

4. Inequality Constraints, Complementary slackness condition, Maximisation and Minimisation, Kuhn-Tucker method: summary

4(A) Inequality constraints

- (i) Two variables, one inequality. To *maximise*  $f(x, y)$  subject to  $g(x, y) \leq b$  we look at the boundary of the region allowed by the inequality. By drawing a sketch of lines of constant  $f$  and constant  $g$  we see that if there is a point where  $\nabla f$  and  $\nabla g$  are parallel, then this point will give a maximum of  $f$  for the region  $g(x, y) \leq b$ . It is necessary that  $\nabla f$  and  $\nabla g$  are parallel and not anti-parallel, i.e. there must be a positive  $\lambda$  with  $\nabla f = \lambda \nabla g$ . If this is the case, the inequality constraint is *binding*.

If  $\nabla f$  and  $\nabla g$  have opposite signs, then we can increase  $f$  by going in the direction of  $\nabla f$  and we are still in the region  $g(x, y) \leq b$ . The boundary of the constraint region is then of no particular significance for finding the maximum of  $f$  subject to  $g(x, y) \leq b$ . In this case we say the constraint is not binding, or the constraint is ineffective. The maximum is then found by looking for the *unconstrained* maximum of  $f(x, y)$

(ii) **Complementary Slackness Condition**

We define a Lagrangian  $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ . If the constraint is binding, then the equations to be solved are

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad g(x, y) = b.$$

If the constraint is not binding, then the equations to be solved are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,$$

and the constraint is ignored. We can neatly capture both of these sets of equations by writing

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \lambda(g(x, y) - b) = 0.$$

The third equation is called the complementary slackness condition. It can either be solved with  $\lambda \neq 0$ , in which case we get the binding constraint conditions, or with  $\lambda = 0$ , in which case the constraint  $g(x, y) - b = 0$  doesn't have to hold, and the Lagrangian  $L = f - \lambda g$  reduces to  $L = f$ . So both cases are taken care of automatically by writing the first order conditions as

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \lambda(g(x, y) - b) = 0.$$

- (iii) **Example:** maximise  $f(x, y) = xy$  subject to  $x^2 + y^2 \leq 1$ .

$$L = xy - \lambda(x^2 + y^2 - 1).$$

Equalities:  $L_x = 0 \rightarrow y = 2\lambda x$ ,  $L_y = 0 \rightarrow x = 2\lambda y$ , complementary slackness gives  $\lambda(x^2 + y^2 - 1) = 0$ . Inequalities are  $\lambda \geq 0$ ,  $x^2 + y^2 \leq 1$ .

Solutions satisfying these equalities and inequalities, are (a),  $\lambda = x = y = 0$ , (b)  $\lambda = 1/2$ ,  $x = y = 1/\sqrt{2}$ , and (c)  $\lambda = 1/2$ ,  $x = y = -1/\sqrt{2}$ . In solution (a), the constraint is not binding, and the solution is a saddle not a maximum. In both the solutions (b) and (c), the constraint is binding, because  $\lambda \neq 0$ , and so the constraint  $g$  takes its maximum value of 1 there. Both (b) and (c) give local maxima, and both give an equal value of 1/2 for  $f$ . Since the constraint is binding, we can use the bordered Hessian method to establish that these solutions really are maxima. Since  $\nabla g = (2x, 2y)$ , and  $L_{xx} = -2\lambda$ ,  $L_{xy} = 1$ ,  $L_{yy} = -2\lambda$ , for solution 2,  $\lambda = 1/2$ ,  $x = y = 1/\sqrt{2}$ , the bordered Hessian is

$$\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$$

which has  $LPM_3 = 8$  which has the same sign as  $(-1)^n = (-1)^2 = 1$ . Since  $n = 2$ ,  $m = 1$  we only need to examine  $LPM_3$  and so the bordered Hessian is negative definite, corresponding to a local maximum. Similarly, the third solution,  $\lambda = 1/2$ ,  $x = y = -1/\sqrt{2}$  is also a local maximum.

