bar{\rho} c_p u_r \vartheta - \bar{\rho} c_p \kappa_T \frac{d\vartheta}{dr} \right) dS = < \bar{\rho} c_p u_r \vartheta - \bar{\rho} c_p \kappa_T \frac{d\vartheta}{dr} >_r \ \mathrm{Wm}^{-2}. \tag{4.3}$$

The radial component of the averaged heat flux can be divided into two parts according to (3.12a),

$$I(r) = I^a(r) + I^{conv}(r), \quad I^a = -k \frac{d\bar{T}}{dr}. \tag{4.4}$$

At the ICB, there are two contributions to the input heat flux into the outer core, the heat conducted out of the inner core, I_{icb}, and the latent heat, I_l, released as a consequence of inner core freezing. Following the notation of BR95, if $r_{icb}(\theta, \phi, t)$ is the location of the inner core boundary, the rate of release of latent heat energy there is $h_l \rho \dot{r}_{icb}$ per unit area, and the rate of release of light material is $\rho \Delta \xi \dot{r}_{icb}$ also per unit area. Here h_l is the latent heat energy per unit mass and $\Delta \xi$ is the jump in ξ across the ICB; numerical values are discussed in section 6 below.

The location of the ICB, $r = r_{icb}$, is determined by the melting point of iron which is a function of p and ξ. BR95 show that its rate of change at the ICB is governed by $\dot{S} = -c_p \Delta_2 \dot{r}_{icb} / r_{icb}$, Δ_2 being a parameter dependent on the melting point (see BR95, or RJC03 equation (3.16)). Using (2.18), the rate of release of latent heat at the ICB is

$$I_l = -\frac{h_l \bar{\rho} r_{icb}}{\Delta_2 c_p \bar{T}} (\bar{T} \dot{\bar{S}} + c_p \dot{\vartheta}) = -\bar{\rho} c_p \kappa_T \frac{d\vartheta}{dr} - I_{icb} + I^a(r_{icb}) \approx -\bar{\rho} c_p \kappa_T \frac{d\vartheta}{dr} \quad \text{on} \quad r = r_{icb} \tag{4.5}$$

which is the thermal boundary condition on the ICB at $r = r_{icb}$. In general, there will be a jump in the temperature gradient at the ICB, and the thermal conductivity may be different across the ICB also. Nevertheless, Buffett et al. (1992) and Labrosse et al. (2001) point out that only small errors arise in assuming the inner core is adiabatically stratified, so we assume the conducted heat flux is the same on both sides of the ICB, and we take $I_{icb} \approx I^a(r_{icb})$, simplifying (4.5). Note that $\dot{S} = \dot{\bar{S}} + c_p\dot{\vartheta}/\bar{T}$ and both terms are of comparable magnitude, as the very large $\bar{S}$ changes only very slowly on the core evolution timescale, while the much smaller $c_p\vartheta/\bar{T}$ changes more rapidly on the dynamo timescale. Similarly, the rate of release of light material at the ICB is given by

$$-\frac{\Delta\xi\bar{\rho}r_{icb}}{\Delta_2 c_p\bar{T}}(\bar{T}\dot{\bar{S}} + c_p\dot{\vartheta}) = -\bar{\rho}\kappa_T\frac{d\xi}{dr} \quad \text{on} \quad r = r_{icb} \tag{4.6}$$

which is the boundary condition on ξ at the ICB. Although both (4.5) and (4.6) contain time-derivatives of ϑ, Glatzmaier and Roberts (1996) found them to be numerically stable.

On the CMB, the heat flux is determined by mantle convection. It may be possible to get information about the distribution of the heat flux through the CMB, I_{cmb}, from seismic measurements see e.g. Olson, 2003, but in principle it should be determined by mantle convection calculations. Assuming that this total heat flux is greater than the heat conducted down the adiabat at the CMB, if we adopt as a simple model a uniform distribution of CMB heat flux

$$\bar{\rho}(r_{cmb})c_p\kappa_T\frac{\partial\vartheta}{\partial r} = -I_{cmb} + I^a(r_{cmb}) = -I^{conv}_{cmb}, \quad \text{on} \quad r = r_{cmb}. \tag{4.7}$$

Actually, there has been a substantial amount of work on models with a non-uniform heat flux across the core, recently reviewed by Olson (2003), and this idea can easily be applied in our framework by an appropriate modification of (4.7). Since there can be no flux of light material across the CMB,

$$\frac{\partial\xi}{\partial r} = 0 \quad \text{on} \quad r = r_{cmb}. \tag{4.8}$$

This completes the anelastic boundary conditions. Possible choices for the magnetic boundary conditions are either to assume an insulating inner core and mantle, or to give a more accurate representation of the field inside the core by solving the magnetic diffusion problem in the solid inner core.

Global energy balance

It is entirely possible that the total heat flux generated by all processes inside the CMB does not exceed the heat flux $I^a(r_{cmb})$ conducted down the adiabat. If this occurs, the core is thermally stably stratified near the CMB, see Gubbins et al. (1982), and for discussions with compositional effects included see Labrosse et al. (1997) and Lister & Buffett (1998).

In this case, there is no thermal convection in the outer part of the core. Conditions then depend critically on whether there is compositional convection. If even a small amount of compositional convection is present, the core may be stirred and may remain close to adiabatic and close to uniform composition. If there is no compositional convection, then there is a value of r, at r_* say, above which there is no convection. In this case, the basic assumption that the core is adiabatically stratified throughout breaks down, and for $r > r_*$ the core temperature is governed by the heat conduction law, $\nabla\cdot\bar{\rho}c_p\kappa\nabla\bar{T} = 0$. There is a superadiabatic heat flux carried by the convection for $r < r_*$, which falls to zero at $r = r_*$, where all the heat flux can be carried by conduction. In this case, the dynamo is entirely driven in the lower part of the convection zone. Note that this is a very different scenario from that which is common in most Boussinesq models, (though see Sarson *et al.*, 1997), where convection occurs throughout the core.

In the case where there is compositional convection, the stirring ensures that the basic reference state is still governed by (2.1), but there is still a value of $r = r_*$ at which the total heat flux equals that conducted down the adiabat. Now we still have to supply heat to maintain the adiabatic gradient for $r > r_*$ and this is done by the turbulent convection induced by the compositional convection transporting heat backwards from the mantle to the core (Loper, 1978). The Nusselt number, defined as the heat transported by convection plus conduction divided by the heat conducted down the adiabat is therefore less than unity in the region $r > r_*$. It is not at present possible to say whether this occurs or not from purely thermodynamic arguments. Current estimates suggest that it does not, but if these estimates overestimate the latent heat release by even a small amount, (or underestimate the thermal conductivity slightly) then an inverted ocean could exist. Whether or not the 'inverted ocean' (Braginsky, 1999) exists, it is clear that the thermal heat flux, and hence the work done by convection, will be substantially greater near the ICB than near the CMB unless there is a large contribution from radioactivity.

Since the boundary conditions (4.5)-(4.8) and the equations (2.28) and (2.29) involve the gradients of ϑ and ξ, an arbitrary constant can be added without affecting the solution, which is taken up in the momentum equation by adding to the pressure gradient. Since the reference temperature is determined by the melting point of iron at the ICB, the average of ϑ should be taken to vanish over the ICB. Similarly, the average of ξ over the ICB can also be taken as zero; it then follows from (2.29) that the time-averaged average of ξ over any spherical surface vanishes. This then removes the arbitrariness in the definitions of ϑ and ξ.

We now consider the volume integral of our energy equations over the whole fluid outer core. Since the radial component of velocity vanishes at the ICB and CMB (we neglect the work done by the very slow movement of the ICB and CMB due to secular cooling and growth of the inner core, as this is not useful work for the dynamo). In the overall energy balance the only fluxes that remain are the flux of heat through the ICB and the CMB, the Poynting flux $-\mathbf{I}^m$ and the exchange of mechanical energy between the outer and inner core. The Poynting flux controls the passage of magnetic energy in and out of the outer core. If there is significant ohmic dissipation in the inner core, this must be driven by the Poynting flux through the ICB. Likewise, any ohmic dissipation in the mantle must be similarly supplied. Since the volume of the inner core is small, we neglect the ohmic dissipation there, and similarly only a small part of the mantle is likely to conduct significantly so we ignore these Poynting fluxes here. The potential part of the field in the mantle will have no long term average Poynting flux. Similarly, the exchange of kinetic energy between the outer and inner core through viscous and magnetic stresses at the ICB has no long term average.

The flux of light material at the ICB due to core freezing gives no contribution to the overall energy balance because μ is zero at the ICB. With our definition (3.2) the gravitational energy source appears as the $\bar\rho\mu\dot{\bar\xi}$ term in the overall energy balance. If μ were chosen to have a different constant, the gravitational energy would be partitioned differently between the ICB boundary term and the $\bar\rho\mu\dot{\bar\xi}$ term, but the sum is of course unchanged. The latent heat I_l integrated over the ICB is $\mathcal{Q}_l$. This must be added to the rather small heat flux coming from the solid inner core, $\mathcal{Q}_{icb} = 4\pi r_{icb}^2 I_{icb}$. The global energy budget is therefore

$$\mathcal{Q}_{cmb} - \mathcal{Q}_{icb} - \mathcal{Q}_l = \mathcal{Q}_{int} + \mathcal{Q}_g, \quad \text{where} \quad \mathcal{Q}_{int} = \int_{V_{oc}} (H^R - \bar\rho\bar T\dot{\bar S})\, dv, \quad \mathcal{Q}_g = \int_{V_{oc}} -\bar\rho\mu\dot{\bar\xi}\, dv, \tag{4.9}$$

V_{oc} being the volume of the outer core. Here $\mathcal{Q}_{cmb}$ is the heat flux I_{cmb} integrated over the surface of the CMB. Note that $\mu < 0$, $\dot{\bar\xi} > 0$, so the release of gravitational energy from the gradual differentiation of the outer core, $\mathcal{Q}_g$, adds to the heat flux coming out of the CMB, as does the cooling of the core, giving $\dot{\bar S} < 0$. The energy budget over the whole outer core for the individual components is also of interest. The magnetic energy budget is, see (3.6),

$$\int_{V_{oc}} -\mathbf{u}\cdot\mathbf{j}\times\mathbf{B}\, dv = \int_{V_{oc}} Q_j\, dv = \mathcal{Q}_j \tag{4.10}$$

where as before we are time-averaging the Poynting flux and the rate of change of magnetic energy $\partial\varepsilon^m/\partial t$ to zero. This says that the work done by Lorentz forces all ends up as ohmic dissipation. The compositional energy budget is, see (3.5),

$$\mathcal{Q}_g = \int_{V_{oc}} -\bar\rho\mu\dot{\bar\xi}\, dv = \int_{V_{oc}} \left(\bar\rho g\alpha^\xi u_r\xi + \bar\rho g\alpha^\xi\kappa^\xi\frac{d\xi}{dr}\right) dv \tag{4.11}$$

which balances the rate of release of gravitational energy with the rate of working of the compositional buoyancy forces, and a (small) term due to subgrid scale forces. The kinetic energy budget (again neglecting the terms which time-average to zero) is, using (3.9) and (4.11),

$$\int_{V_{oc}} \left[\bar\rho g u_r(\alpha\vartheta + \alpha^\xi \xi) - \bar\rho g\left(\alpha\kappa_T \frac{\partial \vartheta}{\partial r} + \alpha^\xi \kappa_T^\xi \frac{\partial \xi}{\partial r}\right)\right] dv = \int_{V_{oc}} (Q_j + Q_v)\, dv = \mathcal{Q}_j + \mathcal{Q}_v \qquad (4.12)$$

which balances the rate of working of both buoyancy forces with the ohmic and viscous dissipation, with two small terms representing the working of the subgrid scale forces. The heat balance equation completes the set, from (3.3)

$$\mathcal{Q}_{cmb} - \mathcal{Q}_{icb} - \mathcal{Q}_l = \int_{V_{oc}} \left(\bar\rho H^R + Q_j + Q_v - \bar\rho \bar T \dot{\bar S} - \bar\rho g \alpha \vartheta u_r + \bar\rho g \alpha \kappa_T \frac{d\vartheta}{dr}\right) dv \qquad (4.13)$$

