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Dedicated to the memory of Vladimir Geyler

Analysis on Configuration Spaces and Gibbs Cluster Ensembles

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Abstract. The distribution μ of a Gibbs cluster point process in $\mathcal{X} = \mathbb{R}^d$ (with n-point clusters) is studied via the projection of an auxiliary Gibbs measure defined on the space of configurations in $\mathcal{X} \times \mathcal{X}^n$. We show that μ is quasi-invariant with respect to the group $\mathrm{Diff}_0(\mathcal{X})$ of compactly supported diffeomorphisms of $\mathcal{X}$ and prove an integration-by-parts formula for μ. The corresponding equilibrium stochastic dynamics is then constructed by using the method of Dirichlet forms.

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1. INTRODUCTION

Various models of random configurations of "particles" have been instrumental in numerous applications, including statistical mechanics, quantum physics, astrophysics, chemical physics, biology, computer science, economics, finance, and so on (see an extensive bibliography in [10]).

In the past decade, there has been a more specific interest in *analysis* on configuration spaces $\Gamma_{\mathcal{X}}$ (over a topological space $\mathcal{X}$, e.g., Euclidean space $\mathbb{R}^d$). Albeverio, Kondratiev, and Röckner [3, 4] have proposed an approach to configuration spaces regarded as *infinite-dimensional manifolds* by using a suitable probability measure μ on $\Gamma_{\mathcal{X}}$ quasi-invariant with respect to the group $\mathrm{Diff}_0(\mathcal{X})$ of compactly supported diffeomorphisms of $\mathcal{X}$. It can be shown that such a measure μ satisfies an integration-by-parts formula which enables one to construct (using the theory of Dirichlet forms) an associated equilibrium dynamics (stochastic process) on $\Gamma_{\mathcal{X}}$ such that μ is its invariant measure [3, 4, 20]. In turn, the equilibrium process plays an important part in the asymptotic study of statistical-mechanical systems whose spatial distribution is controlled by the measure μ; for instance, this process is a natural candidate to be an "attractor" for motions started from a perturbed (nonequilibrium) configuration. We refer the reader to the papers [2, 4, 5, 18, 22] and the references therein for further discussion of various theoretical aspects and applications.

This approach has been used in [3] for the Poisson measure μ on $\Gamma_{\mathcal{X}}$ and in [4] for a broader class of Gibbs measures. In the Poisson case, the equilibrium dynamics turns out to be given by the well-known independent particle process, i.e., an infinite family of independent (distorted) Brownian motions started at the points of a random Poisson configuration. In the Gibbsian case, the equilibrium dynamics is much more complex owing to interaction between particles.

In the present paper, our aim is to develop the analysis for another class of random spatial structures, namely, the *Gibbs cluster ensembles* (see, e.g., [9, 10]). A cluster process is a simple model that describes grouping ("clustering"), where a sample configuration is represented as the union of independent clusters of points distributed around a background configuration of invisible "centers." Cluster models have been very popular in a host of applications, ranging from neurophysiology (nerve impulses) and ecology (spatial distribution of offspring) to seismology (earthquakes) and cosmology (constellations and galaxies). More recent examples include applications to trapping models of diffusion-limited reactions in chemical kinetics [1, 6], where clusterization can arise due to binding of traps to a substrate (e.g., a polymer chain) or trap generation (e.g., by radiation damage). An exciting range of new applications in physics and biology is related to the dynamics of

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atomic or molecular clusters, which is particularly important in modern nanoscience (see a recent review [8] and further references therein).

In our previous paper [7], we studied the simplest cluster model, the *Poisson cluster process* (with a fixed number n of points in each cluster), where the configuration of cluster centers is obtained according to a Poisson point process. Our technique was based on the representation of the Poisson cluster measure on the configuration space $\varGamma_{\mathcal{X}}$ as the projection of an auxiliary Poisson measure defined on the space $\varGamma_{\mathcal{X}^n}$ of configurations of "droplet" points representing individual n-point clusters. The main advantage of this construction is that it enables one to apply the well-developed apparatus of Poisson measures to the study of the Poisson cluster measure.

In the present work, we consider Gibbs cluster processes in $\mathcal{X} = \mathbb{R}^d$ (with n-point clusters). The underlying (grand canonical) Gibbs measure g is determined by a pair interaction potential $\phi(x, x')$ and a reference measure σ_0 on $\mathcal{X}$. Note that we do not assume that the Gibbs measure is unique, and thus our results are not affected by possible "phase transitions" (i.e., nonuniqueness of g). Under some natural smoothness conditions (on σ_0 and on the distribution η of the generic cluster), we prove the $\mathrm{Diff}_0(\mathcal{X})$-quasi-invariance of the corresponding Gibbs cluster measure μ (Section 5), establish an integration-by-parts formula (Section 6), and construct the associated Dirichlet operator, which leads to the existence of the equilibrium stochastic dynamics on the configuration space $\varGamma_{\mathcal{X}}$ (Section 7).

Unlike the Poisson cluster case, it is now inconvenient to work with the measure arising on the space $\varGamma_{\mathcal{X}^n}$ of droplet configurations, which is hard to characterize. Instead, to be able to pursue our projection approach to a tractable pre-projection measure, we consider the configuration space $\varGamma_{\mathcal{X}\times\mathcal{X}^n}$, where each configuration $\widehat{\gamma} \in \varGamma_{\mathcal{X}\times\mathcal{X}^n}$ is made up of pairs $(x, \bar{y})$ with $x \in \mathcal{X}$ representing the cluster center and $\bar{y} \in \mathcal{X}^n$ the n-point configuration which determines the cluster shape after the translation to the center x. One can show that the corresponding measure $\widehat{\mathsf{g}}$ on $\varGamma_{\mathcal{X}\times\mathcal{X}^n}$ is again Gibbsian, with the product reference measure $\sigma = \sigma_0 \otimes \eta$ and the "cylinder" pair potential $\widehat{\phi}((x, \bar{y}), (x', \bar{y}')) \equiv \phi(x, x')$, where ϕ stands for the original pair potential associated with the measure g. We then project the Gibbs measure $\widehat{\mathsf{g}}$ from the "higher floor" $\varGamma_{\mathcal{X}\times\mathcal{X}^n}$ directly to the configuration space $\varGamma_{\mathcal{X}}$ (thus skipping the intermediate "floor" $\varGamma_{\mathcal{X}^n}$) and show that the resulting measure coincides with the original Gibbs cluster measure μ (Section 4).

We expect that the projection approach can also be applied to the study of more general cluster point processes, for example, those with random size clusters. Such models will be considered elsewhere.

2. GIBBS CLUSTER PROCESSES

By a *cluster process* we mean a random point process $Z = \{Z_k\}$ in $\mathcal{X} = \mathbb{R}^d$ of the form $Z = \{X_i + Y_j^i\}$, where $X = \{X_i\}$ is a background point process in $\mathcal{X}$ distributed according to some probability measure μ_{c} in the configuration space $\varGamma_{\mathcal{X}}$ and the aggregates of random vectors $\{Y_j^i\}$ are independent and identically distributed (i.i.d.) for different i. The aggregates $X_i + \{Y_j^i\}$ are referred to as *clusters* (attached to the "centers" X_i). We consider the case in which each aggregate $\{Y_{\cdot}^i\}$ consists of a fixed number n of random vectors $Y_1^i, \dots, Y_n^i$ with joint distribution function $F(y_1, \dots, y_n)$, which is assumed to be symmetric under permutations of its arguments.

For any function $f\colon \mathcal{X} \to \mathbb{R}$ with bounded support, write

$$\langle f, Z\rangle := \sum_{Z_k\in Z} f(Z_k) \equiv \sum_{X_i\in X}\sum_{j=1}^{n} f(X_i + Y_j^i).$$

Proposition 2.1 (see [10]). *The Laplace functional of the cluster process Z is given by*

$$\mathbb{E}\bigl[\mathrm{e}^{\langle f,\, Z\rangle}\bigr] = \int_{\varGamma_{\mathcal{X}}} \prod_{x\in\gamma} \mathbb{E}_Y\Bigl[\exp\Bigl(\sum_{j=1}^{n} f(x + Y_j)\Bigr)\Bigr]\mu_{\mathrm{c}}(\mathrm{d}\gamma),$$

where $\mathbb{E}_Y$ is the expectation with respect to the distribution of the generic cluster $(Y_1, \dots, Y_n)$.