#### 4(B) Several inequality constraints

We can generalize in a natural way to the case where there is more than one inequality constraint and  $n$  variables  $\mathbf{x} = (x_1, \dots, x_n)$ . To maximise  $f(\mathbf{x})$  subject to

$$g_1(\mathbf{x}) \leq b_1, \quad \dots, \quad g_k(\mathbf{x}) \leq b_k, \quad i = 1, k,$$

we define the Lagrangian

$$L = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \dots - \lambda_k g_k(\mathbf{x})$$

and solve the  $n$  first order equality conditions

$$\frac{\partial L}{\partial x_1} = 0, \dots, \frac{\partial L}{\partial x_n} = 0$$

together with the  $k$  complementary slackness conditions

$$\lambda_1(g_1(\mathbf{x}) - b_1) = 0, \dots, \lambda_k(g_k(\mathbf{x}) - b_k) = 0.$$

These are  $n + k$  equations for the  $n + k$  unknowns  $x_1, \dots, x_n, \lambda_1, \dots, \lambda_k$ . However, only solutions which satisfy the  $2k$  inequalities

$$\lambda_1 \geq 0, \dots, \lambda_k \geq 0, \quad g_1(\mathbf{x}) \leq b_1, \quad \dots, \quad g_k(\mathbf{x}) \leq b_k$$

are admissible. If these don't hold, it means that the stationary point solution of the first order equalities is either not in the allowed region given by the constraints, or has a negative Lagrange multiplier, which is not acceptable for a maximum.

If we have a solution of the first order conditions, we must next check which constraints are binding and which are not. If any  $\lambda_i = 0$ , in the solution, then that constraint is not binding. Only constraints with  $\lambda_i > 0$  correspond to binding constraints. To test whether the stationary point is indeed a maximum, we look at the bordered Hessian constructed from the Lagrangian and the binding constraints. Non-binding constraints are simply ignored when we consider whether the stationary point is a maximum or not. We should also check the nondegenerate constraint conditions for the binding constraints, that is check whether the components of  $\nabla g_i$  form a set of linearly independent vectors.

Example: Maximise  $f(x, y, z) = xyz$  subject to  $x + y + z \leq 3$ ,  $-x \leq 0$ ,  $-y \leq 0$  and  $-z \leq 0$ . Note that we have written the conditions that  $x$ ,  $y$  and  $z$  are positive in the standard form for a maximisation problem.

Solution:

$$L = xyz - \lambda_1(x + y + z - 3) + \lambda_2x + \lambda_3y + \lambda_4z$$

and the first order equalities give

$$L_x = yz - \lambda_1 + \lambda_2 = 0, \quad L_y = xz - \lambda_1 + \lambda_3 = 0, \quad L_z = xy - \lambda_1 + \lambda_4 = 0,$$

$$\lambda_1(x + y + z - 3) = 0, \quad \lambda_2x = 0, \quad \lambda_3y = 0, \quad \lambda_4z = 0.$$

The eight inequalities are that the four  $\lambda_i \geq 0$ , and the four constraints already listed. The solutions are analysed by looking first at whether  $\lambda_1 = 0$  or  $x + y + z - 3 = 0$ . Four types of solution are found, (a)  $x = y = 0$ ,  $z =$  any positive value less than 3,  $\lambda_1 = 0$ , (b)  $x = z = 0$ ,  $y =$  any positive value less than 3,  $\lambda_1 = 0$ , (c)  $y = z = 0$ ,  $x =$  any positive value less than 3,  $\lambda_1 = 0$ , (d)  $x = y = z = 1$ ,  $\lambda_1 = 1$ . All four solutions have  $\lambda_2 = \lambda_3 = \lambda_4 = 0$ . The solution (d) is the most interesting, as it provides the local and global maximum; note that the domain of available  $x$ ,  $y$  and  $z$  is bounded by the constraints. At solution (d) only the constraint  $x + y + z = 3$  is binding, all the others being not binding and so having  $\lambda_2 = \lambda_3 = \lambda_4 = 0$ . To show (d) is a local maximum we can examine the  $4 \times 4$  bordered Hessian with  $m = 1$  constraint,  $x + y + z - 3 = 0$  (discounting the non-binding constraints) and  $n = 3$  variables. We get

$$H_B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

which has  $LPM_4 = -3$  and  $LPM_3 = 2$ . The signs of the  $n - m = 2$  highest  $LPM$ 's therefore alternate in sign, with the biggest,  $LPM_4$ , having the same sign as  $(-1)^n$ . Therefore  $H_B$  is negative definite, and so solution (d) is indeed a local maximum. By inserting solutions (a) to (d) into  $f = xyz$  we see that (d) is also the global maximum. Note also that the NDCQ is trivially satisfied at solution (d), because  $\nabla g = (1, 1, 1)$  is non-zero.