The last term is again work done by subgrid scale forces and will be small in a well-resolved calculation. Note that the dissipations do enter the heat energy budget, even though they don't enter the overall energy budget. The rate of working of the thermal buoyancy has to come out of the heat energy budget and so is a negative term here. The balance between the work done by the Archimedean forces and the dissipation is a global balance, not a local balance, because there are substantial flux terms transporting energy around. The dissipation could possibly be occurring in different places in the core from where the buoyant convection is occurring. If we ignore the subgrid scale terms in (4.11) and (4.12) and define $\mathcal{Q}_a$ to be the rate of working of the thermal convection,

$$\mathcal{Q}_g = \int_{V_{oc}} \bar\rho g \alpha^\xi u_r \xi\, dv, \quad \mathcal{Q}_a = \int_{V_{oc}} \bar\rho g u_r \alpha \theta\, dv, \quad \mathcal{Q}_g + \mathcal{Q}_a = \mathcal{Q}_j + \mathcal{Q}_v \qquad (4.14a, b, c)$$

The 'useful work' done by the dynamo is $\mathcal{Q}_j$, since this is the rate of working of the Lorentz force, which in a steady state equals the rate of ohmic dissipation by (4.10). We expect $\mathcal{Q}_v \ll \mathcal{Q}_j$ (see section 5 below) so the rate at which useful work is done is close to $\mathcal{Q}_g + \mathcal{Q}_a$. The amount of gravitational energy released can be estimated directly from the reference state in terms of the rate of growth of the inner core using (4.11), see section 7 below.

It is also useful to integrate the heat energy equation (3.3), using (3.14b), over a spherical shell of volume V_r extending from the ICB to a spherical surface S_r at radius r. We again neglect the small terms arising from the subgrid scale forces to get

$$\int_{S_r} \bar\rho c_p \left(u_r \vartheta - \kappa_T \frac{\partial \vartheta}{\partial r}\right) ds = 4\pi r^2 I_{conv} = F_{conv} = \mathcal{Q}_{icb} + \mathcal{Q}_l + \int_{S_r} \bar\rho c_p \kappa \frac{\partial \bar T}{\partial r}\, ds$$
$$+ \int_{V_r} \left(\bar\rho H^R + Q_v + Q_j - \bar\rho \bar T \dot{\bar S} - \bar\rho g \alpha u_r \vartheta\right) dv. \qquad (4.15)$$

F_{conv} is the convective part of the total heat flux passing through a sphere of radius r. We first consider a volume V_r with r very close to the ICB. Since $u_r \to 0$ as $r \to r_{icb}$ the first term of the convective heat flux will be small there. Both the conducted heat flux down the adiabat and $\mathcal{Q}_{icb}$ are small there compared to the large latent heat component $\mathcal{Q}_l$. The volume integrals also being small there, we have

$$\int_{S_r} \bar\rho c_p \kappa_T \frac{\partial \vartheta}{\partial r}\, ds \approx \mathcal{Q}_l, \qquad (4.16)$$

so the heat flux in this thermal boundary layer is carried mainly by turbulent diffusion. If κ_T is small, this boundary layer will be thin (see section 5 below). Outside the boundary layer, the flux carried by turbulent diffusion will be small (see section 5 below). The dissipation and Archimedean cooling in (4.15) are relatively small, $O(D)$, and so the convective heat flux is effectively constrained by the reference state to be

$$\int_{S_r} \bar\rho c_p u_r \vartheta\, ds \approx \mathcal{Q}_{icb} + \mathcal{Q}_l + \int_{S_r} \bar\rho c_p \kappa \frac{\partial \bar T}{\partial r}\, ds + \int_{V_r} \left(\bar\rho H^R - \bar\rho \bar T \dot{\bar S}\right) dv + O(D). \qquad (4.17)$$

This shows that the convective heat flux is specified at every point in the core, (4.17) being a nonlinear constraint on the convection. The third term on the R.H.S. of (4.17) is the heat flux deficit term, and is increasingly negative as r increases. It is therefore possible that the whole of the R.H.S. of (4.17) is negative at large enough r, and if so this defines r_* again, the point at which the R.H.S is zero and either convection stops or the Nusselt number becomes less than unity at $r > r_*$.

Dissipation integrals

From (4.14) we see that the dissipation can be expressed in terms of the rate of working of the two buoyancy forces $\mathcal{Q}_a$ and $\mathcal{Q}_g$. Since most of the dissipation in the core is believed to be ohmic, the sum $\mathcal{Q}_a + \mathcal{Q}_g$ is the rate of supply of energy to the dynamo, a quantity of some interest.

The value of $\mathcal{Q}_a$ is usually estimated by considering the entropy flux (see e.g. RJC03, BR95, Gubbins 1977), but see also Lister, 2003, where the equivalence of various different methods is shown. However, in our anelastic liquid approximation the simplified form of the entropy (2.18) allows us to obtain $\mathcal{Q}_a$ directly, by multipying (4.17) by $g\alpha/c_p$ and integrating over the core and using the definition (4.14)

$$\mathcal{Q}_a = \int_{r_{icb}}^{r_{cmb}} \frac{g\alpha}{c_p} \left[\int_{S_r} \bar{\rho} c_p u_r \vartheta \, ds \right] dr \approx \int_{r_{icb}}^{r_{cmb}} \frac{g\alpha}{c_p} \left[\mathcal{Q}_{icb} + \mathcal{Q}_l \right] dr$$

$$+ \int_{r_{icb}}^{r_{cmb}} 4\pi r^2 g\alpha\bar{\rho}\kappa \frac{\partial \bar{T}}{\partial r} \, dr + \int_{r_{icb}}^{r_{cmb}} \frac{g\alpha}{c_p} \left[\int_{r_{icb}}^{r} 4\pi r'^2 (\bar{\rho} H^R - \bar{\rho}\bar{T}\dot{\bar{S}}) \, dr' \right] dr + O(D^2) \tag{4.18}$$

We use (4.18) in section 6 below to estimate the contribution made by thermal convection to the energy input of the dynamo.

Note that in the anelastic liquid aproximation, $\mathcal{Q}_a$, $\mathcal{Q}_g$, and the viscous and ohmic dissipations are all $O(D)$. The factors $g\alpha/c_p$ mean that the three integrals on the right-hand-side of (4.18) are all $O(D)$, and the integrals involving products of the dissipation and $g\alpha/c_p$ are second order in D and therefore negligible in this approximation. We must bear in mind that in the Earth's core D is not that small, and the $O(D^2)$ terms could amount to 10% of $\mathcal{Q}_a$. Nevertheless, in the standard entropy method an estimate has to be made of the temperature at which the dissipation occurs (T_D in RJC03) which is somewhat arbitrary, whereas (4.18) has the advantage that it does not require knowledge of where the dissipation occurs. $\dot{\bar{S}}$ is estimated using the global energy balance

$$-\int_{r_{icb}}^{r_{cmb}} 4\pi r^2 \bar{\rho}\bar{T}\dot{\bar{S}} \, dr = \mathcal{Q}_{cmb} - \mathcal{Q}_{icb} - \mathcal{Q}_l - \mathcal{Q}_G - \mathcal{Q}_R \tag{4.19}$$

and is very uncertainly known.

5. AMPLITUDES OF THE SOLUTION AND NON-DIMENSIONAL EQUATIONS

In section 2 we obtained the equations for the anelastic compressible liquid model of the core. Neglecting the small subgrid scale terms, we consider for simplicity the particular case where c_p, η, $\kappa_T^\xi = \kappa_T$, ν_T and κ are assumed constant. These quantities do not vary much across the outer core according to current estimates, though of course their variation across the core can be included in our framework if desired. Our liquid anelastic equations (2.22), (2.15), (2.28), (2.29)and (2.6) are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(\frac{p}{\bar{\rho}} + \psi) + \mathbf{1}_r g(\alpha\vartheta + \alpha^\xi \xi) - 2\mathbf{\Omega}\times\mathbf{u} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\bar{\rho}\mu_0} + \frac{\mathbf{F}^v}{\bar{\rho}}, \qquad \nabla\cdot\bar{\rho}\mathbf{u} = 0, \tag{5.1,2}$$

$$\frac{\partial \vartheta}{\partial t} + (\mathbf{u}\cdot\nabla)\vartheta + \frac{D}{d}\Big(\vartheta u_r - \kappa_T \frac{\partial \vartheta}{\partial r}\Big) = \frac{1}{\bar{\rho}} \nabla\cdot\bar{\rho}(\kappa_T \nabla\vartheta + \kappa\nabla\bar{T}) + \frac{H_{int}}{c_p} + \eta\frac{[\nabla\times\mathbf{B}]^2}{\mu_0 c_p \bar{\rho}} + \frac{\sigma_{ij}\partial_j u_i}{\bar{\rho} c_p} \tag{5.3}$$

$$\frac{\partial \xi}{\partial t} + \mathbf{u}\cdot\nabla\xi = \frac{1}{\bar\rho}\nabla\cdot\bar\rho\kappa_T\nabla\xi - \dot{\bar\xi}, \tag{5.4}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}. \tag{5.5}$$

To non-dimensionalise these equations we use $d = r_{cmb} - r_{icb}$ as the unit of length, η/d^2 as the unit of time, and η/d as the unit of flow velocity. For our typical values of ρ, g etc. we take the values from the PREM model, (Dziewonski and Anderson, 1981), and denote the values at the midpoint of the layer, $r_0 = (r_{cmb} + r_{icb})/2$ by a subscript 0, see section 6 below. The numerical values used are listed in tables 1 and 2, and details of how these estimates are obtained is given in section 7 below. The compressibility parameter is defined as $D_0 = g_0\alpha_0 d/c_p$. We use $\Delta\bar T = \bar T(r_{icb}) - \bar T(r_{cmb})$ as a unit of adiabatic temperature. The variation of quantities such as g across the layer are nondimensionalised by setting

$$\hat\rho(r) = \frac{\bar\rho}{\rho_0},\quad \hat g(r) = \frac{g}{g_0},\quad \hat H^c(r) = -\frac{\bar T\dot{\bar S}d^2}{c_p\kappa\Delta\bar T},\quad \hat H^R(r) = -\frac{H^R(r)\bar T\dot{\bar S}d^2}{c_p\kappa\Delta\bar T},\quad \hat H(r) = \hat H^c(r) + \hat H^R(r),$$

$$\hat\alpha(r) = \frac{\alpha}{\alpha_0},\quad \hat D(r) = \hat g\hat\alpha = D/D_0,\quad \hat T(r) = (\bar T(r) - \bar T(r_{cmb}))/\Delta\bar T, \tag{5.6}$$

so that the hatted variables contain the variation across the core, and are all dimensionless variables of order unity.