In this paper, we are concerned with *Gibbs cluster processes*, where the measure μ_{c} is given by a (grand canonical) Gibbs measure g on $\varGamma_{\mathcal{X}}$ (see the appendix). To be more specific, let $\phi\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ be a *pair interaction potential.* Let a (nonatomic) *reference measure* σ_0 be given. Consider the set $\mathcal{G}(z\sigma_0, \phi)$ of all Gibbs measures on $\Gamma_{\mathcal{X}}$ associated with the potential ϕ and the scaled reference measure $z\sigma_0$ $(z \in \mathbb{R}_+)$. Suppose that $\mathcal{G}(z\sigma_0, \phi) \neq \varnothing$ (for various sufficient conditions, see, e.g., [23, 21] and the appendix). In what follows, we fix a $\mathsf{g} \in \mathcal{G}(z\sigma_0, \phi)$; moreover, we set $z = 1$ (renormalizing σ_0 if necessary). Note that if $\sigma_0(\mathcal{X}) = \infty$ then any Gibbs measure $\mathsf{g} \in \mathcal{G}(\sigma_0, \phi)$ is concentrated on the infinite configurations.

For a Gibbs cluster process with background Gibbs measure g on $\Gamma_{\mathcal{X}}$, Proposition 2.1 gives

$$\mathbb{E}\bigl[\mathrm{e}^{\langle f,\, Z\rangle}\bigr] = \int_{\Gamma_{\mathcal{X}}} \prod_{x\in\gamma} \mathbb{E}_Y\Bigl[\exp\Bigl(\sum_{j=1}^{n} f(x+Y_j)\Bigr)\Bigr]\mathsf{g}(\mathrm{d}\gamma). \tag{2.1}$$

We assume throughout that the measure σ_0 is uniformly bounded, in that

$$C_K := \sup_{x\in\mathcal{X}} \sigma_0(K-x) < \infty \tag{2.2}$$

for any compact set $K \subset \mathcal{X}$. It is also assumed that the measure σ_0 on $\mathcal{X}$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{X}$,

$$\sigma_0(\mathrm{d}x) = s(x)\,\mathrm{d}x, \tag{2.3}$$

with density $s\colon \mathcal{X} \to \mathbb{R}$ satisfying the condition

$$s(\cdot) > 0 \ \ (\mathrm{d}x\text{-a.e.}) \quad \text{and} \quad s(\cdot)^{1/2} \in H^{1,2}_{\mathrm{loc}}(\mathcal{X}), \tag{2.4}$$

where $H^{1,2}_{\mathrm{loc}}(\mathcal{X})$ stands for the local Sobolev space of order 1 in $L^2_{\mathrm{loc}}(\mathcal{X};\mathrm{d}x)$. In particular, σ_0 is a nonatomic Radon measure on $\mathcal{X}$.

Consider the space $\mathcal{X}^n = \underbrace{\mathcal{X}\times\cdots\times\mathcal{X}}_{n}$ with generic elements denoted by $\bar{x} = (x_1, \dots, x_n)$, $\bar{y} = (y_1, \dots, y_n)$, etc. The probability law of each cluster can be regarded as a measure on $\mathcal{X}^n$, which we denote by η. Assume that this measure is absolutely continuous with respect to the Lebesgue measure on $\mathcal{X}^n$,

$$\eta(\mathrm{d}\bar{y}) = h(\bar{y})\,\mathrm{d}\bar{y}, \qquad \bar{y} = (y_1, \dots, y_n) \in \mathcal{X}^n, \tag{2.5}$$

where the density h satisfies the condition

$$h(\cdot) > 0 \ \ (\mathrm{d}\bar{y}\text{-a.e.}) \quad \text{and} \quad h(\cdot)^{1/2} \in H^{1,2}_{\mathrm{loc}}(\mathcal{X}^n). \tag{2.6}$$

Our objective is to give a description of the Gibbs cluster processes in terms of Gibbs measures on suitable configuration spaces and to study them by using the extensive apparatus of calculus on configuration spaces. To this end, we need a brief introduction to analysis on configuration spaces, which is given in the next section.

3. CONFIGURATION SPACES AND MEASURES

In what follows, we always assume that $\mathcal{X}$ is Euclidean space, $\mathcal{X} = \mathbb{R}^d$. The tangent spaces $T_x\mathcal{X}$, $x \in \mathcal{X}$, are naturally identified with $\mathbb{R}^d$ and equipped with the corresponding Euclidean inner product $(\cdot\,,\cdot)_{T_x\mathcal{X}} = (\cdot\,,\cdot)_{\mathbb{R}^d}$ (denoted also by a single dot $\cdot$). Let ∇ stand for the gradient on $\mathcal{X}$.

The configuration space $\Gamma_{\mathcal{X}}$ over $\mathcal{X}$ is defined as the set of all locally finite subsets (configurations) in $\mathcal{X}$, $\Gamma_{\mathcal{X}} := \{\gamma \subset \mathcal{X} : |\gamma\cap K| < \infty \ \text{ for any compact subset } K \subset \mathcal{X}\}$, where $|B|$ stands for the cardinality of set B. We can identify each configuration $\gamma \in \Gamma_{\mathcal{X}}$ with a positive integer-valued Radon measure, $\gamma \leftrightarrow \sum_{x\in\gamma}\delta_x \in \mathcal{M}(\mathcal{X})$, where δ_x stands for the Dirac measure with unit mass at x, $\sum_{x\in\varnothing}\delta_x := 0$, and $\mathcal{M}(\mathcal{X})$ is the set of all positive Radon measures on the Borel sigma-algebra $\mathcal{B}(\mathcal{X})$. The space $\Gamma_{\mathcal{X}}$ is endowed with the relative topology as a subset of the space $\mathcal{M}(\mathcal{X})$ with the vague topology, i.e., the weakest topology on $\Gamma_{\mathcal{X}}$ such that all mappings

$$\Gamma_{\mathcal{X}} \ni \gamma \mapsto \langle f, \gamma\rangle := \textstyle\int_{\mathcal{X}} f(x)\,\gamma(\mathrm{d}x) \equiv \sum_{x\in\gamma} f(x)$$

are continuous whenever $f \in C_0(\mathcal{X})$ (:= the set of all compactly supported continuous functions on $\mathcal{X}$). Let $\mathcal{B}(\Gamma_{\mathcal{X}})$ be the corresponding Borel sigma-algebra.

For $\gamma \in \Gamma_{\mathcal{X}}$ and $x \in \gamma$, denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in $\mathcal{X}$ such that $\mathcal{O}_{\gamma,x} \cap \gamma = \{x\}$. For any measurable function $F\colon \Gamma_{\mathcal{X}} \to \mathbb{R}$, let $F_x(\gamma,\cdot)\colon \mathcal{O}_{\gamma,x} \to \mathbb{R}$ be given by $F_x(\gamma, y) := F((\gamma\setminus\{x\})\cup\{y\})$. Set $\nabla_x F(\gamma) := \nabla F_x(\gamma, y)\big|_{y=x}$ if $F_x(\gamma,\cdot)$ is differentiable at x.

Following [3], introduce the class $\mathcal{FC}(\Gamma_{\mathcal{X}})$ of functions on $\Gamma_{\mathcal{X}}$ of the form

$$F(\gamma) = f(\langle\varphi_1, \gamma\rangle, \dots, \langle\varphi_N, \gamma\rangle), \qquad \gamma \in \Gamma_{\mathcal{X}}, \tag{3.1}$$

where $N \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^N)$ (:= the set of all C^∞-functions on $\mathbb{R}^N$ bounded together with all derivatives) and $\varphi_1, \dots, \varphi_N \in C_0^\infty(\mathcal{X})$ (:= the set of all compactly supported C^∞-functions on $\mathcal{X}$).

Clearly, any $F \in \mathcal{FC}(\mathit{\Gamma}_{\mathcal{X}})$ is local, i.e., there is a compact set $K_F \subset \mathcal{X}$ such that $F(\gamma) = F(\gamma \cap K_F)$ for all $\gamma \in \mathit{\Gamma}_{\mathcal{X}}$. Thus, for any γ, there are only finitely many nonzero partial derivatives $\nabla_x F(\gamma)$.

For a given (nonatomic) measure θ on $\mathcal{X}$, we can extend the definition of the class $\mathcal{FC}(\mathit{\Gamma}_{\mathcal{X}})$ to the class $\mathcal{FC}_\theta(\mathit{\Gamma}_{\mathcal{X}})$ of functions on $\mathit{\Gamma}_{\mathcal{X}}$ of the form (3.1), where $\varphi_1, \dots, \varphi_N$ are smooth functions such that $\theta(\operatorname{supp}\varphi_i) < \infty$, $i = 1, \dots, N$. Any function $F \in \mathcal{FC}_\theta(\mathit{\Gamma}_{\mathcal{X}})$ is local, i.e., there is a set $K_F \subset \mathcal{X}$ such that $\theta(K_F) < \infty$ and $F(\gamma) = F(\gamma \cap K_F)$ for any $\gamma \in \mathit{\Gamma}_{\mathcal{X}}$, which implies that F is measurable. As in the case of functions in $\mathcal{FC}(\mathit{\Gamma}_{\mathcal{X}})$, for a chosen γ, there are only finitely many nonzero partial derivatives $\nabla_x F(\gamma)$.

