Homogeneous Binary Relational Structures with the same Lattice of Reducts
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Preliminaries

Definition. Let \( M \) be a structure. A structure \( N \) is a reduct of \( M \) if \( N \) has the same domain as \( M \) and all \( \emptyset \)-definable relations in \( N \) are \( \emptyset \)-definable in \( M \).

Intuition. \( N \) is a reduct of \( M \) if \( N \) is a less detailed version of \( M \), or, if \( N \) contains less information than \( M \).

General Question. Given a structure \( M \), what are its reducts?

Remark 1. If two reducts \( N_1, N_2 \) of \( M \) are reducts of each other, they are considered to be the same reduct of \( M \); intuitively they contain the same information.

Remark 2. The reducts of a structure \( M \) form a lattice. For example, the join of two reducts \( N_1 \) and \( N_2 \) is the structure whose relations are those \( \emptyset \)-definable in both \( N_1 \) and \( N_2 \).

A Familiar Structure: \((Q, <)\)

These properties of \( Q \) provide some intuition for the later structures.
- \((Q, <)\) is \( \aleph_0 \)-categorical.
- \((Q, <)\) embeds all linear orders.
- \((Q, <)\) is homogeneous: Any iso\(^m\) \( f : A \rightarrow B, A, B \subset Q \) finite, can be extended to an auto\(^m\) of \( Q \).
- Let \( p(x) \) be a 1-type over a finite parameter set \( a_1, \ldots, a_n \). Let \( A = \{ a \in Q : a \models p(x) \} \). Then \( A = \{ a_i \} \) for some \( i \), or, \( A \cong Q \).

Some relations on \((Q, <)\)

We define three relations:
- \( <_w(a, b; x, y) := a < b \leftrightarrow x < y \)
- \( \text{cyc}(x, y, z) := x < y < z \) \( \lor y < z < x \) \( \lor z < x < y \).
- \( \text{cyc}_w(a, b, c; x, y, z) := \text{cyc}(a, b, c) \leftrightarrow \text{cyc}(x, y, z) \) (‘w’ abbreviates ‘weakened’.)

Reducts of \((Q, <)\)

Theorem. ([Cam76]) The reducts of \((Q, <)\) are: \((Q, <), (Q, <_w), (Q, \text{cyc}), (Q, \text{cyc}_w)\) and \((Q, =)\).

Three other structures

The following structures have the same lattice of reducts as \((Q, <)\):
- The random graph \( \Gamma \), [Tho91]
- The random tournament, [Ben97]
- The generic partial order, [PPP+11]

(These can be defined as satisfying the earlier properties of \( Q \) but with ‘linear order’ changed appropriately.)

Surprisingly, the reducts are defined in the same way: the original binary relation, its ‘weakened version’, a ‘cyclic’ relation, its ‘weakened version’ and the trivial structure.

Question. Is this just a coincidence? Are there other homogeneous binary structures with the same pattern of reducts?