Table 1

d	$r_{cmb} - r_{icb}$	2.26×10^6 m	ρ_{icb}	12166 kg m^{-3}
t	d^2/η	8.1×10^4 years	ρ_{cmb}	9903 kg m^{-3}
$\mathbf{u}$	η/d	9×10^{-7}m s^{-1}	ρ_0	11340 kg m^{-3}
$\mathbf{B}$	$\sqrt{\Omega\bar\rho_0\mu_0\eta}$	1.44×10^{-3} T	g_0	7.805 m s^{-2}
ϑ_0	$(\kappa/\eta)\Delta\bar T$	3.5×10^{-3} K	Ω	7.29×10^{-5} s^{-1}
$\Delta\bar T$	$\bar T_{icb} - \bar T_{cmb}$	1400 K	α_0	1.38×10^{-5} K^{-1}
r_{cmb}		3.48×10^6 m	c_p	860 J kg^{-1} K^{-1}
r_{icb}		1.22×10^6 m	γ	1.458
κ		5×10^{-6} m^2 s^{-1}	h_l	10^6 J kg^{-1}
κ_T	(turbulent)	$\sim\eta = 2$ m^2 s^{-1}	α^ξ	1
η		2 m^2 s^{-1}	$\Delta\xi$	0.02
$\bar T_{cmb}$		4000 K	$\dot{\bar S}$	-4×10^{-16} W kg^{-1} K^{-1}
$\bar T_{icb}$		5400 K	$\dot{\bar\xi}$	3.5×10^{-20} s^{-1}
$\bar T_0$		4875 K	I^{conv}_{cmb}	9×10^{-3} W m^{-2}
$d\bar T/dr\vert_{icb}$		3.23×10^{-4} K m^{-1}	Δ_2	0.04

Parameters used for the non-dimensionalisation of the equations

Table 2

q, q_T	κ/η, κ_T/η	2.5×10^{-6}, ~ 1
P_m	ν_T/η (turbulent)	~ 1
E	$\nu_T/\Omega d^2$ (turbulent)	5.4×10^{-9}
D_0	$d\alpha_0 g_0/c_p$	0.28
Ra	$g_0\alpha\Delta\bar{T}d/\Omega\kappa$	9.3×10^{14}
A	$q^2 Ra$	5840
$d\hat{\bar{\xi}}/d\hat{t}$	$\alpha^\xi\dot{\bar{\xi}}d^2/\kappa\alpha_0\Delta\bar{T}$	1.86
$\hat{L}$	$h_l r_{icb}/\Delta_2 q_T c_p \bar{T}_{icb} d$	2.91
$\hat{C}$	$\alpha^\xi\Delta\xi r_{icb}/\Delta_2 q_T\alpha_0\bar{T}_{icb}d$	3.63
$\hat{H}^c_{icb}$	$-\bar{T}_{icb}\dot{\bar{S}}d^2/c_p\kappa\Delta\bar{T}$	1.83
$\hat{M}$	$I^{conv}_{cmb}d/q_T\kappa\rho_{cmb}c_p\Delta\bar{T}$	0.35
$\hat{\mathcal{Q}}_{icb}$	$-r^2_{icb}\rho_{icb}(d\bar{T}/dr\vert_{icb})/\rho_0 d\Delta\bar{T}$	0.164
$\hat{\mathcal{Q}}_l$	$-r^3_{icb}\rho_{icb}h_l\dot{\bar{S}}/\rho_0 d\Delta_2 c_p^2\kappa\Delta\bar{T}$	1.672

Values of non-dimensional parameters

The unit of magnetic field is $B_0 = \sqrt{\Omega\rho_0\mu_0\eta}$. The superadiabatic temperature fluctuation is much smaller than $\Delta\bar{T}$ and so it is not appropriate to use $\Delta\bar{T}$ here. We note that the driving terms in (5.3) are the internal heating and the heat flux deficit, so balancing $\partial\vartheta/\partial t \sim \kappa\Delta\bar{T}/d^2$, we find the natural unit of superadiabatic temperature is $\kappa\Delta\bar{T}/\eta$. The dimensionless variables are then

$$\hat{r} = \frac{r}{d}, \quad \hat{t} = \frac{\eta t}{d^2}, \quad \hat{\mathbf{u}} = \frac{\mathbf{u}d}{\eta}, \quad \hat{p} = \frac{p}{\Omega\rho_0\eta}, \quad \hat{\psi} = \frac{\psi}{\Omega\rho_0\eta}, \quad \hat{\vartheta} = \frac{\vartheta\eta}{\kappa\Delta\bar{T}}, \quad \hat{\mathbf{B}} = \frac{\mathbf{B}}{\sqrt{\Omega\bar{\rho}_0\mu_0\eta}},$$

$$\hat{\xi} = \frac{\alpha^\xi\eta}{\kappa\alpha_0\Delta\bar{T}}\xi, \quad \frac{d\hat{\bar{\xi}}}{dt} = \frac{\alpha^\xi d^2}{\kappa\alpha_0\Delta\bar{T}}\dot{\bar{\xi}}, \quad \hat{\mathbf{F}}_v = \mathbf{F}_v\frac{d^3}{\nu_T\eta}, \quad \hat{\sigma}_{ij} = \sigma_{ij}\frac{d^2}{\nu_T\bar{\rho}_0\eta}. \tag{5.7}$$

Typical values of $\hat{H}$ and $d\hat{\bar{\xi}}/d\hat{t}$, assuming no radioactivity, are given table 2. They are both slightly greater than unity and have similar values, showing that the compositional driving is indeed comparable to the thermal driving.

Then the non-dimensional equations can be written in the form:

$$E_m\Big(\frac{\partial\hat{\mathbf{u}}}{\partial\hat{t}} + (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}\Big) = -\nabla(\frac{\hat{p}}{\hat{\rho}} + \hat{\psi}) - 2\mathbf{1}_z\times\hat{\mathbf{u}} + \frac{(\nabla\times\hat{\mathbf{B}})\times\hat{\mathbf{B}}}{\hat{\rho}} + A\hat{g}\hat{\alpha}(\hat{\vartheta} + \hat{\xi})\mathbf{1}_r + E\frac{\hat{\mathbf{F}}_v}{\hat{\rho}}, \tag{5.8}$$

$$\frac{\partial\hat{\vartheta}}{\partial\hat{t}} + (\hat{\mathbf{u}}\cdot\nabla)\hat{\vartheta} = q_T\frac{1}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\hat{\vartheta} + \frac{1}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\hat{T} + \hat{H} + \frac{D_0}{A}\frac{\hat{D}}{\hat{\rho}}[\nabla\times\hat{\mathbf{B}}]^2$$
$$+\frac{ED_0}{A}\frac{\hat{D}}{\hat{\rho}}\hat{\sigma}_{ij}\partial_j\hat{u}_i - D_0\hat{D}(\hat{\vartheta}\hat{u}_r - q_T\frac{\partial\hat{\vartheta}}{\partial\hat{r}}), \tag{5.9}$$

$$\frac{\partial\hat{\xi}}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\nabla\hat{\xi} = q_T\frac{1}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\hat{\xi} - \frac{d\hat{\bar{\xi}}}{d\hat{t}}, \tag{5.10}$$

$$\frac{\partial \hat{\mathbf{B}}}{\partial \hat{t}} = \nabla^2 \hat{\mathbf{B}} + \nabla \times (\hat{\mathbf{u}} \times \hat{\mathbf{B}}). \tag{5.11}$$

Here the following dimensionless parameters are used: the Ekman number, E, the magnetic Ekman number, E_m, the Archimedean number A, the turbulent Roberts number, q_T, the magnetic Prandtl number P_m and the dissipation number D_0:

$$E = \frac{\nu_T}{\Omega d^2}, \quad E_m = \frac{\eta}{\Omega d^2} = \frac{E}{P_m}, \; q_T = \frac{\kappa_T}{\eta}, \; A = \frac{g_0 \alpha_0 \Delta \bar{T} d \kappa}{\Omega \eta^2}, \quad P_m = \frac{\nu_T}{\eta}, \quad D_0 = \frac{g_0 \alpha_0 d}{c_p}. \tag{5.12}$$

We use the Archimedean number

$$A = \frac{g_0 \alpha_0 \Delta \bar{T} d \kappa}{\Omega \eta^2} = q^2 Ra \approx 6 \times 10^3, \quad \text{where} \qquad q = \frac{\kappa}{\eta}, \qquad Ra = \frac{g_0 \alpha_0 \Delta \bar{T} d}{\Omega \kappa} \tag{5.13}$$

instead of the more conventional Rayleigh number because this has a 'sensible' value, whereas the conventional Rayleigh number has a very large value because q is very small (see also Kono and Roberts, 2001; Gubbins, 2001). Note that although κ plays an important role in defining our units, q does not appear in the final equations at all, only q_T, which will be much larger in the core.