4. GIBBS CLUSTER PROCESSES VIA GIBBS MEASURES

In this section, we give a description of the Gibbs cluster processes in terms of auxiliary Gibbs measures on configuration spaces of Cartesian powers of $\mathcal{X} = \mathbb{R}^d$.

Along with the space $\mathcal{X}^n$, consider the set of all n-point subsets of $\mathcal{X}$, $\mathcal{X}^{(n)} := \{A \subset \mathcal{X} : |A| = n\}$. Note that $\mathcal{X}^{(n)}$ can be represented as the quotient space

$$\mathcal{X}^{(n):} = \widetilde{\mathcal{X}}^n / S_n, \quad \text{where} \quad \widetilde{\mathcal{X}}^n := \{\bar{x} = (x_1, \dots, x_n) \in \mathcal{X}^n : x_i \neq x_j \ \text{ for } \ i \neq j\}$$

and S_n is the group of permutations of n symbols.

Let $p\colon \widetilde{\mathcal{X}}^n \to \mathcal{X}^{(n)}$ be the natural projection,

$$p(\bar{x}) := \{x_1, \dots, x_n\}, \qquad \bar{x} = (x_1, \dots, x_n). \tag{4.1}$$

Further, define the mapping $q\colon \mathcal{X} \times \widetilde{\mathcal{X}}^n \to \mathcal{X}^{(n)}$ by the formula

$$q(x, \bar{y}) := p(x + \bar{y}) = \{x + y_1, \dots, x + y_n\}, \qquad x \in \mathcal{X}, \ \ \bar{y} \in \widetilde{\mathcal{X}}^n. \tag{4.2}$$

Introduce the product measure

$$\sigma = \sigma_0 \otimes \eta \tag{4.3}$$

on the space $\mathcal{X}^{n+1} = \mathcal{X} \times \mathcal{X}^n$. According to (2.3) and (2.5), this measure is absolutely continuous, $\sigma(\mathrm{d}x, \mathrm{d}\bar{y}) = m(x, \bar{y})\, \mathrm{d}x\, \mathrm{d}\bar{y}$, $x \in \mathcal{X}$, $\bar{y} \in \mathcal{X}^n$, with the density given by

$$m(x, \bar{y}) = s(x)\, h(\bar{y}), \qquad x \in \mathcal{X}, \ \ \bar{y} \in \mathcal{X}^n. \tag{4.4}$$

Furthermore, it follows from (2.4) and (2.6) that $m(x, \bar{y})$ satisfies the condition

$$m(\cdot\,, \cdot) > 0 \quad (\mathrm{d}(x, \bar{y})\text{-a.e.}) \quad \text{and} \quad m(\cdot\,, \cdot)^{1/2} \in H^{1,2}_{\mathrm{loc}}(\mathcal{X} \times \mathcal{X}^n).$$

Consider the configuration space $\mathit{\Gamma}_{\mathcal{X} \times \mathcal{X}^n}$, with the generic elements $\widehat{\gamma} = \{(x, \bar{y}) : x \in \mathcal{X}, \bar{y} \in \mathcal{X}^n\}$. Define a pair potential $\widehat{\phi}$ on $\mathcal{X} \times \mathcal{X}^n$ by the formula

$$\widehat{\phi}((x, \bar{y}), (x', \bar{y}')) := \phi(x, x'), \qquad x, x' \in \mathcal{X}, \ \ \bar{y}, \bar{y}' \in \mathcal{X}^n. \tag{4.5}$$

For a given Gibbs measure g on the configuration space $\mathit{\Gamma}_{\mathcal{X}}$, we construct a new measure $\widehat{\mathsf{g}}$ on the space $\mathit{\Gamma}_{\mathcal{X} \times \mathcal{X}^n}$ as the distribution of random configurations $\widehat{\gamma} \in \mathit{\Gamma}_{\mathcal{X} \times \mathcal{X}^n}$ obtained from Gibbsian configurations $\gamma \in \mathit{\Gamma}_{\mathcal{X}}$ (governed by the Gibbs measure g) by attaching independent random vectors $\bar{y} = \bar{y}_x$ (with distribution η) to each point $x_i \in \gamma$,

$$\gamma \mapsto \widehat{\gamma} := \{(x, \bar{y}) : x \in \gamma, \ \bar{y} \in \mathcal{X}^n\}. \tag{4.6}$$

Geometrically, a construction of this kind can be regarded as random i.i.d. pointwise translations of Gibbs configurations from $\mathcal{X}$ to the "plane" $\mathcal{X} \times \mathcal{Y}$ ($\mathcal{Y} = \mathcal{X}^n$).

Remark 4.1. A vector $\bar{y}$ in each pair $(x, \bar{y}) \in \mathcal{X} \times \mathcal{X}^n$ can also be interpreted as a *mark* attached to the point $x \in \mathcal{X}$, and thus $\widehat{\gamma}$ becomes a *marked configuration* with the mark space $\mathcal{Y} = \mathcal{X}^n$ (see [10, 17]).

One can show that the distribution of random configurations $\widehat{\gamma}$ constructed in (4.6) is given by a Gibbs measure

$$\widehat{\mathsf{g}} \in \mathcal{G}(\sigma, \widehat{\phi}\,), \tag{4.7}$$

i.e., with the intensity measure σ given by (4.3) and the potential $\widehat{\phi}$ given by (4.5). This follows by verifying the definition of a Gibbs measure involving the Ruelle equation (see the appendix). The calculations are straightforward but tedious, and we omit the details.

Remark 4.2. The measure $\widehat{\mathsf{g}}$, originally defined on the configurations $\widehat{\gamma}$ of the form (4.6), naturally extends to a probability measure on the entire space $\mathit{\Gamma}_{\mathcal{X} \times \mathcal{X}^n}$.

Remark 4.3. Since the measure η is continuous, we have $\eta(\mathcal{X}^n \setminus \widetilde{\mathcal{X}}^n) = 0$. Hence, $\widehat{\gamma} \subset \mathcal{X} \times \widetilde{\mathcal{X}}^n$ for $\widehat{\mathsf{g}}$-almost all (a.a.) configurations $\widehat{\gamma}$.

For any set $K \subset \mathcal{X}$, write

$$\widehat{\mathcal{X}}_K := \{(x, \bar{y}) \in \mathcal{X} \times \widetilde{\mathcal{X}}^n : p(\bar{y}) \cap (K - x) \neq \varnothing\}. \tag{4.8}$$

The following result is crucial for our purposes.

Proposition 4.1. *Let $K \subset \mathcal{X}$ be a compact set. In this case,*

$$\sigma(\widehat{\mathcal{X}}_K) < \infty \tag{4.9}$$

and, for $\widehat{\mathsf{g}}$-a.a. configurations $\widehat{\gamma} \in \Gamma_{\mathcal{X}^{n+1}}$,

$$\left|\widehat{\gamma} \cap \widehat{\mathcal{X}}_K\right| < \infty. \tag{4.10}$$

Proof. Let $\mathbf{1}_A(\cdot)$ be the indicator function of a given set A. Using (4.8), note that

$$\mathbf{1}_{\widehat{\mathcal{X}}_K}(x, \bar{y}) \leqslant \sum_{i=1}^{n} \mathbf{1}_K(x + y_i). \tag{4.11}$$

Then, using (2.2), (2.5), and (4.11), we obtain

$$\begin{aligned} \sigma(\widehat{\mathcal{X}}_K) &= \int_{\mathcal{X} \times \mathcal{X}^n} \mathbf{1}_{\widehat{\mathcal{X}}_K}(x, \bar{y})\, \sigma(\mathrm{d}x, \mathrm{d}\bar{y}) \leqslant \sum_{i=1}^{n} \int_{\mathcal{X}^n} \left(\int_{\mathcal{X}} \mathbf{1}_K(x + y_i)\, \sigma_0(\mathrm{d}x) \right) \eta(\mathrm{d}\bar{y}) \\ &= \sum_{i=1}^{n} \int_{\mathcal{X}^n} \sigma_0(K - y_i)\, \eta(\mathrm{d}\bar{y}) \leqslant n\, C_K\, \eta(\mathcal{X}^n) < \infty, \end{aligned}$$

and (4.9) is proved. The bound (4.10) now follows by the general theory of Gibbs measures.

We can extend the mapping q defined by (4.2) to any configuration $\widehat{\gamma} \in \Gamma_{\mathcal{X}^{n+1}}$ by setting

$$q(\widehat{\gamma}) := \bigcup_{(x,\, \bar{y}) \in \widehat{\gamma}} (p(\bar{y}) + x) \subset \mathcal{X}. \tag{4.12}$$

Remark 4.4. For $\widehat{\mathsf{g}}$-a.a. $\widehat{\gamma}$, the sets $p(\bar{y}) + x$ are mutually disjoint for different $(x, \bar{y}) \in \widehat{\gamma}$. This holds because, for any $m = 2, 3, \dots$ and $\sigma^{\otimes m}$-a.a. m-tuples $((x_1, \bar{y}_1), \dots, (x_m, \bar{y}_m))$, $(x_i, \bar{y}_i) \in \mathcal{X} \times \widetilde{\mathcal{X}}^n$, we have $(p(\bar{y}_i) + x_i) \cap (p(\bar{y}_j) + x_j) = \varnothing$ if $i \neq j$.