#### 4(C) Mixed equality and inequality constraints

In general we can have both inequality constraints and equality constraints. To maximise  $f(\mathbf{x})$  subject to

$$g_1(\mathbf{x}) \leq b_1, \quad \dots, \quad g_k(\mathbf{x}) \leq b_k, \quad i = 1, k,$$

and

$$h_1(\mathbf{x}) = c_1, \quad \dots, \quad h_m(\mathbf{x}) = c_m, \quad i = 1, m,$$

we use the complementary slackness conditions to provide the equations for the Lagrange multipliers corresponding to the inequalities, and the usual constraint equations to give the Lagrange multipliers corresponding to the equality constraints. Thus

$$L = f - \sum_{i=1}^k \lambda_i (g_i - b_i) - \sum_{i=1}^m \mu_i (h_i - c_i)$$

and the equalities are

$$\frac{\partial L}{\partial x_1} = 0, \dots, \frac{\partial L}{\partial x_n} = 0$$

together with the  $k$  complementary slackness conditions

$$\lambda_1 (g_1(\mathbf{x}) - b_1) = 0, \dots, \lambda_k (g_k(\mathbf{x}) - b_k) = 0,$$

and the  $m$  equality constraints

$$\mu_1 (h_1(\mathbf{x}) - c_1) = 0, \dots, \mu_m (h_m(\mathbf{x}) - c_m) = 0.$$

This gives  $n + m + k$  equations for  $n$  variables  $x_1, x_2, \dots, x_n$ ,  $k$  Lagrange multipliers  $\lambda_i$  and  $m$  Lagrange multipliers  $\mu_i$ . These equality conditions go along with  $2k$  inequalities that any maximum must satisfy,  $\lambda_i \geq 0$  and  $g_i \leq b_i$ . These equations constitute the first order conditions. The second order conditions are found from the bordered Hessian, where any non-binding inequality constraint is ignored, and the binding inequality constraints are treated in the same way as the equality constraints.

#### 4(D) Constrained Minimisation

If we want to *minimise*  $f(\mathbf{x})$  subject to the inequality constraints, we need to write the inequality constraints in the form

$$g_1(\mathbf{x}) \geq b_1, \quad \dots, \quad g_k(\mathbf{x}) \geq b_k, \quad i = 1, k.$$

The key point is the  $\geq$ , different from the  $\leq$  occurring when we want to maximise  $f$ . Why do we need to change the sign of the inequality? Because we want to keep the Lagrange multipliers always positive, so we need  $\nabla f$  and  $\nabla g$  to point in the same direction. Now at a binding constraint,  $\nabla f$  points in the direction of increasing  $f$ , whereas the minimum lies in the direction of decreasing  $f$ . This is why the sign of the inequality must be written the other way round. Draw a few sketches of contours of  $f(x, y)$  and  $g(x, y)$  to convince yourself of this.

Example: Minimise  $f = 2y - x^2$  subject to  $x^2 + y^2 \leq 1$ ,  $x \geq 0$ ,  $y \geq 0$ .

Solution: we must write the first constraint as  $1 - x^2 - y^2 \geq 0$  because we want to minimise  $f$ , not maximise it. The other constraints are already in the correct form for minimisation.

$$L = 2y - x^2 - \lambda_1(1 - x^2 - y^2) - \lambda_2 x - \lambda_3 y.$$

The first order equalities are then

$$-2x + 2\lambda_1 x - \lambda_2 = 0, \quad 2 + 2\lambda_1 y - \lambda_3 = 0$$

$$\lambda_1(1 - x^2 - y^2), \quad \lambda_2 x = 0, \quad \lambda_3 y = 0.$$

As usual, all the  $\lambda_i \geq 0$ . Solutions are  $x = y = 0$ ,  $\lambda_1 = \lambda_2 = 0$ ,  $\lambda_3 = 2$  and  $x = 1$ ,  $y = 0$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 0$ ,  $\lambda_3 = 2$ . This second solution is a local and global minimum. Note that this second solution has two binding constraints,  $x^2 + y^2 \leq 1$  and  $y \geq 0$ . Since  $\nabla(x^2 + y^2) = (2, 0)$  and  $\nabla y = (0, 1)$  at the critical point, we note that these are two linearly independent vectors (not parallel), so the NDCQ are satisfied here.