The boundary conditions (4.1)-(4.2) non-dimensionalise in an obvious way, but (4.5)-(4.8) introduce new dimensionless combinations to become

$$\frac{\partial \hat{\vartheta}}{\partial \hat{r}} = \hat{L}\Big(\frac{\partial \hat{\vartheta}}{\partial \hat{t}} - \hat{H}^c_{icb}\Big), \quad \frac{\partial \hat{\xi}}{\partial \hat{r}} = \hat{C}\Big(\frac{\partial \hat{\vartheta}}{\partial \hat{t}} - \hat{H}^c_{icb}\Big) \quad \text{on} \quad \hat{r} = \hat{r}_{icb} = r_{icb}/d, \tag{5.14a,b}$$

$$\frac{\partial \hat{\vartheta}}{\partial \hat{r}} = -\hat{M}, \quad \frac{\partial \hat{\xi}}{\partial \hat{r}} = 0 \quad \text{on} \quad \hat{r} = \hat{r}_{cmb} = r_{cmb}/d, \tag{5.15a,b}$$

where

$$\hat{L} = \frac{h_l r_{icb}}{\Delta_2 q_T c_p \bar{T}_{icb} d}, \; \hat{C} = \frac{\alpha^{\xi} \Delta \xi r_{icb}}{q_T \alpha_0 \bar{T}_{icb} \Delta_2 d}, \quad \hat{M} = \frac{(I_{cmb} - I^a(r_{cmb}))d}{q_T \kappa \bar{\rho} c_p \Delta T}. \tag{5.16a,b,c}$$

Even though the Archimedean number A, unlike the Rayleigh number, is not enormous, it is still significantly larger than unity. If $\hat{\mathbf{u}}$ were of order unity, this would make the magnetic Reynolds number of order unity, which is insufficient for dynamo action. We therefore expect the typical velocity, and hence the magnetic Reynolds number, will scale as some positive power of A. We now attempt to estimate the typical magnitudes of the various terms in these equations in terms of A. This is inevitably somewhat speculative, as it depends on the nature of the convection. We define our dimensionless typical velocity, which is also the magnetic Reynolds number, and the typical temperature fluctuation as

$$\hat{V} = R_m = \left\{\frac{1}{V_{oc}} \int_{V_{oc}} \hat{\mathbf{u}}^2 \, dv\right\}^{1/2}, \; \hat{\Theta} = \left\{\frac{1}{V_{oc}} \int_{V_{oc}} \hat{\vartheta}^2 \, dv\right\}^{1/2}, \text{ where } V_{oc} = \frac{4\pi(r^3_{cmb} - r^3_{icb})}{3d^3}. \tag{5.17}$$

The Archimedean force in (5.8) is balanced by the Coriolis force in some locations and by the Lorentz force in others, and the contributions from thermal and compositional convection are similar. This is the "MAC" balance of Magnetic, Archimedean and Coriolis force in the core. If we take the curl of (5.8) to obtain a vorticity equation, the balance between Coriolis and buoyancy force takes the form $-2(\hat{\mathbf{z}} \cdot \nabla)\hat{\mathbf{u}} \sim A \nabla \times \hat{g} \hat{\alpha} (\hat{\vartheta} + \hat{\xi}) \mathbf{1}_r$ and in rapidly rotating convection where there are tall thin columns aligned parallel to $\mathbf{\Omega}$ which are of length d but width $\ell << d$, this gives an estimate of $\hat{V} = (d/2\ell) A \hat{\Theta}$. However, in magnetoconvection the magnetic field expands the width of the rolls significantly (see e.g. Jones, Mussa and Worland, 2003) and so d/ℓ may not be much different from unity in the core. In high Rayleigh number, high Prandtl number Bénard convection, small plumes occur, which fill only a small fraction of the space available. If this is the case in the core, then the balance of Coriolis and buoyancy force will occur only in the plumes where the typical temperature fluctuation $\hat{\Theta}_{max}$ may be much greater than $\hat{\Theta}$. It is also conceivable that the velocity may be much greater in the plumes than outside them, but for simplicity we ignore this possibility here. Then $\hat{V} = (d/2\ell)A\hat{\Theta}_{max}$, and if the plumes occupy only a filling fraction f of the core, $\hat{\Theta} = f\hat{\Theta}_{max}$, so

$$\hat{V} \sim \frac{Ad}{2f\ell}\hat{\Theta}. \tag{5.18}$$

The temperature equation (5.9) is forced by the two large terms on the right-hand side, the heat flux deficit $(1/\hat{\rho})\nabla \cdot \hat{\rho}\nabla\hat{T}$ and the internal heating $\hat{H}$, which are of order unity on our scaling, the dissipation terms being comparatively small. To see how these terms are balanced, we consider the dimensionless form of (4.15), which is the time-averaged heat flux equation, and omitting terms of order D

$$\hat{\rho}\int_{S_r} \hat{u}_r\hat{\vartheta}\, dS - \hat{\rho}\int_{S_r} q_T \frac{\partial\hat{\vartheta}}{\partial\hat{r}}\, dS = -\hat{\rho}_{icb}\int_{S_{icb}} q_T \frac{\partial\hat{\vartheta}}{\partial\hat{r}}\, dS + \hat{\rho}\int_{S_r} \frac{d\hat{T}}{d\hat{r}}\, dS - \hat{\rho}_{icb}\int_{S_{icb}} \frac{d\hat{T}}{d\hat{r}}\, dS + \int_{V_r} \hat{\rho}\hat{H}\, dv \tag{5.19}$$

where S_r is the spherical surface of radius $\hat{r}$ and V_r is the volume of the core between this surface and the ICB. The right-hand-side of (5.19) is also of order unity. Since $\hat{V}$ is the magnetic Reynolds number, the existence of a dynamo suggests that $\hat{V}/q_T$, the Peclet number, must be quite large, so the convective term is expected to dominate the diffusive term on the left-hand-side of (5.19) outside of any thermal boundary layers (note that the derivative $d/d\hat{r}$ in the diffusive term can be taken outside the integral, so this does not introduce a small length scale). We introduce the quantity

$$\Gamma(\hat{r}) = \frac{< \hat{u}_r\hat{\vartheta} >_{\hat{r}}}{\hat{V}\hat{\Theta}}, \tag{5.20}$$

which measures the correlation between the radial velocity and the temperature fluctuation on each spherical surface. With perfect correlation,

$$\frac{1}{V_{oc}}\int_{\hat{r}_{icb}}^{\hat{r}_{cmb}} 4\pi\hat{r}^2\Gamma\, d\hat{r} = 1, \tag{5.21}$$

and this integral can never exceed unity. In thermal convection, we expect the correlation between $\hat{u}_r$ and $\hat{\vartheta}$ to be high, because hot fluid rises and cold fluid sinks. Starchenko and Jones (2002) assumed the correlation $\Gamma \sim 1$ for their estimates of typical velocities in planetary dynamos. The experiments of Aubert *et al.* (2001) also suggested a high value of Γ in rapidly rotating convection, but when Coriolis and Lorentz forces are strong compared to viscous forces it is possible that Γ could be significantly less than unity, and more numerical experiments are needed to settle this issue. Taking $\hat{\Gamma}$ as an average value away from the boundaries, we obtain from (5.19)

$$\hat{\Gamma}\hat{V}\hat{\Theta} \sim 1, \tag{5.22}$$

which combined with (5.18) gives

$$\hat{V} = R_m \approx \left(\frac{Ad}{2\hat{\Gamma}f\ell}\right)^{1/2}, \quad \hat{\Theta} \approx \left(\frac{2f\ell}{\hat{\Gamma}Ad}\right)^{1/2} \tag{5.23}$$

In an extreme case, one might imagine that the correlation Γ is so small that the turbulent diffusion term takes over the main role of transporting the heat, in which case if $\kappa_T \sim O(1)$, then $\hat{\Theta} \sim O(1)$, giving

$$\hat{V} = R_m \approx \frac{Ad}{2f\ell}, \tag{5.24}$$

a rather large value of R_m. While observations of the secular variation suggest that the typical value is less than this, this estimate could be reduced if the velocity was much smaller outside of the plumes. Note that the dominance of the convective term over the diffusive term on the left-hand-side of (5.19) is required for our method of estimating the rate of working of thermal buoyancy forces, (4.18).

With $A \approx 6\times10^3$ and the factors $\hat{\Gamma}$, f and ℓ/d all unity, (5.23) gives a magnetic Reynolds number of around 50, which is somewhat on the low side, and (5.24) gives $R_m \approx 3000$, somewhat higher than most estimates. A plausible compromise is that (5.23) applies with the three factors $\hat{\Gamma}$, f and ℓ/d, which are all likely to be less than unity, multiplying together to give a value of around 0.01. We also estimate that the combination $f\ell/\hat{\Gamma}d \sim 1$ and then we have

$$R_m \approx 500, \quad \hat{\Theta} \approx 0.02, \quad A \approx 6\times10^3. \tag{5.25}$$

This value of R_m corresponds to a typical velocity $V \approx 4\times10^{-4}$ ms^{-1}, or 12 km/yr, which is the order of magnitude of the 'westward drift' velocity measured by the secular variation. The value of $\hat{\Theta}$ means the typical temperature fluctuation driving the thermal part of the convection is only about 10^{-4} K.

The magnetic field is even more difficult to estimate. Naively, the MAC balance in (5.8) suggests that the typical field strength $\hat{B} \sim A^{1/2}$, but this is much larger than the typical field strength found in simulations. The difficulty is that the Lorentz force acts locally and the typical current $\hat{j} \sim \hat{B}d/\mu_0\delta$, where the length scale δ is significantly less than d. Flux expulsion arguments (Galloway, Proctor and Weiss, 1978) suggest that $\delta/d \approx R_m^{-1/2}$, but this is based on kinematic calculations which ignore the effect of the Lorentz force on the flow, and so it may not be valid in the core.

From (5.9) we see that both ohmic and viscous dissipations are of order D_0, which can be loosely identified with the 'Carnot efficiency factor'. However, the viscous dissipation has the additional factor of E which suggests that it can only be significant for the small-scale components of the velocity, such as those occurring in Ekman boundary layers. Since it is likely that the total magnetic dissipation will be considerably larger than the total viscous dissipation, it has to balance the work done by the buoyancy forces, $\mathcal{Q}_a + \mathcal{Q}_g$ from (4.14). For the core, $\mathcal{Q}_a$ and $\mathcal{Q}_g$ are of the same order so we expect that $\hat{\mathbf{j}}^2$ must be of order A to achieve balance between $\mathcal{Q}_a$ and $\mathcal{Q}_j$ (see (5.9)). $\hat{\mathbf{j}}^2$ could of course be locally higher, as it is only the volume averaged ohmic dissipation that has to balance $\mathcal{Q}_a + \mathcal{Q}_g$. We may therefore expect that

$$|\hat{\mathbf{j}}\times\hat{\mathbf{B}}| \sim R_m, \quad |\hat{\mathbf{j}}|^2 \sim A, \quad |\hat{\mathbf{j}}| \sim |\hat{\mathbf{B}}|d/\delta. \tag{5.26}$$

Using (5.26), $\delta/d \approx 0.1$, a plausible estimate for the scale on which ohmic dissipation is occurring. This then gives $\hat{B} \approx 7$, corresponding to around 0.01T, about the field strength found in simulations. We should emphasise though that these are crude estimates, and in reality there will be a spectral distribution of magnetic field energy over the various wavelengths, and the dissipation must be found by integrating over the entire spectrum (see RJC03 for details).