Remark 4.5. In fact, Remarks 4.3 and 4.4 hold because all diagonals in the space $(\mathcal{X} \times \mathcal{X}^n)^m$ have $\sigma^{\otimes m}$-measure zero.

Let us point out that, in general, the set $q(\widehat{\gamma})$ may be not locally finite. Thus, formula (4.12) defines the mapping $q\colon \Gamma_{\mathcal{X}^{n+1}} \to \Sigma_{\mathcal{X}}$ into the space $\Sigma_{\mathcal{X}}$ of configurations in $\mathcal{X}$ *with accumulation points* (i.e., the space of all countable subsets of $\mathcal{X}$). However, we have the following result.

Proposition 4.2. *For $\widehat{\mathsf{g}}$-a.a. configurations $\widehat{\gamma} \in \Gamma_{\mathcal{X}^{n+1}}$, we have*

$$q(\widehat{\gamma}) \in \Gamma_{\mathcal{X}}. \tag{4.13}$$

Proof. If a configuration $\gamma = q(\widehat{\gamma})$ is not locally finite, then there is a compact set $K \subset \mathcal{X}$ such that $|\gamma \cap K| = \infty$, and hence $\left|\widehat{\gamma} \cap \widehat{\mathcal{X}}_K\right| = \infty$. However, according to Proposition 4.1, this is impossible for $\widehat{\mathsf{g}}$-a.a. configurations $\widehat{\gamma}$. Hence, (4.13) follows.

Next, we introduce the measure μ on $\Sigma_{\mathcal{X}}$ as the image of g under the mapping q,

$$\mu := q^* \mathsf{g}, \tag{4.14}$$

i.e., $\mu(A) := \mathsf{g}\,(q^{-1}(A))$, $A \subset \Sigma_{\mathcal{X}}$, or, equivalently, $\int_{\Sigma_{\mathcal{X}}} F(\gamma)\, \mu(\mathrm{d}\gamma) = \int_{\Gamma_{\mathcal{X}^{n+1}}} F(q(\widehat{\gamma}))\, \widehat{\mathsf{g}}\,(\mathrm{d}\widehat{\gamma})$.

Proposition 4.3. *The measure μ is concentrated on the configuration space $\Gamma_{\mathcal{X}}$, $\mu(\Gamma_{\mathcal{X}}) = 1$.*

Proof. This follows from (4.13).

Our next goal is to identify μ with the distribution of the Gibbs cluster process (see Section 2). We need the following technical lemma.

Lemma 4.4. *Let $F\colon \mathcal{X}\times\mathcal{X}^n\to\mathbb{R}$ be a continuous bounded function such that $\sigma(\operatorname{supp}F)<\infty$. Then*

$$\int_{\Gamma_{\mathcal{X}^{n+1}}}\exp\Big(\sum_{(x,\,\bar y)\in\widehat\gamma}F(x,\bar y)\Big)\widehat{\mathsf g}(\mathrm d\widehat\gamma)=\int_{\Gamma_{\mathcal X}}\prod_{x\in\gamma}\Big(\int_{\mathcal X^n}\mathrm e^{F(x,\,\bar y)}\,\eta(\mathrm d\bar y)\Big)\,\mathsf g(\mathrm d\gamma). \tag{4.15}$$

Proof. By an approximation argument (via the thermodynamic limit in the coordinate $x\in\mathcal X$), it suffices to prove the lemma for any function F such that $\operatorname{supp}F=K\times\mathcal X^n$, where $K\subset\mathcal X$ is a compact set. For (finite) configurations $\zeta\in\Gamma_K$, denote $S(\zeta):=\int_{\Gamma_{\mathcal X\setminus K}}\mathrm e^{-W(\zeta,\,\gamma')}\,\mathsf g(\mathrm d\gamma')$, where $W(\zeta,\gamma')$ is the interaction energy between the configuration $\zeta\subset K$ and an "external" configuration $\gamma'\subset\mathcal X\setminus K$ (see the appendix). Similarly, for finite configurations $\widehat\zeta\in\Gamma_{K\times\mathcal X^n}$, set $\widehat S(\widehat\zeta):=\int_{\Gamma_{(\mathcal X\setminus K)\times\mathcal X^n}}\mathrm e^{-\widehat W(\widehat\zeta,\,\widehat\gamma')}\,\widehat{\mathsf g}\,(\mathrm d\widehat\gamma\,')$. Recalling the projection structure of the measure $\widehat{\mathsf g}$ constructed in (4.7), one can see that

$$\widehat S(\widehat\zeta)=S(p_1(\widehat\zeta)),\qquad \widehat\zeta\in\Gamma_{K\times\mathcal X^n}, \tag{4.16}$$

where $p_1:\Gamma_{K\times\mathcal X^n}\to\Gamma_{\mathcal X}$ is the mapping induced by the "coordinate" projection $K\times\mathcal X^n\ni(x,\bar y)\mapsto x\in K$. Further, setting $\mathcal S(\zeta):=\mathrm e^{-E(\zeta)}\,S(\zeta)$, $\zeta\in\Gamma_K$, by (4.16), we have $\mathcal S(p_1(\widehat\zeta))=\mathrm e^{-E(\widehat\zeta)}\,\widehat S(\widehat\zeta)$, $\widehat\zeta\in\Gamma_{K\times\mathcal X^n}$.

Now, using the definition of Lebesgue–Poisson measure (see the appendix, equation (8.1)) as applied to the measure $\widehat{\mathsf g}$ and the volume $\Lambda:=K\times\mathcal X^n$ (which is finite with respect to the measure $\sigma=\sigma_0\otimes\eta$), together with the Ruelle equation (see the appendix, equation (8.2)), we obtain

$$\begin{aligned}\int_{\Gamma_{\mathcal X^{n+1}}}\exp\Big(\sum_{(x,\,\bar y)\in\widehat\gamma}F(x,\bar y)\Big)\widehat{\mathsf g}\,(\mathrm d\widehat\gamma)&=\int_{\Gamma_{K\times\mathcal X^n}}\prod_{(x,\,\bar y)\in\widehat\zeta}\mathrm e^{F(x,\,\bar y)}\,\mathcal S(p_1(\widehat\zeta))\,\lambda_\sigma(\mathrm d\widehat\zeta)\\&=\sum_{m=0}^\infty\frac1{m!}\int_{(K\times\mathcal X^n)^m}\prod_{k=1}^m\mathrm e^{F(x_k,\,\bar y_k)}\,\mathcal S(p_1(\{(x_1,\bar y_1),\dots,(x_m,\bar y_m)\}))\,\sigma(\mathrm dx_k,\mathrm d\bar y_k)\\&=\sum_{m=0}^\infty\frac1{m!}\int_{(K\times\mathcal X^n)^m}\prod_{k=1}^m\mathrm e^{F(x_k,\bar y_k)}\,\mathcal S(\{(x_1,\dots,x_m\})\,\sigma_0(\mathrm dx_k)\,\eta(\mathrm d\bar y_k)\\&=\sum_{m=0}^\infty\frac1{m!}\int_{K^m}\mathcal S(\{x_1,\dots,x_m\})\prod_{k=1}^m\Big(\int_{\mathcal X^n}\mathrm e^{F(x_k,\,\bar y_k)}\,\eta(\mathrm d\bar y_k)\Big)\sigma_0(\mathrm dx_1)\cdots\sigma_0(\mathrm dx_m).\end{aligned} \tag{4.17}$$

Assembling this expression back to the configurational form (i.e., as an integral over the space Γ_K with respect to the Lebesgue–Poisson measure λ_{σ_0}), we can represent the right-hand side of (4.17) as

$$\begin{aligned}\int_{\Gamma_K}\mathcal S(\zeta)\prod_{x\in\zeta}\Big(\int_{\mathcal X^n}\mathrm e^{F(x,\,\bar y)}\,\eta(\mathrm d\bar y)\Big)\lambda_{\sigma_0}(\mathrm d\zeta)\qquad&\\=\int_{\Gamma_K}\mathrm e^{-E(\zeta)}\Big(\int_{\Gamma_{\mathcal X\setminus K}}\mathrm e^{-W(\zeta,\,\gamma')}\,\mathsf g(\mathrm d\gamma')\Big)&\prod_{x\in\zeta}\Big(\int_{\mathcal X^n}\mathrm e^{F(x,\,\bar y)}\,\eta(\mathrm d\bar y)\Big)\lambda_{\sigma_0}(\mathrm d\zeta)\\=\int_{\Gamma_{\mathcal X}}\prod_{x\in\gamma}\Big(\int_{\mathcal X^n}\mathrm e^{F(x,\,\bar y)}\,\eta(\mathrm d\bar y)\Big)\mathsf g(\mathrm d\gamma),&\end{aligned}$$

where, in the last equality, we have again used the Ruelle equation applied to the measure $\mathsf g$ and the σ_0-finite volume K. Thus, the lemma is proved.

