Examples II

E2.1. A child plays with 6 letters of the alphabet: E, E, P, P, P, R. What is the probability that with a random permutation of the letters he will obtain the word ‘PEPPER’?

Solution. A sample space consists of $P_6 = 6! = 720$ permutations of 6 letters. Assuming that all such permutations are equally likely, we obtain

$$P(\text{PEPPER}) = \frac{2! \cdot 3!}{6!} = \frac{12}{720} = \frac{1}{60},$$

where the term $2! \cdot 3!$ accounts for possible permutations of the letters E, E and P, P, P in the word.

E2.2. There are 30 students in a class. What is the probability that no two students have the same birthday?

Solution. To answer the question, we assume that there are no twins in the class and also that no-one was born on 29 February (so we ignore leap years). Since the students have been gathered in the class irrespective of their birthday, it is reasonable to assume that all birthdays are equally likely (thus we ignore the fact that birth rates are not exactly uniform over the year). Therefore, if we list the students in alphabetical order then there are $|\Omega| = (365)^{30}$ possible sequences of birthdays, all of which are equally likely.

We are interested in the event $B = \{\text{No two students have the same birthday}\}$. To evaluate the probability $P(B)$ we have to count the number of outcomes in $B$. To this end, we think of building up the outcomes in $B$ successively, one birthday at a time. The first student can have any of the 365 birthdays. The second student must avoid the first student’s birthday and hence has 364 choices. Continuing in this way, each student has one fewer choice until the 30th student must avoid the previous 29 birthdays and thus has 336 choices. Applying the Basic Principle of Counting, we see that

$$|B| = 365 \cdot 364 \cdots 336.$$  

The last answer can also be obtained by noting that there are 365 days in the calendar year and we are going to pick 30 of them to ‘stand in line’, so the number of possibilities is given by the number of all arrangements of 30 out of 365,

$$|B| = A_{365}^{30} = \frac{365!}{335!} = 365 \cdot 364 \cdots 336.$$  

Hence we obtain

$$P(B) = \frac{|B|}{|\Omega|} = \frac{365 \cdot 364 \cdots 336}{365^{30}} = 0.2937$$

Remark. At first sight, the last answer may be surprising: 30 is less than $1/12$ of the 365 days on the calendar, but the chances to find at least two students with the same birthday are better than 7 out of 10. This ‘paradox’ evaporates when we realise that there are $C_2^{30} = (30 \cdot 29)/2 = 435$ pairs of students, and each pair have a common birthday with probability $1/365$.

Let $p_k$ be the probability that $k$ people all have different birthdays. Clearly,

$$p_1 = 1 \quad \text{and} \quad p_{k+1} = p_k(365 - k)/365 \quad (1 \leq k \leq 365).$$

Using this recursion it is easy to evaluate $p_k$ for any given $k$. For instance,

$$p_2 = \frac{364}{365} = 0.99726, \quad p_3 = \frac{364}{365} \cdot \frac{363}{365} = 0.99180, \quad \text{etc.}$$
This probability becomes less than 0.5 already for \( k = 23 \). For \( k = 40 \) it decreases to 0.10877, and for \( k = 100 \) we have \( \frac{1}{3^{100}} = 0.000000333 \ldots \)

**E2.3.** Six married couples are seated at random at a round table. Compute the probability that all wives sit next to their husbands.

**Solution.** Let us enumerate the 12 seats at the table (say, clockwise). Then the number of all permutations of 12 people at a table equals \( P_{12} = 12! \).

Consider the event \( A = \{ \text{All wives sit next to their husbands} \} \). Note that if \( A \) occurs then the 12 seats may be broken down into consecutive pairs in such a way that each of the six couples (wife and husband) occupies one pair of seats. Obviously, such a partition may be either

\[
(1, 2), (3, 4), (5, 6), (7, 8), (9, 10), (11, 12) \quad (1)
\]
or

\[
(12, 1), (2, 3), (4, 5), (6, 7), (8, 9), (10, 11). \quad (2)
\]

To obtain \(|A|\), let us first allocate 6 husbands to the pairs of seats — this can be done in \( P_6 = 6! \) ways for each of the cases (1) and (2). After that, let each husband choose one of the two allocated seats (this is done in \( 2^6 \) ways), and his wife then take the remaining seat. As a result,

\[
|A| = 2 \times 6! \times 2^6,
\]

so the required probability is given by

\[
P(A) = \frac{6! \cdot 2^7}{12!} = \frac{2}{10395} = 0.00019...
\]

**E2.4.** According to the ‘central dogma’ in molecular biology, proposed in 1957 by Francis Crick and George Gamov, the genetic code specifies the amino acid in a protein by a *non-ordered* sequence of three nucleotides (called a *triplet*). Each nucleotide can be one of the four kinds, \( T \) (thymine), \( A \) (adenine), \( C \) (cytosine), or \( G \) (guanine), with repetitions permitted. How many amino acids can be coded in this manner?