Based on these estimates, we can estimate the space densities of the kinetic, the magnetic and the heat energies:

$$\frac{\rho_0\mathbf{u}^2}{2} \sim E_m\left(\frac{R_m^2}{2A}\right)\left(\frac{\kappa}{\eta}\alpha_0\Delta\bar{T}\right)\rho_0 g_0 d \approx 10^{-3}Jm^{-3}, \tag{5.27}$$

$$\frac{\mathbf{B}^2}{2\mu_0} \sim \hat{B}^2\rho_0\Omega\eta = \frac{\hat{B}^2}{2A}\left(\frac{\kappa}{\eta}\alpha_0\Delta\bar{T}\right)\rho_0 g_0 d \approx 40Jm^{-3}, \tag{5.28}$$

$$c_p\rho_0\vartheta \sim \hat\Theta\frac{\kappa}{\eta}\Delta\bar T c_p\rho_0 = \frac{\hat\Theta}{D_0}\Big(\frac{\kappa}{\eta}\alpha_0\Delta\bar T\Big)\rho_0 g_0 d \approx 7\times 10^2 Jm^{-3}. \tag{5.29}$$

All these energies are very much smaller than $\rho_0 g_0 d = 2\times 10^{11} Jm^{-3}$ which is of the order of the gravitational potential difference between the upper and the bottom boundaries. This is because $(\kappa/\eta)\alpha\Delta\bar T$ is a small parameter. The magnetic energy is much larger than the kinetic energy, $E_m << 1$, and the magnetic energy is of order D_0 compared to the thermal energy, and so in the Boussinesq limit $D_0 \to 0$ both kinetic and magnetic energies become small compared to the thermal energy. Therefore the ohmic and viscous heating terms disappear from the temperature equation in the Boussinesq limit.

6. EARTH CORE MODELS BASED ON PREM

Our analysis of the geodynamo equations has shown that there are some key quantities, such as the heat flux deficit. Here we give some numerical estimates for them based on the Preliminary Reference Earth Model, PREM, (Dziewonski and Anderson, 1981). RJC03 also evaluated the model using the ak135 data (Kennett *et al.*, 1995), but the differences were insignificant compared to uncertainties in the thermodynamic variables. Polynomial interpolation formulae for g and ρ are given in PREM. The coefficient of expansion $\alpha = \gamma c_p/u_p^2$, where u_p is the core sound speed, and u_p is also given as an interpolation formula in PREM. (2.1b) then gives for $\bar T$,

$$\bar T = T_{icb}\exp\Big\{-\int_{r_{icb}}^{r} \gamma g/u_p^2\, dr\Big\}, \tag{6.1}$$

which enables us to evaluate $\bar T$ as another interpolated polynomial. The value of γ in the core is uncertain. RJC03 gave a formula based on Laio *et al.* (2000) which leads to a value of γ ranging from about 1.15 at the ICB to 1.3 at the CMB. However, *ab initio* quantum calculations (Vočadlo *et al.* 2003) suggest a fairly constant value near 1.5. This higher value increases the temperature variation across the core. We adopted a 1400 K drop across the core with an anchor point of 5,400 K at the ICB, which gives a value of $\gamma = 1.458$. We took $c_p = 860$ J kg^{-1} K^{-1}, $\eta = 2$ m^2 s^{-1} and $\kappa = 5\times 10^{-6}$ m^2 s^{-1}, consistent with the Wiedemann-Franz law (see BR95).

The corresponding values of D, ρ, α, g and $\bar T$ at the midpoint $(r_{cmb} + r_{icb})/2$ are given in table 1. The corresponding dimensionless parameters are given in table 2. The variation of these quantities with $\hat r$ is shown in figure 1. Note that the dissipation number D varies strongly across the core, rising to $0.28\times 1.91 = 0.54$ near the CMB, while the density only has a 20% variation across the core. The variation of α across the core, almost always ignored in existing dynamo models, is actually quite substantial. The dimensionless heat flux deficit, $\nabla\cdot\hat\rho\nabla\hat T/\hat\rho$ is shown in figure 2. It does not vary much across the core. The second internal heat source term in (5.9) is $\hat H^c$, the contribution from core cooling. From (5.6), this is simply proportional to $\bar T$, so its form can be seen in figure 1. The constant of proportionality depends on the heat flux at the CMB and is very uncertain. With our values of the thermal conductivity and temperature structure, the adiabatic heat flux at the CMB is 0.041 W m^{-2} which corresponds to a total of 6.2TW at the CMB. The corresponding figure at the ICB is only 0.3TW, which assuming the thermal conductivity does not change significantly across the ICB means that $\mathcal{Q}_{icb} = 0.3$TW also.

For our standard model we assume that the cooling rate is $\dot{\bar S} \approx -4\times 10^{-16}$ W kg^{-1} K^{-1}. Since $\dot{\bar S} = c_p\dot{\bar T}_{cmb}/\bar T_{cmb}$ this corresponds to the CMB cooling at the rate of 1 K per 17 million years. In (4.19), $\mathcal{Q}_l$ and $\mathcal{Q}_g$ are controlled by $\dot{\bar S}$ through the time-averaged parts of (4.5) and (4.6), i.e. omitting the $\dot\vartheta$ terms,

$$\dot r_{icb} = -\frac{\dot{\bar S} r_{icb}}{c_p\Delta_2},\quad \dot{\bar\xi} = -\frac{4\pi r_{icb}^3\bar\rho_{icb}\Delta\xi\dot{\bar S}}{M_{oc}\Delta_2 c_p},\quad \mathcal{Q}_l = -\frac{4\pi r_{icb}^3 h_l\bar\rho_{icb}\dot{\bar S}}{\Delta_2 c_p},\quad \mathcal{Q}_g = -\int_{r_{icb}}^{r_{cmb}} 4\pi r^2\bar\rho\mu\dot{\bar\xi}\, dr \tag{6.2}$$

where $M_{oc} = 1.84\times10^{24}$ kg is the mass of the outer core. We evaluated the integrals in (4.19) and (6.2) numerically to obtain the total heat flux at the CMB, $\mathcal{Q}_{cmb} = 7.6$TW, corresponding to a convected heat flux of 0.009 W m^{-2}, only about 20% of the heat conducted down the adiabat. This is not far from the values assumed by Glatzmaier and Roberts (1995). The values used for the thermodynamic quantities h_l, α^{ξ}, $\Delta\xi$ and Δ_2 are listed in table 1. They are essentially as in RJC03, except that we have reverted to the Gubbins *et al.* (1979) value $h_l = 10^6$ J kg^{-1}, in view of the increase in γ. The corresponding value for the rate of growth of the inner core is then $\dot{M}_{ic}/M_{ic} \approx 3.5\times10^{-17}$ s^{-1}, slightly higher than the value adopted by Glatzmaier and Roberts (1996a). If the growth of the inner core were assumed steady, this would give the age of the inner core as 0.9Gyrs, though this is only an order of magnitude estimate, see Labrosse (2003) for a more detailed model. With these estimates, the heat flux into the outer core from the inner core is 0.3TW, the latent heat released at the ICB is 3.2TW, the rate of core cooling is 3.4TW and the gravitational energy released is 0.6TW (See BR95 and RJC03 for details). All these uncertain estimates go into the calculation of $\hat{H}^c$, but the value at the ICB is listed in table 2. If there is significant radioactivity in the core, as argued by RJC03, the estimates change again.

We also computed a low heat flux model in which the cooling rate is reduced to $\dot{\bar{S}} \approx -3\times10^{-16}$ W kg^{-1} K^{-1}. This gives a slightly older inner core. The CMB heat flux is then only 5.8TW, slightly less then the 6.2TW conducted down the adiabat, so in this model the convective heat flux is slightly negative at -0.003W m^{-2} at the CMB. In this low heat flux model, the heat flux into the outer core from the inner core is still 0.3TW, the latent heat released is 2.4TW, the rate of core cooling is 2.6TW and the gravitational energy is released at 0.5TW.

We can now evaluate $\mathcal{Q}_a$ using our two core models and the approximate form (4.18) suggested by the anelastic liquid approximation. Of the three integrals in (4.18) the first and last are positive while the second, related to the heat flux deficit, is negative. For our standard model, the first and second integrals almost cancel, and the estimate for $\mathcal{Q}_a$ is 0.6TW, similar to the estimate using the usual entropy balance arguments in RJC03, and similar to the rate of gravitational energy release. Assuming the bulk of the dissipation is ohmic rather than viscous, (4.14c) implies that the ohmic dissipation is around 1.2TW. The low heat flux model with a stably stratified layer near the CMB and 5.8TW only passing through the CMB reduces $\mathcal{Q}_a$ to 0.3TW and $\mathcal{Q}_g$ to 0.5TW, giving a lower ohmic dissipation of 0.8TW.

7. COMPARISON WITH BOUSSINESQ MODELS

What are the main differences between BA and the anelastic liquid approximation ALA concerning energy? In the BA approximation we neglect the rate of working of the expanding liquid parcel (the term before the last one in the R.H.S. of (3.3)). In consequence, only the heat energy is conserved. The flow does not supply the heat transport equation with any cooling or heating. In the ALA the whole energy is conserved and two terms of cooling, the heat flux deficit and the Archimedean cooling, appear in the heat transport equations. Also, heating arises due to converting the magnetic and the kinetic energies into heat.

The Boussinesq limit $D_0 \to 0$

In practice, only the non-spherically symmetric part of the superadiabatic temperature plays a role in the momentum equation (5.8), since any spherically symmetric part of the Archimedean force can be included in the pressure force. In consequence, we can subtract off any spherically symmetric $\check{\vartheta}(r)$ and similarly any spherically symmetric $\check{\xi}(r)$ and define new variables

$$\tilde{\vartheta} = \hat{\vartheta} - \check{\vartheta}(\hat{r}), \quad \tilde{\xi} = \hat{\xi} - \check{\xi}(\hat{r}). \tag{7.1a, b}$$

We can use this freedom to simplify the dimensionless temperature and composition equations (5.9) and (5.10). The natural choice is to set $d\check{\vartheta}(r)/d\hat{r}$ and $d\check{\xi}(r)/d\hat{r}$ to be solutions of

$$\frac{q_T}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\check{\vartheta} + \frac{1}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\hat{T} + \hat{H} = 0, \quad \frac{q_T}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\check{\xi} - \frac{d\hat{\bar{\xi}}}{d\hat{t}} = 0, \tag{7.2a, b}$$

and to choose homogeneous boundary conditions

$$\frac{\partial\tilde{\vartheta}}{\partial\hat{r}}=\hat{L}\frac{\partial\tilde{\vartheta}}{\partial\hat{t}},\quad \frac{\partial\tilde{\xi}}{\partial\hat{r}}=\hat{L}\frac{\partial\tilde{\vartheta}}{\partial\hat{t}},\quad \text{on}\quad \hat{r}=\hat{r}_{icb},\quad \frac{\partial\tilde{\vartheta}}{\partial\hat{r}}=0,\quad \frac{\partial\tilde{\xi}}{\partial\hat{r}}=0,\quad \text{on}\quad \hat{r}=\hat{r}_{cmb}. \qquad (7.3a,b,c,d)$$

Equations (7.2) are first order equations for

$$\beta^{\vartheta}(\hat{r})=\frac{\partial\check{\vartheta}}{\partial\hat{r}},\quad \beta^{\xi}(\hat{r})=\frac{\partial\check{\xi}}{\partial\hat{r}}. \qquad (7.4a,b)$$