Theorem 4.5. *The measure μ on $\Gamma_{\mathcal X}$ defined by* (4.14) *coincides with the distribution of the Gibbs cluster process.*

Proof. Let us compute the Laplace transform of the measure μ. For any $f\in C_0(\mathcal X)$, we have

$$\int_{\Gamma_{\mathcal X}}\mathrm e^{\langle f,\gamma\rangle}\,\mu(\mathrm d\gamma)=\int_{\Gamma_{\mathcal X^{n+1}}}\mathrm e^{\langle f,\,q(\widehat\gamma)\rangle}\,\widehat{\mathsf g}(\mathrm d\widehat\gamma)=\int_{\Gamma_{\mathcal X\times\mathcal X^n}}\exp\Big(\sum_{(x,\,\bar y)\in\widehat\gamma}F(x+\bar y)\Big)\widehat{\mathsf g}(\mathrm d\widehat\gamma), \tag{4.18}$$

where $F(\bar y) := \sum_{y\in\bar y} f(y)$. If $K := \operatorname{supp} f$, then $\operatorname{supp} F = \widehat{\mathcal X}_K$ (see (4.8)). Hence, $\sigma(\operatorname{supp} F) < \infty$ by Proposition 4.1, which implies that the number of nonzero terms in the sum in (4.18) is finite for $\widehat{\mathsf g}$-a.a. $\widehat\gamma$. Applying formula (4.15) to the right-hand side of (4.18), we obtain

$$\int_{\Gamma_{\mathcal X}} \mathrm e^{\langle f,\gamma\rangle}\,\mu(\mathrm d\gamma) = \int_{\Gamma_{\mathcal X}} \prod_{x\in\gamma}\Bigl[\int_{\mathcal X^n} \exp\Bigl(\sum_{y\in\bar y} f(x+y)\Bigr)\eta(\mathrm d\bar y)\Bigr]\mathsf g(\mathrm d\gamma),$$

which coincides with expression (2.1) for the Laplace transform of the Gibbs cluster process.

5. QUASI-INVARIANCE

Let $Q\colon L^2(\Gamma_{\mathcal X},\mu)\to L^2(\Gamma_{\mathcal X^{n+1}},\widehat{\mathsf g})$ be the isometry defined by the mapping q, i.e., $(QF)(\widehat\gamma) := F(q(\widehat\gamma))$, $\widehat\gamma\in\Gamma_{\mathcal X^{n+1}}$, and let $Q^*\colon L^2(\Gamma_{\mathcal X^{n+1}},\widehat{\mathsf g})\to L^2(\Gamma_{\mathcal X},\mu)$ be the adjoint operator.

Let $\operatorname{Diff}_0(\mathcal X)$ be the group of compactly supported diffeomorphisms of $\mathcal X$ (i.e., each $\varphi\in\operatorname{Diff}_0(\mathcal X)$ is reduced to the identity mapping outside a compact set, $\operatorname{supp}\varphi$). For any $\varphi\in\operatorname{Diff}_0(\mathcal X)$, introduce the diffeomorphism $\widehat\varphi$ of $\mathcal X\times\mathcal X^n$ by the rule $\widehat\varphi(x;y_1,\dots,y_n) := (x;\varphi(x+y_1)-x,\dots,\varphi(x+y_n)-x)$.

Remark 5.1. Let us point out that $\widehat\varphi\notin\operatorname{Diff}_0(\mathcal X\times\mathcal X^n)$. Indeed, $\operatorname{supp}\widehat\varphi = \widehat{\mathcal X}_K$, where $K := \operatorname{supp}\varphi$. Note that $\widehat{\mathcal X}_K$ is not compact; however, $\sigma(\widehat{\mathcal X}_K)<\infty$, which is sufficient for our purposes.

The transformations φ and $\widehat\varphi$ can be lifted to the "diagonal" transformations of the configuration spaces $\Gamma_{\mathcal X}$ and $\Gamma_{\mathcal X\times\mathcal X^n}$, where $\varphi(\{\dots,x,y,z,\dots\}) := \{\dots,\varphi(x),\varphi(y),\varphi(z),\dots\}$ and $\widehat\varphi(\{\dots,(x,\bar x),(y,\bar y),(z,\bar z),\dots\}) := \{\dots,\widehat\varphi(x,\bar x),\widehat\varphi(y,\bar y),\widehat\varphi(z,\bar z),\dots\}$. The next assertion is obvious.

Lemma 5.1. *For any* $\widehat\gamma\in\Gamma_{\mathcal X}$,

$$\varphi(q(\widehat\gamma)) = q(\widehat\varphi(\widehat\gamma)), \tag{5.1}$$

and $Q(F\circ\varphi) = (QF)\circ\widehat\varphi$ *for* $F\in L^2(\Gamma_{\mathcal X},\mu)$.

Note that the measure η is quasi-invariant with respect to the action of $\operatorname{Diff}_0(\mathcal X^n)$, i.e., for any $\psi\in\operatorname{Diff}_0(\mathcal X^n)$, the measure $\psi^*\eta := \eta\circ\psi^{-1}$ is absolutely continuous with respect to η. Denote by $\rho(\psi,\bar y)$ the corresponding Radon–Nikodym density, $\psi^*\eta(\mathrm d\bar y) = \rho(\psi,\bar y)\,\eta(\mathrm d\bar y)$, $\bar y\in\mathcal X^n$. In particular, for a given $\varphi\in\operatorname{Diff}_0(\mathcal X)$, consider the diffeomorphism $\widehat\varphi_x$ ($x\in\mathcal X$) defined by

$$\widehat\varphi_x(\bar y) = (\varphi(x+y_1)-x,\dots,\varphi(x+y_n)-x),\qquad \bar y = (y_1,\dots,y_n)\in\mathcal X^n. \tag{5.2}$$

Write $\rho_\varphi(x,\bar y) := \rho(\widehat\varphi_x,\bar y)$. Then, using the structure of the measure $\sigma=\sigma_0\otimes\eta$, one can see that σ is quasi-invariant with respect to $\widehat\varphi$, and the corresponding Radon–Nikodym density is equal to ρ_φ, $(\widehat\varphi^*\sigma)(\mathrm dx,\mathrm d\bar y) = \rho_\varphi(x,\bar y)\,\sigma(\mathrm dx,\mathrm d\bar y)$.

Proposition 5.2. *The Gibbs measure* $\widehat{\mathsf g}$ *is quasi-invariant with respect to the action of diffeomorphisms* $\widehat\varphi$ *on* $\Gamma_{\mathcal X^{n+1}}$ *with the Radon–Nikodym density* $R^{\widehat\varphi}_{\widehat{\mathsf g}} := \mathrm d(\widehat\varphi^*\,\widehat{\mathsf g})/\mathrm d\,\widehat{\mathsf g}$ *given by the formula*

$$R^{\widehat\varphi}_{\widehat{\mathsf g}}(\widehat\gamma) = \prod_{(x,\,\bar y)\in\widehat\gamma}\rho_\varphi(x,\bar y)\cdot\exp\Bigl(\int_{\mathcal X\times\mathcal X^n}(1-\rho_\varphi(x,\bar y))\,\sigma(\mathrm dx,\mathrm d\bar y)\Bigr). \tag{5.3}$$

Proof. This follows from the general theory of Gibbs measures similar to [4, 15]. Observe that the density ρ_φ is equal to 1 outside the set $\widehat{\mathcal X}_K$, where $K=\operatorname{supp}\varphi$, and $\sigma(\widehat{\mathcal X}_K)<\infty$ (cf. Remark 5.1), which implies that the product on the right-hand side of (5.3) contains only finitely many terms not equal to 1 (for $\widehat{\mathsf g}$-a.a. $\widehat\gamma$). Thus, $R^{\widehat\varphi}_{\widehat{\mathsf g}}(\widehat\gamma)$ is well defined for $\widehat{\mathsf g}$-a.a. $\widehat\gamma$.

Remark 5.2. $R^{\widehat\varphi}_{\widehat{\mathsf g}}$ is local, i.e., $R^{\widehat\varphi}_{\widehat{\mathsf g}}(\widehat\gamma) = R^{\widehat\varphi}_{\widehat{\mathsf g}}(\widehat\gamma\cap\widehat{\mathcal X}_K)$ for $\widehat{\mathsf g}$-a.a. $\widehat\gamma$.