**Solution.** The total number of different triplets constructed of the letters \( T, A, C, \) and \( G \) can be easily computed. Indeed, there are

- 4 one-letter triplets, like \( TTT, AAA, \) etc.
- \( C_4^1 \cdot C_4^3 = 12 \) two-letter triplets, like \( TTA, TAA, \) etc.
- \( C_4^3 = 4 \) three-letter triplets, like \( TAC, ACG, \) etc.

Altogether, this gives \( 4 + 12 + 4 = 20 \) triplets.

Another, more systematic approach can be as follows. Consider four cells arranged in a row, marked \( T, A, C \) and \( G \), and suppose that there are three indistinguishable particles each of which can be placed into any cell. Clearly, any such allocation yields a triplet and, conversely, every triplet corresponds to a certain placement of the particles. For example, if all the particles are placed into cell \( T \), this yields the triplet \( TTT \); if cells \( T, A, C \) are filled in with one particle each and cell \( G \) remains empty, this corresponds to the triplet \( TAC \), etc.

The number of all possible allocations of the particles can be computed by the following trick. Let us represent the particles by stars and indicate the 4 cells by the 4 spaces between 5 vertical bars. Thus \( |**| |*| |*| \) is used as a symbol for the triplet \( TTC \). Such a symbol necessarily starts and ends with a bar, but the remaining \( 5 - 2 = 3 \) bars and 3 stars can appear in an arbitrary order. In this way it becomes clear that the required number of all possible triplets is given by the number of ways of selecting 3 places out of 3 + 3 = 6, namely

\[
C_3^6 = \frac{6!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = 20.
\]
E2.5. Three dice are rolled. If no two show the same face, what is the probability that one is an ace (i.e., shows a ‘6’)?

Solution. We work in the sample space $\Omega$ with $\omega = (i, j, k)$, where $i, j, k$ may take any values from 1 to 6. Note that $|\Omega| = 6^3$.

Consider the events $H = \{\text{No two dice show the same face}\}$, $A = \{\text{One die is an ace}\}$. We have to find the conditional probability

$$P(A | H) = \frac{P(A \cap H)}{P(H)}. \quad (3)$$

It is easy to see that

$$P(H) = \frac{A_6^3}{6^3} = \frac{6 \cdot 5 \cdot 4}{6^3}. \quad (4)$$

Taking into account that the event $A \cap H$ occurs if one and only one face is ‘6’, we get

$$P(A \cap H) = \frac{3A_5^2}{6^3} = \frac{3 \cdot (5 \cdot 4)}{6^3}. \quad (5)$$

Substituting (4) and (5) into (3), we obtain

$$P(A | H) = \frac{3 \cdot 5 \cdot 4}{6 \cdot 5 \cdot 4} = \frac{1}{2}. \quad (6)$$

E2.6. To encourage Elmer’s promising tennis career, his father offers him a prize if Elmer wins at least two tennis sets in a row in a three-set series to be played with his father ($F$) and the club champion ($C$) alternately: either $FCF$ or $CFC$, according to Elmer’s choice. The champion is a better player than Elmer’s father. Which series should Elmer choose in order to increase his chances to win the prize?

Solution. Suppose that Elmer wins a tennis set against his father and the champion with probabilities $p_1$ and $p_2$, respectively, so that $p_1 > p_2$. Assume also that the sets are independent of each other. Let $A_i$ be the event that Elmer wins the $i$th set in a series ($i = 1, 2, 3$), then the event $D = \{\text{Elmer wins the prize}\}$ is represented as

$$D = (A_1 \cap A_2) \cup (A_2 \cap A_3).$$

By the formula for the probability of the union of two events, we have

$$P(D) = P((A_1 \cap A_2) \cup (A_2 \cap A_3)) = P(A_1 \cap A_2) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3). \quad (6)$$

By assumption, the events $A_1$, $A_2$, $A_3$ are independent, hence the probabilities of intersections in (6) are factorised to yield

$$P(D) = P(A_1) \cdot P(A_2) + P(A_2) \cdot P(A_3) - P(A_1) \cdot P(A_2) \cdot P(A_3). \quad (7)$$

For the series $FCF$ we have

$$P(A_1) = p_1, \quad P(A_2) = p_2, \quad P(A_3) = p_1,$$

and equation (7) yields

$$P(D) = p_1p_2 + p_2p_1 - p_1p_2p_1 = 2p_1p_2 - p_1^2p_2. \quad (8)$$

Similarly, for the series $CFC$ we have

$$P(A_1) = p_2, \quad P(A_2) = p_1, \quad P(A_3) = p_2,$$

so from (7) we get

$$P(D) = p_2p_1 + p_1p_2 - p_2p_1p_2 = 2p_1p_2 - p_1p_2^2. \quad (9)$$
Comparing expressions (8) and (9), we see that the second series is preferable, because

\[ 2p_1p_2 - p_1^2p_2^2 > 2p_1p_2 - p_1^2p_2 \]

(recall that \( p_2 < p_1 \)).