We now get the temperature and composition equations into the form which most closely resembles the Boussinesq equations, and is therefore most suitable for comparison with existing BA numerical simulations,

$$\frac{\partial\tilde{\vartheta}}{\partial\hat{t}}+(\hat{\mathbf{u}}\cdot\nabla)\tilde{\vartheta}+\beta^{\vartheta}(\hat{r})\hat{u}_r=\frac{q_T}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\tilde{\vartheta}-D_0\hat{D}\hat{u}_r(\tilde{\vartheta}+\check{\vartheta})$$

$$+D_0\hat{D}q_T\left(\frac{\partial\tilde{\vartheta}}{\partial\hat{r}}+\frac{\partial\check{\vartheta}}{\partial\hat{r}}\right)+\frac{D_0}{A}\frac{\hat{D}}{\hat{\rho}}[\nabla\times\hat{\mathbf{B}}]^2+\frac{ED_0}{A}\frac{\hat{D}}{\hat{\rho}}\hat{\sigma}_{ij}\partial_j\hat{u}_i, \qquad (7.5)$$

which in the limit $D_0\to 0$ and $\hat{\rho}$ constant reduces to the usual Boussinesq equation but with β^{ϑ} a function of position; this was the form used by Sarson *et al.* (1997). The composition equation becomes

$$\frac{\partial\tilde{\xi}}{\partial\hat{t}}+\hat{\mathbf{u}}\cdot\nabla\tilde{\xi}+\beta^{\xi}(\hat{r})\hat{u}_r=\frac{q_T}{\hat{\rho}}\nabla\cdot\hat{\rho}\nabla\tilde{\xi} \qquad (7.6)$$

These equations are supplemented by (5.8) with the tilde variables replacing the standard variables, and the induction equation (5.11) is unchanged. The continuity equation (5.2) becomes

$$\nabla\cdot\hat{\mathbf{u}}=0. \qquad (7.7)$$

Since this Boussinesq form of the equations is homogeneous, there is a linear problem associated with it, so a critical value of A could be found at which convection onsets. However, since this is a linearization about a state which does not satisfy the zero order temperature equation, it is not clear that this linear problem is very meaningful physically.

Evaluation of β^{ϑ} and β^{ξ}

We must now evaluate the quantities $\check{\vartheta}$ and $\check{\xi}$. From (7.3a) β^{ϑ} must satisfy both (5.14a) and (5.15a). To understand why two boundary conditions appear, we note that (7.2a) is related to the heat flux equation (4.15). In the Boussinesq limit, the $\mathcal{Q}_j$, $\mathcal{Q}_v$ and $gu_r\alpha\theta$ terms are $O(D)$ and are negligible, but we retain the surface integral of $\bar{\rho}c_p\kappa d\bar{T}/dr$, which comes from the important heat flux deficit term. We also ignore radioactivity, so (4.15) becomes

$$F_{conv}=\mathcal{Q}_{icb}+\mathcal{Q}_l+4\pi r^2\bar{\rho}c_p\kappa\frac{d\bar{T}}{dr}+\int_{V_r}-\bar{\rho}\bar{T}\dot{\bar{S}}\,dv. \qquad (7.8)$$

We divide (7.8) by $F_0=4\pi\rho_0c_p\kappa d\Delta T$, the unit of surface heat flux, to obtain

$$\hat{F}_{conv}=\hat{\mathcal{Q}}_{icb}+\hat{\mathcal{Q}}_l+\hat{r}^2\hat{\rho}\frac{d\hat{T}}{d\hat{r}}+\hat{H}^c_{icb}\int_{\hat{r}_{icb}}^{\hat{r}}\frac{\hat{\rho}\bar{T}}{\bar{T}_{icb}}\hat{r}^2\,d\hat{r}, \qquad (7.9)$$

where

$$\hat{F}_{conv}=F_{conv}/F_0,\quad \hat{\mathcal{Q}}_{icb}=\mathcal{Q}_{icb}/F_0=-\hat{r}^2_{icb}\hat{\rho}_{icb}\frac{d\hat{T}}{d\hat{r}}\Big|_{icb},\quad \hat{\mathcal{Q}}_l=\mathcal{Q}_l/F_0=q_T\hat{r}^2_{icb}\hat{\rho}_{icb}\hat{H}^c_{icb}\hat{L}. \qquad (7.10)$$

Now we integrate (7.2a) from $\hat{r}_{icb}$ to $\hat{r}$ to obtain

$$-q_T \hat{r}^2 \hat{\rho} \frac{d\check{\vartheta}}{d\hat{r}} = -\hat{r}_{icb}^2 \hat{\rho}_{icb} \frac{d\hat{T}}{d\hat{r}}\Big|_{icb} - q_T \hat{r}_{icb}^2 \hat{\rho}_{icb} \frac{d\check{\vartheta}}{d\hat{r}}\Big|_{icb} + \hat{r}^2 \hat{\rho} \frac{d\hat{T}}{d\hat{r}} + \hat{H}_{icb}^c \int_{\hat{r}_{icb}}^{\hat{r}} \frac{\hat{\rho}\bar{T}}{\bar{T}_{icb}} \hat{r}^2 \, d\hat{r} \tag{7.11}$$

and using (7.10) and the boundary conditions (7.3a) and (5.14a)

$$-q_T \hat{r}^2 \hat{\rho} \frac{d\check{\vartheta}}{d\hat{r}} = \hat{\mathcal{Q}}_{icb} + \hat{\mathcal{Q}}_l + \hat{r}^2 \hat{\rho} \frac{d\hat{T}}{d\hat{r}} + \hat{H}_{icb}^c \int_{\hat{r}_{icb}}^{\hat{r}} \frac{\hat{\rho}\bar{T}}{\bar{T}_{icb}} \hat{r}^2 \, d\hat{r} \tag{7.12}$$

and comparing with equation (7.9) we have

$$-q_T \hat{r}^2 \hat{\rho} \frac{d\check{\vartheta}}{d\hat{r}} = -q_T \hat{r}^2 \hat{\rho} \beta^\vartheta = \hat{F}_{conv}. \tag{7.13}$$

This solution for β^ϑ used (5.14a), but for consistency, we should obtain the same result using boundary condition (5.15a). In the BA, the heat flux equation is

$$\hat{\mathcal{Q}}_{cmb} = \hat{\mathcal{Q}}_{icb} + \hat{\mathcal{Q}}_l + \hat{\mathcal{Q}}_{int}, \tag{7.14}$$

where in the absence of radioactivity $\hat{\mathcal{Q}}_{int}$ is the secular cooling heat flux. Comparing with (4.9) we note that $\hat{\mathcal{Q}}_g$ is ignored in the BA. Integrating (7.2a) from $\hat{r}$ to $\hat{r}_{cmb}$ we obtain

$$-q_T \hat{r}^2 \hat{\rho} \frac{d\check{\vartheta}}{d\hat{r}} = -\hat{r}_{cmb}^2 \hat{\rho}_{cmb} \frac{d\hat{T}}{d\hat{r}}\Big|_{cmb} - q_T \hat{r}_{cmb}^2 \hat{\rho}_{cmb} \frac{d\check{\vartheta}}{d\hat{r}}\Big|_{cmb} + \hat{r}^2 \hat{\rho} \frac{d\hat{T}}{d\hat{r}} + \hat{H}_{icb}^c \int_{\hat{r}_{icb}}^{\hat{r}} \frac{\hat{\rho}\bar{T}}{\bar{T}_{icb}} \hat{r}^2 \, d\hat{r} \tag{7.15}$$

and using the CMB boundary conditions (7.3c), (5.15a) with (5.16c) we obtain

$$-q_T \hat{r}^2 \hat{\rho} \frac{d\check{\vartheta}}{d\hat{r}} = \hat{\mathcal{Q}}_{cmb} - \hat{\mathcal{Q}}_{int} + \hat{r}^2 \hat{\rho} \frac{d\hat{T}}{d\hat{r}} + \hat{H}_{icb}^c \int_{\hat{r}_{icb}}^{\hat{r}} \frac{\hat{\rho}\bar{T}}{\bar{T}_{icb}} \hat{r}^2 \, d\hat{r}, \tag{7.16}$$

which is identical to (7.12) in view of (7.14). We must be careful when solving (7.2a) to recall that the value of $\hat{M}$ in table 2 has been calculated with the $\mathcal{Q}_g$ term in the CMB heat flux, i.e. using (4.9) rather than (7.14). In consequence, (5.15a) will not be satisfied exactly if the first order ODE (7.2a) for β^ϑ is integrated subject to (5.14a) unless the value of $\hat{M}$ in table 2 is adjusted by removing the $\mathcal{Q}_g$ term.

The solution of (7.2b) causes no such problems, because mass conservation ensures that the rate of injection of light material at the ICB must equal the time-averaged rate of growth of ξ,

$$-4\pi r_{icb}^2 \bar{\rho} \kappa_T \frac{d\xi}{dr}\Big|_{icb} = \int_{V_{oc}} \rho \dot{\bar{\xi}} \, dv. \tag{7.17}$$

So using (5.14b) and (7.3b) $d\check{\xi}/d\hat{r} = -\hat{C}\hat{H}_{icb}^c$ at the ICB, then given (7.3d), (5.15b) is automatically exactly satisfied in both the BA and the ALA. Integrating (7.2b) from $\hat{r}_{icb}$ to $\hat{r}$ gives

$$-q_T \hat{r}^2 \hat{\rho} \beta^\xi = -q_T \hat{r}^2 \hat{\rho} \frac{d\check{\xi}}{d\hat{r}} = \hat{F}_m = q_T \hat{r}_{icb}^2 \hat{\rho}_{icb} \hat{H}_{icb}^c \hat{C} - \frac{d\hat{\bar{\xi}}}{dt} \int_{\hat{r}_{icb}}^{\hat{r}} \hat{\rho} \hat{r}^2 \, d\hat{r}, \tag{7.18}$$

where $\hat{F}_m$ is the dimensionless mass flux of light material (in units of $4\pi\rho_0\alpha_0\kappa d\Delta T/\alpha^\xi$) through a spherical surface of radius $\hat{r}$.

Equations (7.13), (7.9) and (7.18) give a prescription for β^ϑ and β^ξ which can be evaluated in terms of PREM, or any other reference Earth model.