Remark 5.3 [explicit form of R]. According to (4.4) and (5.2), we have

$$\rho_\varphi(x,\bar y) = \frac{h(\varphi^{-1}(y_1+x)-x,\dots,\varphi^{-1}(y_n+x)-x)}{h(y_1,\dots,y_n)}\prod_{i=1}^n J(\varphi)^{-1}(y_i),$$

where $J(\varphi)$ is the Jacobian determinant of φ. Then $R^{\widehat\varphi}_{\widehat{\mathsf g}}(\widehat\gamma)$ can be calculated by using formula (5.3). In particular, if $h(y_1,\dots,y_n) = \prod_{i=1}^n f(y_i)$ (i.e., the components of the random vector $(Y_1,\dots,Y_n)$

are i.i.d.), then we have $\rho_\varphi(x,\bar y)=\prod_{i=1}^n J(\varphi)^{-1}(y_i)f(\varphi^{-1}(y_i+x)-x)\,/\prod_{i=1}^n f(y_i)$ and $R^{\widehat{\varphi}}_{\widehat{\mathsf g}}(\widehat{\gamma})=C\prod_{(x,\,\bar y)\in\widehat{\gamma}}\prod_{i=1}^n J(\varphi)^{-1}(y_i)f(\varphi^{-1}(y_i+x)-x)\,/\prod_{i=1}^n f(y_i)$, where C is the normalizing constant, $C=\exp\Big\{\int_{\mathcal X\times\mathcal X^n}(1-\rho_\varphi(x,\bar y))\,\sigma(\mathrm dx,\mathrm d\bar y)\Big\}$.

Theorem 5.3. *The Gibbs cluster measure μ is quasi-invariant with respect to the action of* $\mathrm{Diff}_0(\mathcal X)$ *on $\varGamma_{\mathcal X}$. The corresponding Radon–Nikodym density is given by $R^\varphi_\mu=Q^*R^{\widehat{\varphi}}_{\widehat{\mathsf g}}$.*

Proof. Let us first note that, because of (5.1), the measure $\varphi^*\mu$ is the image of the measure $(\widehat\varphi)^*\,\mathsf g$ under the mapping q. Therefore, the absolute continuity $(\widehat\varphi)^*\,\mathsf g\ll\mathsf g$ implies that $\varphi^*\mu\ll\mu$. Moreover,

$$\begin{aligned}\int_{\varGamma_{\mathcal X}}F(\gamma)\,\varphi^*\mu(\mathrm d\gamma)&=\int_{\varGamma_{\mathcal X\times\mathcal X^n}}QF(\widehat\gamma)\,(\widehat\varphi)^*\,\widehat{\mathsf g}(\mathrm d\widehat\gamma)\\&=\int_{\varGamma_{\mathcal X\times\mathcal X^n}}QF(\widehat\gamma)\,R^{\widehat\varphi}_{\widehat{\mathsf g}}(\widehat\gamma)\,\widehat{\mathsf g}\,(\mathrm d\widehat\gamma)=\int_{\varGamma_{\mathcal X}}F(\gamma)\,\Big(Q^*R^{\widehat\varphi}_{\widehat{\mathsf g}}\Big)(\gamma)\,\mu(\mathrm d\gamma),\end{aligned}$$

which completes the proof.

Remark 5.4. Observe that, because of the structure of the diffeomorphism $\widehat\varphi$, the density $R^{\widehat\varphi}_{\widehat{\mathsf g}}$ does not depend on the pair potential ϕ. The density R^φ_μ does depend on ϕ via the projection Q^*.

Remark 5.5. The Gibbs cluster measure μ on the configuration space $\varGamma_{\mathcal X}$ can be used to construct a unitary representation U of the diffeomorphism group $\mathrm{Diff}_0(\mathcal X)$ by operators in $L^2(\varGamma_{\mathcal X},\mu)$, given by the formula

$$U_\varphi F(\gamma)=\sqrt{R^\varphi_\mu(\gamma)}\,F(\varphi^{-1}(\gamma)),\qquad F\in L^2(\varGamma_{\mathcal X},\mu).\tag{5.4}$$

Such representations, which can be defined for arbitrary quasi-invariant measures on $\varGamma_{\mathcal X}$, play a significant role in the representation theory of the group $\mathrm{Diff}_0(\mathcal X)$ [14, 24] and in quantum field theory [12, 13]. An important question is whether the representation (5.4) is irreducible. According to [24], this is equivalent to the $\mathrm{Diff}_0(\mathcal X)$-ergodicity of the measure μ, which in our case is equivalent to the ergodicity of the measure $\mathsf g$ with respect to the group of transformations $\widehat\varphi$ $(\varphi\in\mathrm{Diff}_0(\mathcal X))$. The latter is an open question.

6. INTEGRATION-BY-PARTS FORMULA

Let $v\in\mathrm{Vect}_0(\mathcal X)$ (:= the space of compactly supported smooth vector fields on $\mathcal X$), and define a vector field $\hat v_x$ on $\mathcal X^n$ by the formula $\hat v_x(\bar y):=(v(y_1+x),\dots,v(y_n+x))$, $\bar y=(y_1,\dots,y_n)\in\mathcal X^n$. Observe that the measure σ satisfies the following integration-by-parts formula,

$$\int_{\mathcal X\times\mathcal X^n}\nabla^{\hat v_x}f(x,\bar y)\,\sigma(\mathrm dx,\mathrm d\bar y)=-\int_{\mathcal X\times\mathcal X^n}f(x,\bar y)\,\beta^{\hat v}(x,\bar y)\,\sigma(\mathrm dx,\mathrm d\bar y),\qquad f\in C_0^\infty(\mathcal X\times\mathcal X^n),$$

where $\nabla^{\hat v_x}$ is the derivative along the vector field $\hat v_x$,

$$\beta^{\hat v}(x,\bar y):=(\beta_\eta(\bar y),\hat v_x(\bar y))_{T_{\bar y}\mathcal X^n}+\operatorname{div}\hat v_x(\bar y)\tag{6.1}$$

is the logarithmic derivative of $\sigma(\mathrm dx,\mathrm d\bar y)=m(x,\bar y)\,\mathrm dx\,\mathrm d\bar y$ along $\hat v_x$, and $\beta_\eta(\bar y):=\nabla h(\bar y)/h(\bar y)$, $\bar y\in\mathcal X^n$. This fact leads to the following theorem.

Theorem 6.1. *The Gibbs cluster measure μ satisfies the integration-by-parts formula*

$$\int_{\varGamma_{\mathcal X}}\sum_{x\in\gamma}\nabla_xF(\gamma)\cdot v(x)\,\mu(\mathrm d\gamma)=-\int_{\varGamma_{\mathcal X}}F(\gamma)\,B^v_\mu(\gamma)\,\mu(\mathrm d\gamma),\qquad F\in\mathcal{FC}(\varGamma_{\mathcal X}),\tag{6.2}$$

where $B^v_\mu(\gamma):=Q^*\langle\beta^{\hat v},\widehat\gamma\rangle$ and the logarithmic derivative $\beta^{\hat v}$ is defined in (6.1).

Proof. It is clear that $\operatorname{supp}\hat v=\widehat{\mathcal X}_K$, where $K:=\operatorname{supp}v$, and thus we have $\sigma(\operatorname{supp}\hat v)<\infty$ by Proposition 4.1. Note that, for any $F\in\mathcal{FC}(\varGamma_{\mathcal X^{n+1}})$, the following integration-by-parts formula holds:

$$\int_{\varGamma_{\mathcal X^{n+1}}}\sum_{(x,\,\bar y)\in\widehat\gamma}\nabla_{\bar y}F(\widehat\gamma)\cdot\hat v_x(\bar y)\,\widehat{\mathsf g}\,(\mathrm d\widehat\gamma)=-\int_{\varGamma_{\mathcal X^{n+1}}}F(\widehat\gamma)\,B^{\hat v}_{\widehat{\mathsf g}}(\widehat\gamma)\,\widehat{\mathsf g}\,(\mathrm d\widehat\gamma),\tag{6.3}$$

where

$$B_{\widehat{\mathsf{g}}}^{\hat{v}}(\widehat{\gamma}) := \langle \beta^{\hat{v}}, \widehat{\gamma} \rangle. \tag{6.4}$$

Note that $B_{\widehat{\mathsf{g}}}^{\hat{v}}$ is well defined because $\sigma(\operatorname{supp} \hat{v}) < \infty$, which implies formula (6.3) in a standard way (cf. [3, 4]).