**Remark.** That Elmer should choose the series \( CFC \) may seem surprising. Common sense suggests to choose \( FCF \), in order to reduce the number of sets to be played with the champion. However, calculations show that this is wrong.

**E2.7.** In the circuit on Fig. 1 each switch is closed with probability \( p \), independently of all other switches. (‘Closed’ means that electric current can pass through the switch.) Find the probability that a flow of current is possible between points A and B.

![Figure 1: Circuit diagram](image)

**Solution.** Let us first find the probability of the event

\[ (A \xrightarrow{12} B) = \{ \text{Current can flow through switches 1 and 2} \} \]

(see Fig. 1). Clearly, this is possible if both switches are closed. Using independence of switches, we find that the probability of this event is

\[ P(A \xrightarrow{12} B) = p \times p = p^2. \]

Similarly,

\[ P(A \xrightarrow{3} B) = p, \quad P(A \xrightarrow{45} B) = p^2. \]

We are interested in the event \( (A \rightarrow B) = \{ \text{Current from A to B is possible} \} \), that is,

\[ (A \rightarrow B) = (A \xrightarrow{12} B) \cup (A \xrightarrow{3} B) \cup (A \xrightarrow{45} B). \]

It is easier to find the probability of the complementary event \( (A \rightarrow B)^c \) (that current between A and B is **not** possible), which can only occur if all the three routes are blocked. This yields

\[ P((A \rightarrow B)^c) = (1 - P(A \xrightarrow{12} B)) \times (1 - P(A \xrightarrow{3} B)) \times (1 - P(A \xrightarrow{45} B)) \]

\[ = (1 - p^2)(1 - p)(1 - p^2). \]

Here we have again used independence of switches, which implies that each route is blocked or unblocked independently of other routes.
Finally, we get
\[ P(A \rightarrow B) = 1 - P((A \rightarrow B)^c) = 1 - (1 - p)(1 - p^2)^2. \]

**E2.8.** Urn I contains two white and four red balls, whereas urn II contains two white and one red ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II.

(a) What is the probability that the ball selected from urn II is white?
(b) What is the conditional probability that the transferred ball was white, given that a white ball has been selected from urn II?

**Solution.** Let \( A_1 \) be the event that the transferred ball was white, and \( A_2 \) the event that the ball selected from urn II was white.

(a) By the formula of total probability we have
\[
P(A_2) = P(A_2 \mid A_1) \cdot P(A_1) + P(A_2 \mid A_1^c) \cdot P(A_1^c)
= \frac{3}{4} \cdot \frac{2}{6} + \frac{2}{4} \cdot \frac{4}{6} = \frac{7}{12}.
\]

(b) Using the Bayes formula and the result of part (a), we obtain
\[
P(A_1 \mid A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)} = \frac{P(A_2 \mid A_1) \cdot P(A_1)}{P(A_2)} = \frac{\frac{3}{4} \cdot \frac{2}{6}}{\frac{7}{12}} = \frac{3}{7}.
\]

**Remark.** Note that \( \frac{3}{7} > \frac{2}{5} \), so that \( P(A_1 \mid A_2) > P(A_1) \). That is to say, the appearance of a white ball from urn II makes it more likely that a ball drawn from urn I and transferred to urn II was white (which is intuitively clear).

**E2.9.** In the Californian gubernatorial election in 1982, several TV stations predicted, on the basis of questioning people when they exited the polling place, that Tom Bradley, then mayor of Los Angeles, would win the election against his opponent, George Deukmejian. When the votes were counted, however, Bradley lost by a considerable margin. What happened?

**Solution.** To give our explanation, we need some notation and some numbers. Consider the event
\[ B = \{ \text{A person chosen at random votes for Bradley} \} \]
and suppose that \( P(B) = 0.45 \) (so that Bradley is likely to lose the election). There were only two candidates, so this makes the probability of voting for Deukmejian \( P(B^c) = 0.55 \). Let
\[ A = \{ \text{The voter tells how he/she voted} \} \]
and suppose that \( P(A \mid B) = 0.4 \) whereas \( P(A \mid B^c) = 0.3 \). That is, 40\% of the Bradley voters will respond compared to 30\% of the Deukmejian voters.

Now we can compute the fraction of voters in our sample who voted for Bradley, that is the probability \( P(B \mid A) \). Indeed, using the Bayes formula we obtain
\[
P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B) \cdot P(A \mid B)}{0.45 \cdot 0.4 + 0.55 \cdot 0.3} = \frac{0.18}{0.18 + 0.165} = 0.5217.
\]
Therefore, from our sample it looks as if Bradley will win. The point here is that the difference in the response rates makes our sample not representative of the population as a whole.