Interpretation of β^ϑ and β^ξ, and relation to existing Boussinesq models

We can now interpret $\breve{\vartheta}$ and its gradient β^ϑ, which from (7.5) is related to the static temperature distribution used in Boussinesq models. Let

$$T_s = \breve{\theta}\frac{\kappa\Delta T}{\eta} \tag{7.19}$$

be the dimensional static temperature distribution, and then (7.13) becomes

$$-4\pi r^2 \kappa_T c_p \bar{\rho}\frac{dT_s}{dr} = F_{conv} \tag{7.20}$$

so dT_s/dr is the temperature gradient which would conduct the convective heat flux in the presence of turbulent conductivity. T_s is very small, typically 10^{-3} K, so the temperature drop across the layer that drives the Boussinesq convection is of this order. Note that it would be a very serious error to confuse the actual temperature drop across the core ($\sim$ 1400 K) with that which must be used in the equivalent Boussinesq model ($\sim 10^{-3}$ K).This substantiates our remark in the introduction that the relationship of BA variables to actual core variables is not immediately apparent without an anelastic analysis; for example if T is the actual temperature in the core, the Boussinesq fluctuation $\tilde{\vartheta}$ is given by

$$\tilde{\vartheta} = \frac{\eta}{\kappa\Delta T}(T - \bar{T} - T_s) \tag{7.21}$$

where $\bar{T}$ is the adiabatic temperature in the core, and $T_s(r)$ is the solution of (7.20). T_s is not particularly relevant physically, but it is useful to compare existing Boussinesq models with their anelastic counterparts.

Busse *et al.*(2003) define T_s and the corresponding Rayleigh numbers

$$T_s = T_o - \frac{\beta_B d^2 \hat{r}^2}{2} + \frac{T_\Delta \hat{r}_{cmb}\hat{r}_{icb}}{\hat{r}}, \quad R_e = \frac{g_0\alpha_0 T_\Delta d^3}{\hat{r}_0 \nu_T \kappa_T}, \quad R_i = \frac{g_0 \alpha_0 \beta_B d^5}{\hat{r}_0 \nu_T \kappa_T}, \tag{7.22a, b, c}$$

where $\hat{r}_0 = (\hat{r}_{cmb} + \hat{r}_{icb})/2$. The β_B term, and the corresponding Rayleigh number R_i are due to a supposed uniform internal heating, and the T_Δ term, with R_e, would be due to a uniform heat flux passing through the core without internal heating. Formula (7.22) is therefore able to accommodate both internal heating and a uniform heat flux. As we see below, it can also approximately accommodate the heat flux deficit, but this term corresponds to a *negative* R_i. The dynamo benchmark (Christensen *et al.* 2001), and many other models, set $R_i = 0$. Busse *et al.* (2003) (and references therein) show results for $R_e = 0$ with positive R_i, uniform heating models. How does (7.22) relate to our models? Our T_s would be obtained by integrating (7.12), the constant of integration being arbitrary, and in general this will not be of the form (7.22). However, by making some crude approximations, (7.12) can be brought into rough correspondence with (7.22). From (2.1b), if g is assumed proportional to r, and α and c_p are assumed constant, then $d\bar{T}/dr$ is proportional to r (see $\hat{D}$ in figure 1 for how good this approximation is). Similarly, if we assume $\hat{\rho}$ is constant ($\hat{\rho} = 1$) and $\bar{T}$ is constant inside the integral in (7.12) then

$$\hat{r}^2\hat{\rho}\frac{d\hat{T}}{d\hat{r}} \approx -\frac{\hat{r}^3 D_0 T_0}{\hat{r}_0 \Delta T}, \qquad \int_{\hat{r}_{icb}}^{\hat{r}} \frac{\hat{\rho}\bar{T}}{\bar{T}_{icb}}\hat{r}^2\, d\hat{r} \approx \frac{1}{3}(\hat{r}^3 - \hat{r}^3_{icb}). \tag{7.23}$$

With these approximations (7.22) and (7.12) can be written

$$-\frac{dT_s}{d\hat{r}} = \beta_B d^2 \hat{r} + \frac{T_\Delta \hat{r}_{icb}\hat{r}_{cmb}}{\hat{r}^2} \approx \frac{\kappa\Delta T}{\kappa_T}\Big(\frac{1}{3}\hat{H}^c_{icb} - \frac{D_0 T_0}{\hat{r}_0 \Delta T}\Big)\hat{r} + \frac{\kappa\Delta T}{\kappa_T}(\hat{\mathcal{Q}}_{icb} + \hat{\mathcal{Q}}_l - \frac{1}{3}\hat{H}^c_{icb}\hat{r}^3_{icb})\frac{1}{\hat{r}^2} \tag{7.24}$$

which shows that both expressions now have the same $\hat{r}$ dependence. This has been achieved because the heat flux deficit has (approximately) the same form as a uniform negative heat sink. This could be anticipated from figure 2, where the heat flux deficit is shown as being approximately constant. Identifying the coefficients of $\hat{r}$ and $\hat{r}^{-2}$, we get expressions for β_B and T_Δ which can then be inserted in (7.22b) and (7.22c) to get

$$R_e = \frac{C_e A E^{-1}}{q_T^2}, \quad R_i = \frac{C_i A E^{-1}}{q_T^2}, \quad C_e = \frac{(\hat{\mathcal{Q}}_{icb} + \hat{\mathcal{Q}}_l - \frac{1}{3}\hat{H}^c_{icb}\hat{r}^3_{icb})}{\hat{r}_0 \hat{r}_{icb}\hat{r}_{cmb}}, \quad C_i = \frac{1}{\hat{r}_0}\Big(\frac{1}{3}\hat{H}^c_{icb} - \frac{D_0 T_0}{\hat{r}_0 \Delta T}\Big). \tag{7.25}$$

Using the values listed in table 2, $C_e = 2.0$ and $C_i = -0.32$. This negative value arises because the heat flux deficit term outweighs the heat source due to cooling. A model with $C_e = 0$ and C_i positive is therefore not a good model of the current Earth, but it could form the basis of a reasonable model of the Earth before the inner core formed, though the different geometry (full sphere rather than spherical shell) would have to be taken into account. In terms of the modified Rayleigh number $Ra = ER_e$ defined in (5.13), for our model $Ra \approx 1.2\times10^4$ if q_T is taken as unity. This is significantly larger than the value of $Ra = 100$ used for example in the dynamo benchmark (Christensen *et al.* 2001).

Can these estimates be modified to take compositional convection into account? From (5.8) we see that if we add the Boussinesq form of the equations for $\tilde{\vartheta}$ and $\tilde{\xi}$, (7.5) and (7.6), to form a new variable $\tilde{C} = \tilde{\vartheta} + \tilde{\xi}$ this would be possible, provided the boundary condition equations (7.3) could be expressed entirely in terms of $\tilde{C}$. Unfortunately, this is not possible because of the $\hat{L}$ terms appearing in (7.3). However, if these terms were omitted, then the compositional terms could be included in (7.22) by replacing $\hat{\mathcal{Q}}_l$ by $\hat{\mathcal{Q}}_l(\hat{L} + \hat{C})/\hat{L}$ and $\hat{H}^c_{icb}$ by $\hat{H}^c_{icb} - d\hat{\bar{\xi}}/dt$. For our table 2 model, this gives $C_e = 4.52$ and $C_i = -0.92$. The injected mass flux at the ICB has more than doubled C_e, but the stabilisation produced by the secular increase in the light component has almost exactly eliminated the driving by secular cooling, leaving the enhanced negative value of R_i due to the heat flux deficit.

In figure 3 we show the dimensionless convective heat flux $\hat{F}_{conv}$ as a function of $\hat{r}$ using (7.9). The heavy solid line is for our table 2 model, and the light solid line is for the low heat flux model. The unit of heat flux for our table 2 values is 1.94TW. Note that for the low heat flux model, beyond $\hat{r} \approx 1.4$ the convected flux is negative, so the temperature gradient must be slightly subadiabatic there. Also shown are the corresponding values for the commonly used Boussinesq uniform flux model (dashed lines corresponding to no heat flux deficit or secular cooling terms) and also the corresponding values for the uniform heating model (dotted lines). These are shown both for the standard model (thick lines) and for the low heat flux model (thinner lines). The difference between the models with the heat flux deficit included and those without is quite striking. The heavy and light dot-dashed lines are for $\hat{F}_m$, the dimensionless mass flux across a spherical surface of radius $\hat{r}$. The unit here is 3.1×10^4 kg s^{-1}. The mass flux, and hence the gradient β^ξ, both fall to zero at $\hat{r}_{cmb}$. In consequence, the thermally stable region can itself be divided into two regions; close to $\hat{r} \approx 1.4$ the destabilising ξ gradient will be larger than the stabilising thermal gradient, i.e. $|\beta^\xi| > |\beta^T|$, so we can be reasonably confident there will be strong mixing. Closer to the CMB this will no longer be the case. Convection can still occur through double diffusive convection (salt-fingering) but whether this will be sufficient to effectively mix the core and reduce the gradients to close to their adiabatic values is unclear.

8. CONCLUSIONS

Equations (5.8) to (5.11), with (5.2), constitute our anelastic liquid equations. The coefficients in these equations can be derived from the PREM model, as indicated in section 7, once γ, c_p and κ have been chosen, though the secular cooling term $\hat{H}$ requires $\dot{\bar{S}}$, or equivalently the CMB heat flux, to be specified. In principle, the CMB heat flux is determined by mantle convection. The boundary conditions (5.14) and (5.15) require further thermodynamic quantities, whose exact magnitudes are still somewhat controversial, to be specified. However, from a numerical point of view, these coefficients can be expressed as simple polynomial interpolation formulae, and will not significantly add to the computational work required for numerical solution.

Glatzmaier and Roberts (1996) did not non-dimensionalise their equations, and while BA modellers have usually used non-dimensionalised equatons, their system differs in significant ways from this anelastic liquid approach. They noted that the ICB boundary conditions (5.14), (and their Boussinesq equivalents (7.3)) contain time-derivatives, but to our knowledge these have not been included in Boussinesq models, and no systematic exploration of their effect has been made. Since the dimensionless parameter $\hat{L}$ is of order unity these time-derivatives are likely to be significant. Our non-dimensionalisation has avoided the use of poorly known quantities such such the turbulent diffusivities or the molecular diffusion as fundamental units. This has the advantage that our Archimedean number, which replaces the Rayleigh number, can be estimated much more reliably. It does still depend on quantities which are not known accurately, such as the molecular thermal conductivity and the temperature jump across the core, but it is unlikely to be wrong by much more than a factor 2, whereas the Rayleigh number, depending on the turbulent diffusion, is uncertain to many orders of magnitude.

What are the prospects for obtaining numerical solutions to our equations which are realistic for the core? The three quantities for which Boussinesq models have particular difficulties are the Ekman number, the Rayleigh number and the Roberts number q_T. The very low value of the Ekman number in the core is not attainable by any current numerical simulation, and this situation is unlikely to change in the near future. However, Rotvig and Jones (2002) have shown that in plane layer geometry, where lower values of E can be attained than is possible in spherical geometry, there is evidence that a Taylor state is being approached in which the large scale flows are becoming independent of viscosity. If this can be extended to spherical geometry as computer power increases, it may be possible to get to values of E sufficiently small to be reasonably close to the asymptotic regime.