Write $\Phi(\gamma) := \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot v(x)$. It is obvious that $Q\,\Phi(\widehat{\gamma}) = \sum_{(x,\,\bar{y}) \in \widehat{\gamma}} \nabla_{\bar{y}} QF(\widehat{\gamma}) \cdot \hat{v}_x(\bar{y})$. Thus,

$$\begin{aligned}\int_{\Gamma_{\mathcal{X}}} \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot v(x)\, \mu(\mathrm{d}\gamma) &= \int_{\Gamma_{\mathcal{X}^{n+1}}} Q\,\Phi(\widehat{\gamma})\, \widehat{\mathsf{g}}\,(\mathrm{d}\widehat{\gamma}) = \int_{\Gamma_{\mathcal{X}^{n+1}}} \sum_{(x,\,\bar{y}) \in \widehat{\gamma}} \nabla_{\bar{y}}\, QF(\widehat{\gamma}) \cdot \hat{v}_x(\bar{y})\, \widehat{\mathsf{g}}\,(\mathrm{d}\widehat{\gamma}) \\ &= -\int_{\Gamma_{\mathcal{X}^{n+1}}} QF(\widehat{\gamma})\, B_{\widehat{\mathsf{g}}}^{\hat{v}}(\widehat{\gamma})\, \widehat{\mathsf{g}}\,(\mathrm{d}\widehat{\gamma}) = -\int_{\Gamma_{\mathcal{X}}} F(\gamma)\, Q^* B_{\widehat{\mathsf{g}}}^{\hat{v}}(\gamma)\, \mu(\mathrm{d}\gamma),\end{aligned}$$

and the theorem is proved.

Formula (6.2) can be extended to more general vector fields on $\Gamma_{\mathcal{X}}$, i.e., mappings $\Gamma_{\mathcal{X}} \ni \gamma \mapsto V(\gamma) \in T_\gamma \Gamma_{\mathcal{X}} := \bigoplus_{x \in \gamma} T_x \mathcal{X}$ (of course, $T_x \mathcal{X} = \mathbb{R}^d$; however, we prefer to keep the manifold-like notation here). For any such V of the form $V(\gamma) = (V(\gamma)_x)_{x \in \gamma}$, $V(\gamma)_x = \sum_{j=1}^N G_j(\gamma)\, v_j(x) \in T_x \mathcal{X}$, where $G_j \in \mathcal{FC}(\Gamma_{\mathcal{X}})$ and $v_j \in \mathrm{Vect}_0(\mathcal{X})$, $j = 1, \dots, N$, set $B_\mu^V(\gamma) := \big(Q^* B_{\widehat{\mathsf{g}}}^{QV}\big)(\gamma)$, where $B_{\widehat{\mathsf{g}}}^{QV}(\widehat{\gamma})$ is the logarithmic derivative of $\widehat{\mathsf{g}}$ along $QV(\widehat{\gamma}) := V(q(\widehat{\gamma}))$ (see [3]). Note that QV is a vector field on $\Gamma_{\mathcal{X}^{n+1}}$ because of the obvious equality $T_{\widehat{\gamma}} \Gamma_{\mathcal{X}^{n+1}} = T_{q(\widehat{\gamma})} \Gamma_{\mathcal{X}}$. Clearly, $B_\mu^V(\gamma) = \sum_{j=1}^N \Big(G_j(\gamma) B_\mu^{v_j}(\gamma) + \sum_{x \in \gamma} \nabla_x G_j(\gamma) \cdot v_j(x) \Big)$.

Theorem 6.2. *For arbitrary* $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{X}})$ *and* V *as above, we have*

$$\begin{aligned}&\int_{\Gamma_{\mathcal{X}}} \sum_{x \in \gamma} \nabla_x F_1(\gamma) \cdot V(\gamma)_x\, F_2(\gamma)\, \mu(\mathrm{d}\gamma) \\ &\qquad = -\int_{\Gamma_{\mathcal{X}}} F_1(\gamma) \sum_{x \in \gamma} \nabla_x F_2(\gamma) \cdot V(\gamma)_x\, \mu(\mathrm{d}\gamma) - \int_{\Gamma_{\mathcal{X}}} F_1(\gamma)\, F_2(\gamma)\, B_\mu^V(\gamma)\, \mu(\mathrm{d}\gamma).\end{aligned}$$

Proof. The proof can be obtained by a straightforward generalization of the arguments used in the proof of Theorem 6.1.

7. THE DIRICHLET FORM AND EQUILIBRIUM STOCHASTIC DYNAMICS

Let us introduce the pre-Dirichlet form $\mathcal{E}_\mu$ associated with μ, defined on functions $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{X}})$ by the formula

$$\mathcal{E}_\mu(F_1, F_2) := \int_{\Gamma_{\mathcal{X}}} \sum_{x \in \gamma} \nabla_x F_1(\gamma) \cdot \nabla_x F_2(\gamma)\, \mu(\mathrm{d}\gamma).$$

We also consider the pre-Dirichlet form $\mathcal{E}_{\widehat{\mathsf{g}}}$ associated with the Gibbs measure $\widehat{\mathsf{g}}$, defined on the space $\mathcal{FC}(\Gamma_{\mathcal{X}^{n+1}}) \subset L^2(\Gamma_{\mathcal{X}^{n+1}}, \widehat{\mathsf{g}}\,)$ as

$$\mathcal{E}_{\widehat{\mathsf{g}}}(\Phi_1, \Phi_2) := \int_{\Gamma_{\mathcal{X}^{n+1}}} \sum_{\hat{x} \in \widehat{\gamma}} (\nabla_{\hat{x}}\, \Phi_1(\widehat{\gamma}), \nabla_{\hat{x}}\, \Phi_2(\widehat{\gamma}))_{T_{\hat{x}} \mathcal{X}^{n+1}}\, \widehat{\mathsf{g}}\,(\mathrm{d}\widehat{\gamma}). \tag{7.1}$$

As is known (see [3]), expression (7.1) can be reduced to the form

$$\mathcal{E}_{\widehat{\mathsf{g}}}(\Phi_1, \Phi_2) = \int_{\Gamma_{\mathcal{X}^{n+1}}} H_{\widehat{\mathsf{g}}}\, \Phi_1(\widehat{\gamma})\, \Phi_2(\widehat{\gamma})\, \widehat{\mathsf{g}}\,(\mathrm{d}\widehat{\gamma}), \qquad \Phi_1, \Phi_2 \in \mathcal{FC}(\Gamma_{\mathcal{X}^{n+1}}), \tag{7.2}$$

where $H_{\widehat{\mathsf{g}}}$ is the Dirichlet operator of the Gibbs measure $\widehat{\mathsf{g}}$. One can readily see that both $\mathcal{E}_{\widehat{\mathsf{g}}}$ and $H_{\widehat{\mathsf{g}}}$ can be defined on the bigger space $\mathcal{FC}_\sigma(\Gamma_{\mathcal{X}^{n+1}})$ (cf. the end of Section 3).

Theorem 7.1. *For* $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{X}})$,

$$\mathcal{E}_\mu(F_1, F_2) = \int_{\Gamma_{\mathcal{X}}} H_\mu F_1(\gamma)\, F_2(\gamma)\, \mu(\mathrm{d}\gamma), \tag{7.3}$$

where $H_\mu := Q^* H_{\widehat{\mathbf{g}}} Q$.

Proof. Note that if $F_1, F_2 \in \mathcal{FC}(\Gamma_{\mathcal{X}})$ then $QF_1, QF_2 \in \mathcal{FC}_\sigma(\Gamma_{\mathcal{X}^{n+1}})$. Furthermore, it can be shown by a direct calculation that

$$\mathcal{E}_\mu(F_1, F_2) = \mathcal{E}_{\widehat{\mathbf{g}}}(QF_1, QF_2), \tag{7.4}$$

and (7.3) follows.

Formula (7.3) implies that the form $\mathcal{E}_\mu$ is closable. It follows from the properties of the *carré du champ* $\sum_{x\in\gamma} \nabla_x F_1(\gamma) \cdot \nabla_x F_2(\gamma)$ that its closure is a quasi-regular local Dirichlet form (see [20]) on the bigger state space $\ddot{\Gamma}_{\mathcal{X}}$ consisting of all $\mathbb{Z}_+$-valued Radon measures on $\mathcal{X}$. By the general theory of Dirichlet forms (see [19]), this implies the following result.