#### 4(E) The Kuhn-Tucker method

This is an alternative, and slightly simpler method for dealing with the common case where there are positivity constraints, that is constraints of the form

$$x_i \geq 0 \quad \text{or} \quad -x_i \leq 0.$$

This is common in practice, where often variables are only meaningful when positive, e.g. production rates cannot be negative.

The typical problem is maximise  $f(\mathbf{x})$  subject to  $g_i(\mathbf{x}) \leq b_i$ ,  $i = 1, k$ , and  $x_i \geq 0$ ,  $i = 1, n$ . The standard method for this problem would be to use the  $n + k$  Lagrange multipliers corresponding to the  $n + k$  constraints, and proceed as before. This works, but is rather cumbersome, and the Kuhn-Tucker method is neater.

We define the Kuhn-Tucker Lagrangian

$$\bar{L} = f - \sum_{i=1}^k \lambda_i (g_i - b_i)$$

just ignoring the positivity constraints. Then the usual Lagrangian

$$L = \bar{L} + \mu_1 x_1 + \cdots + \mu_n x_n.$$

Now for the classical Lagrangian we know  $\frac{\partial L}{\partial x_i} = 0$  for stationary points, so  $\frac{\partial \bar{L}}{\partial x_i} + \mu_i = 0$ . The complementary slackness equations for the  $-x_i \leq 0$  constraints are just  $\mu_i x_i = 0$ , so we have

$$x_i \frac{\partial \bar{L}}{\partial x_i} = 0, \quad i = 1, n. \quad (1)$$

The complementary slackness conditions on the remaining  $\lambda_i$  constraints are

$$\lambda_i (g_i - b_i) = 0,$$

which can be written

$$\lambda_i \frac{\partial \bar{L}}{\partial \lambda_i} = 0, \quad i = 1, k. \quad (2)$$

Equations (1) and (2) are the  $n + k$  first order equalities in the Kuhn-Tucker method. They are nicely symmetric in  $x_i$  and  $\lambda_i$  which makes them more memorable. Note also because we don't introduce the  $\mu_i$  explicitly, we only need  $n + k$  equations instead of  $2n + k$  equations, a significant saving.

In addition, since the  $\mu_i \geq 0$ , we must have the inequalities

$$\frac{\partial \bar{L}}{\partial x_i} \leq 0,$$

so the full set of  $2n + 2k$  inequalities that a solution of the first order conditions must satisfy are

$$\frac{\partial \bar{L}}{\partial x_i} \leq 0, \quad x_i \geq 0, \quad i = 1, n; \quad \lambda_i \geq 0, \quad g_i \leq b_i, \quad i = 1, k.$$

Example: Maximise  $f = x - y^2$  subject to  $x^2 + y^2 \leq 4$ ,  $x \geq 0$ ,  $y \geq 0$  using the Kuhn-Tucker method.

Solution:  $\bar{L} = x - y^2 - \lambda(x^2 + y^2 - 4)$ . The Kuhn-Tucker equalities are

$$x \frac{\partial \bar{L}}{\partial x} = x - 2\lambda x^2 = 0, \quad (i)$$

$$y \frac{\partial \bar{L}}{\partial y} = -2y^2 - 2\lambda y^2 = 0, \quad (ii)$$

$$\lambda(x^2 + y^2 - 4) = 0. \quad (iii)$$

The Kuhn-Tucker inequalities are

$$\frac{\partial \bar{L}}{\partial x} = 1 - 2\lambda x \leq 0, \quad \frac{\partial \bar{L}}{\partial y} = -2y - 2\lambda y \leq 0, \quad x \geq 0, \quad y \geq 0, \quad \lambda \geq 0, \quad x^2 + y^2 \leq 4.$$

Analysis of these shows that  $x = 2$ ,  $y = 0$ ,  $\lambda = 1/4$  is the only admissible solution, and is the local and global maximum.