The very large value of the Rayleigh number $Ra = g_0 \alpha_0 \Delta \bar{T} d / \Omega \kappa \sim 9 \times 10^{14}$ is not really relevant, because the typical temperature fluctuation creating the buoyancy is only $\vartheta_0 = q \Delta T$ which is much smaller. Even then, it is usual to use the turbulent value of κ in the definition of Ra if sensible results are to be obtained, so then if $q_T = 1$, Ra is reduced by another factor q down to A. Incidentally, this explains the paradox about the Nusselt number mentioned in the introduction. Although the "raw" Rayleigh number defined in (5.13) is enormous, and would lead to a huge Nusselt number, this Rayleigh number is actually irrelevant to core convection, and the much smaller Archimedean number is the relevant parameter, and this leads to the modest values of the Nusselt number found in the core. Having said that the Archimedean number $A \sim 6000$ is relatively modest, it is actually somewhat larger than the equivalent Ra values currently used (or indeed numerically possible) in simulations. However, it is not impossible that faster machines and better numerical methods could one day achieve the required Archimedean number, at least for moderate q_T.

This leaves us with the turbulent Roberts number $q_T = \kappa_T / \eta$. It has proved extremely difficult to get this significantly below 1 in numerical dynamo simulations, though very recent calculations (Christensen and Tilgner, 2004) have achieved $q_T \approx 0.15$ in some parameter regimes. This is unfortunate, because it is hard to predict the nature of convection and the dynamo in the small q_T limit. We might expect thin thermal boundary layers to develop at the ICB and CMB, and the convection in the interior could consist of thin plumes arising out of these boundary layers. How far such plumes would penetrate before mixing is very uncertain. It is also rather unlikely that the turbulent diffusion is isotropic (Braginsky and Meytlis, 1990). There are many reasons why it would be of interest to reduce the value of q_T and investigate the effect of the smaller scales of flow on the convection and the dynamo process, but this is still a formidable challenge.

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APPENDIX A

Papers dealing with the Earth's core frequently use many different symbols. Here we give a list of those used, together with their dimensions (blank for dimensionless quantities), and the equation in which they first appear. Occasionally the definition is just above, rather than just below the relevant equation. We also use subscripts, superscripts and accents. Sometimes these are particular to the defined quantity, e.g. c_p is specific heat at constant pressure, but we list below those that attached to several different quantities and are generally applicable.

Accents, subscripts and superscripts

$\bar{}$	Adiabatic reference value, e.g. $\bar{T}$ is adiabatic reference temperature
$\hat{}$	Dimensionless variable., e.g. $\hat{\mathbf{u}} = \mathbf{u}d/\eta$ is dimensionless velocity
$\breve{}$	Quantity averaged over turbulent subgrid scale
$\check{}$	Equivalent Boussinesq static quantity (section 7)
$\tilde{}$	Boussinesq variable with $\check{}$ part subtracted off (section 7
a	Archimedean (buoyancy), e.g. $\mathcal{Q}_a$ is rate of working of buoyancy forces
cmb	Value at core-mantle boundary
$conv$	Part due to convection, e.g. I_{conv} is part of heat flux carried by convection
g	Gravitational, e.g. ε^g is the gravitational energy density

icb	Value at inner core boundary
int	Internal source, i.e. radioactivity and secular cooling, (but not ohmic or viscous dissipation)
k	Kinetic, e.g. ε^k is the kinetic energy density
l	Latent heat, e.g. $\mathcal{Q}_l$ is rate of latent heat energy release
m	Magnetic
r	Radial component
R	Radioactivity, e.g. H^R is rate of release of radioactivity in the core
T	Turbulent value or turbulent part
ϑ	Thermal, e.g. $\mathbf{I}_\vartheta$ is temperature flux
0	Value at $(r_{icb} + r_{cmb})/2$. Exception: μ_0 is magnetic permeability

Definition of symbols

Symbol	Meaning	Equation first used	Units
A	Archimedean number	(5.12)	
α	Coefficient of thermal expansion	(2.1f)	K^{-1}
α_i^ξ	Isothermal compositional expansion coefficient	(2.16a)	
α^ξ	Adiabatic compositional expansion coefficient	(2.8)	
$\mathbf{B}$	Magnetic field	(2.3a)	T
$\hat{B}$	Typical magnetic field strength	(5.28)	T
β^ϑ	Equivalent Boussinesq static temperature gradient $\partial\check{\vartheta}/\partial\hat{r}$	(7.4a)	
β^ξ	Equivalent Boussinesq static composition gradient $\partial\check{\xi}/\partial\hat{r}$	(7.4b)	
β^B	Coefficient in definition of T_s	(7.22a)	$K\ m^{-2}$
$\hat{C}$	Coefficient for composition boundary condition	(5.14b)	
$\tilde{C}$	Boussinesq co-density $\tilde{\vartheta} + \tilde{\xi}$	below (7.25)	
C_e, C_i	Coefficients in Boussinesq Rayleigh numbers	(7.25)	
c_p	Specific heat at constant pressure	(2.1f)	$J\ kg^{-1}\ K^{-1}$
c_v	Specific heat at constant volume	(2.16)	$J\ kg^{-1}\ K^{-1}$
d	$r_{cmb} - r_{icb}$	(2.1b)	m
D	Dissipation number	(2.1b)	
δ	Magnetic dissipation length scale	(5.26)	m
Δ_2	Dimensionless quantity derived from melting point curve	(4.5)	
$\mathbf{E}$	Electric field	(2.6a)	$V\ m^{-1}$
E	Ekman number	(5.12)	
E_m	Magnetic Ekman number	(5.12)	
ε	Energy density	(3.1)	$J\ m^{-3}$
η	Magnetic diffusivity	(3.6)	$m^2\ s^{-1}$
$\mathbf{F}^v$	Viscous force density	(2.7a)	$N\ m^{-3}$
f	Convective plume filling factor	(5.18)	
F	Flux through a surface of radius r	(4.15)	W
g	Acceleration due to gravity	(2.1a)	$m\ s^{-2}$
G	Gravitational constant	(2.4b)	$kg^{-1}\ m^3\ s^{-2}$
γ	Grüneisen parameter	(2.1c)	

Γ	Correlation coefficient for convected heat flux	(5.20)	
H	Internal heat source	(2.5a),(3.13)	W kg^{-1}
H'	Internal heating including secular cooling	(2.20),(3.14b)	W kg^{-1}
$\hat{H}^c$	Dimensionless secular cooling heat source	(5.6)	
$\hat{H}^R$	Dimensionless radioactive heat source	(5.6)	
h^ξ	Heat of reaction	(2.11)	J kg^{-1}
h_l	Latent heat	(4.5)	J kg^{-1}
$\mathbf{j}$	Current density	(2.3a)	A m^{-2}
$\mathbf{I}$	Energy flux of various quantities	(3.9)	W m^{-2}
k	Thermal conductivity	(2.5a)	W m^{-1} K^{-1}
k^ξ	Mass diffusion coefficient	(2.5b)	kg m^{-1} s^{-1}
κ	Thermal diffusivity	(2.5)	m^2 s^{-1}
κ^ξ	Mass diffusivity	(2.5)	m^2 s^{-1}
$\hat{L}$	Coefficient in the ICB thermal boundary condition	(5.14a)	
ℓ	Typical width of convection rolls	(5.18)	m
$\hat{M}$	Coefficient in the CMB thermal boundary condition	(5.15a)	
M_{oc}	Mass of outer core	(6.2)	kg
M_{ic}	Mass of inner core	after (6.2)	kg
μ_0	Magnetic permeability	(2.6c)	H m^{-1}
μ	Chemical potential	(3.2)	m^2 s^{-2}
ν_T	Turbulent kinematic viscosity	(2.30)	m^2 s^{-1}
$\mathbf{\Omega}$	Rotation rate of mantle (assumed constant)	(2.3a)	s^{-1}
$\mathbf{\Omega}_{IC}$	Rotation rate of inner core	(4.1)	s^{-1}
P,(p)	Pressure (fluctuating pressure)	(2.1a),(2.2a)	Pa
P_m	Magnetic Prandtl number	(5.12)	
$\Psi, (\psi)$	Gravitational potential (fluctuating potential)	(2.2)	m^2 s^{-2}
q,q_T	Roberts number κ/η and turbulent Roberts number	(5.13)	
Q	Rate of energy release per unit volume	(3.6)	W m^{-3}
$\mathcal{Q}$	Rate of energy production in whole core	(4.9)	W
r	Distance from Earth's centre	(2.1a)	m
ρ	Density	(2.1a)	kg m^{-3}
ρ'	Density perturbation	(2.2b)	kg m^{-3}
Ra	Rayleigh number	(5.13)	
R_m	Magnetic Reynolds number	(5.17)	
R_e,R_i	Boussinesq Rayleigh numbers	(7.22)	
S,(s)	Specific entropy (entropy fluctuation)	(2.1e),(2.2f)	J K^{-1} kg^{-1}
S_r	Spherical surface of radius r	(4.3)	m^2
σ	Electrical conductivity	(2.6b)	S m^{-1}
σ_{ij}	Viscous stress tensor	(2.30)	N m^{-2}
t	Time	(2.3)	s
T	Temperature	(2.2c)	K
$\bar{T}$	Temperature of adiabatic state	(2.1b)	K
ϑ	Temperature fluctuation due to convection	(2.2c)	K
ΔT	ICB-CMB temperature difference	(5.6)	K
T_s	Equivalent Boussinesq static temperature	(7.19)	K
T_Δ	Equivalent Boussinesq static temperature drop across core	(7.22a)	K
$\mathbf{u}$	velocity	(2.3a)	m s^{-1}
u_p	Sound speed in the core	(6.1)	m s^{-1}
$\hat{V}$	Typical dimensionless velocity	(5.17)	m s^{-1}
V_{oc}	Volume of outer core	(4.9)	m^3
V_r	Volume from ICB to sphere of radius r	(4.15)	m^3

$\Xi,\bar{\xi}$	Mass fraction of light element (adiabatic value)	(2.2e)
ξ	Fluctuation mass fraction of light element	(2.2e)
$\Delta\xi$	Jump of ξ across the ICB	(4.5)

FIGURE CAPTIONS

Figure 1. Form of the dimensionless core variables $\hat{D} = D/D_0$, $\hat{g} = g/g_0$, $\hat{\alpha} = \alpha/\alpha_0$, $\hat{\rho} = \bar{\rho}/\rho_0$, $\bar{T}/T_0$ as a function of $\hat{r}$. By construction all curves pass through 1 at $\hat{r} = (\hat{r}_{icb} + \hat{r}_{cmb})/2$. For the dimensional values at that point see table 1.

Figure 2. The dimensionless heat flux deficit, $\nabla \cdot \hat{\rho}\nabla\hat{T}/\hat{\rho}$ as a function of $\hat{r}$.

Figure 3. The dimensionless convective heat flux $\hat{F}_{conv}$ as a function of $\hat{r}$ for the standard model (table 1 values, heavy type curves) and for a low heat flux model (lighter type curves). The dashed curves give the form for constant heat flux released at the ICB and no internal heating or heat flux deficit. The dotted curves are for uniform internal heating with no latent heat release at the ICB and no heat flux deficit. The dot - dash curves are the dimensionless mass flux of light material, $\hat{F}_m$.