Theorem 7.2. *There exists a conservative diffusion process* $\mathbf{M}$ *on* $\ddot{\Gamma}_{\mathcal{X}}$,

$$\mathbf{M} = \Bigl(\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\geqslant 0},\ (\Theta_t)_{t\geqslant 0},\ (\mathcal{X}_t)_{t\geqslant 0},\ (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_{\mathcal{X}}}\Bigr),$$

properly associated with $\mathcal{E}_\mu$, *i.e., for all* (μ*-versions of*) $F \in L^2(\ddot{\Gamma}_{\mathcal{X}}, \mu)$ *and all* $t > 0$, *the function*

$$\gamma \mapsto p_t F(\gamma) := \int_\Omega F(\mathbf{X}_t)\, \mathrm{d}\mathbf{P}_\gamma \quad (\gamma \in \ddot{\Gamma}_{\mathcal{X}})$$

is an $\mathcal{E}_\mu$*-quasi-continuous version of* $\exp(-tH_\mu)\, F$. *The process* $\mathbf{M}$ *is unique up to* μ*-equivalence. In particular,* $\mathbf{M}$ *is* μ*-symmetric* (*i.e.,* $\int F_1\, p_t F_2\, \mathrm{d}\mu = \int F_2\, p_t F_1\, \mathrm{d}\mu$ *for all measurable functions* $F_1, F_2 \colon \ddot{\Gamma}_{\mathcal{X}} \to \mathbb{R}_+$) *and has* μ *as an invariant measure.*

Remark 7.1. Formula (7.2) implies that the "pre-projection" form $\mathcal{E}_{\widehat{\mathbf{g}}}$ is closable. According to the general theory of Dirichlet forms on configuration spaces, its closure is a quasi-regular local Dirichlet form on $\ddot{\Gamma}_{\mathcal{X}^{n+1}}$, and therefore generates a diffusion process $\mathbf{M}^{(n+1)}$ on $\ddot{\Gamma}_{\mathcal{X}^{n+1}}$ (see [4, 19, 20]). However, it is not clear in what sense the process $\mathbf{M}$ constructed in Theorem 7.2 can be obtained directly via the projection of $\mathbf{M}^{(n+1)}$ from $\ddot{\Gamma}_{\mathcal{X}^{n+1}}$ onto $\ddot{\Gamma}_{\mathcal{X}}$.

8. APPENDIX: GIBBS MEASURES ON CONFIGURATION SPACES

Here we briefly discuss the definition and some properties of Gibbs measures on $\Gamma_{\mathcal{X}}$ with pair potentials. For details, see the classical books [11, 21, 23], and also [4, 15–17].

Let $\phi\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ be a measurable symmetric function (called *pair interaction potential*) such that $\phi(x,y) \in \mathbb{R}$ for all $x \neq y$. Consider the space $\Gamma_{\mathcal{X},0}$ of finite configurations in $\mathcal{X}$ and define the *energy* $E\colon \Gamma_{\mathcal{X},0} \to \mathbb{R}\cup\{+\infty\}$ by $E(\zeta) := \sum_{\{x,\,y\}\subset\zeta} \phi(x,y)$ ($\zeta \in \Gamma_{\mathcal{X},0}$), $E(\varnothing) := 0$. For $\zeta \in \Gamma_{\mathcal{X},0}$ and any $\gamma \in \Gamma_{\mathcal{X}}$, the *interaction energy* between ζ and γ is defined by

$$W(\zeta,\gamma) := \begin{cases} \displaystyle\sum_{x\in\gamma,\, y\in\zeta} \phi(x,y) & \text{if } \displaystyle\sum_{x\in\gamma,\, y\in\zeta} |\phi(x,y)| < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

If θ is a given nonatomic Radon measure on $\mathcal{X}$, then the space $\Gamma_{\mathcal{X},0}$ can be endowed with a parametric family of *Lebesgue-Poisson measures* $\lambda_{z\theta}$ (where the parameter $z > 0$ is called the *activity*) defined by the formula

$$\int_{\Gamma_\Lambda} F(\zeta)\, \lambda_{z\theta}(\mathrm{d}\zeta) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{\Lambda^m} F(\{x_1, \dots, x_m\})\, \theta(\mathrm{d}x_1) \cdots \theta(\mathrm{d}x_m), \tag{8.1}$$

for each $\Lambda \subset \mathcal{X}$ such that $\theta(\Lambda) < \infty$.

There are many equivalent definitions of Gibbs measures. The following assertion is most convenient for our purposes.

Definition 8.1. We say that a probability Borel measure g on $\varGamma_{\mathcal{X}}$ is a (*grand canonical*) *Gibbs measure* corresponding to an intensity measure $z\theta$ and a pair potential ϕ if it satisfies the *Ruelle equation*

$$\int_{\varGamma_{\mathcal{X}}} F(\gamma)\,\mathsf{g}(\mathrm{d}\gamma) = \int_{\varGamma_{\varLambda}} \Bigl(\int_{\varGamma_{\mathcal{X}\setminus\varLambda}} F(\zeta\cup\gamma)\,\mathrm{e}^{-E(\gamma)-W(\gamma,\zeta)}\,\mathsf{g}(\mathrm{d}\gamma)\Bigr)\lambda_{z\theta}(\mathrm{d}\zeta), \tag{8.2}$$

for any bounded measurable function $F \geqslant 0$ on $\varGamma_{\mathcal{X}}$ and any $\varLambda\subset\mathcal{X}$, $\theta(\varLambda)<\infty$.

Let $\mathcal{G}(z\theta,\phi)$ be the class of all such measures. In the simple "free" case of $\phi\equiv 0$, it can be shown that the unique grand canonical Gibbs measure coincides with the Poisson measure $\pi_{z\theta}$ with intensity measure $z\theta$. In the general situation, there are various types of conditions which ensure that the class $\mathcal{G}(z\theta,\phi)$ is not empty (see [23, 21, 11]). The most suitable condition for our purposes is a certain modification of the standard assumptions (proposed and discussed in [15–17]). More precisely, suppose that the interaction potential ϕ satisfies the following conditions.

(S) (*Stability*). There exists $B\geqslant 0$ such that

$$E^{\phi}_{\varLambda}(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(x,y) \geqslant -B|\gamma| \tag{8.3}$$

for any compact set $\varLambda\subset\mathcal{X}$ and any $\gamma\in\varGamma_{\varLambda}$.

(I) (*Integrability*).

$$C := \operatorname{ess\,sup}_{x\in\mathcal{X}} \int_{\mathcal{X}} |\mathrm{e}^{-\phi(x,y)}-1|\,\theta(\mathrm{d}y) < \infty. \tag{8.4}$$

(F) (*Finite range*). There exists $R>0$ such that

$$\phi(x,y) = 0 \quad \text{if} \ \ |x-y|>R. \tag{8.5}$$

Theorem 8.1. 1) *Let assumptions* (S), (I) *and* (F) *hold and let $z>0$ be such that*

$$z < (2\mathrm{e}^{2B+1}C)^{-1}, \tag{8.6}$$

where the constants B and C are as in (S) *and* (I), *respectively. Then there is a Gibbs measure $\mathsf{g}\in\mathcal{G}(z\sigma,\phi)$ such that, for each $m\in\mathbb{N}$ and for any measurable symmetric function $F\colon\mathcal{X}^m\to[0,\infty]$,*

$$\int_{\varGamma_{\mathcal{X}}} \sum_{\{x_1,\dots,x_m\}\subset\gamma} F(x_1,\dots,x_m)\,\mathsf{g}(\mathrm{d}\gamma) = \frac{1}{m!}\int_{\mathcal{X}^m} F(x_1,\dots,x_m)\,\kappa^m_{\mathsf{g}}(x_1,\dots,x_m)\,\theta(\mathrm{d}x_1)\cdots\theta(\mathrm{d}x_m), \tag{8.7}$$

where κ^m_{g} is a nonnegative measurable symmetric function on $\mathcal{X}^m$ (*the m-point correlation function of the measure* g) *which satisfies the following uniform estimate*:

$$\kappa^m_{\mathsf{g}}(x_1,\dots,x_m) \leqslant a^m, \qquad (x_1,\dots,x_m)\in\mathcal{X}^m, \tag{8.8}$$

where the constant $a>0$ is independent of m (the Ruelle bound).

2) *Let ϕ be a nonnegative potential which satisfies assumptions* (I) *and* (F). *Then, for each $z>0$, there exists a Gibbs measure $\mathsf{g}\in\mathcal{G}(z\sigma,\phi)$ such that its correlation functions κ^m_{g} satisfy the Ruelle bound* (8.8).

This theorem was proved in [15–17] for general Riemannian manifolds (satisfying some natural conditions). For $\mathcal{X}=\mathbb{R}^d$, the existence of Gibbs measures satisfying the Ruelle bound (for all z) is known for arbitrary (not necessarily positive) interaction potentials ϕ under additional conditions of *superstability* and *lower regularity* (see [23]). Let us present two classical examples of symmetric translation-invariant potentials (i.e., $\phi(x,y)=\phi_0(x-y)=\phi_0(y-x)$) satisfying these conditions.

Example 8.1 (Lennard–Jones type potential). Here $\phi_0\in C^2(\mathbb{R}^d\setminus\{0\})$, $\phi_0\geqslant 0$ on $\mathbb{R}^d$, and $\phi_0(x)=c|x|^{-\alpha}$ for $|x|\leqslant r_1$, $\phi_0(x)=0$ for $|x|>r_2$, where $c>0$, $\alpha>0$, and $0<r_1<r_2<\infty$.

Example 8.2 (Lennard–Jones "6–12" potential). For $x\in\mathbb{R}^3$, set $\phi_0(x) := c(|x|^{-12}-|x|^{-6})$, $c>0$.

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