Markov Processes Relat. Fields **6**, 473–516 (2000)

Markov Processes and Related Fields

Annealed Moment Lyapunov Exponents for a Branching Random Walk in a Homogeneous Random Branching Environment*

S. Albeverio[1], L.V. Bogachev[2], S.A. Molchanov[3] and E.B. Yarovaya[4]

[1] Institut für Angewandte Mathematik, Universität Bonn, 53115 Bonn, Germany. E-mail: albeverio@uni-bonn.de
[2] Faculty of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia. E-mail: lvb@mech.math.msu.su
[3] Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA. E-mail: smolchan@uncc.edu
[4] Laboratory of Biostatistics, National Research Center for Preventive Medicine, 101953 Moscow, Russia. E-mail: yarov@cardpl.msk.ru

Received March 17, 2000

Abstract. We consider a continuous-time branching random walk on the lattice $\mathbf{Z}^d$ $(d \geq 1)$ evolving in a random branching environment. The motion of particles proceeds according to the law of a simple symmetric random walk. The branching medium formed of Markov birth-and-death processes at the lattice sites is assumed to be spatially homogeneous. We are concerned with the long-time behavior of the quenched moments m_n $(n \in \mathbf{N})$ for the local and total particle populations in a "frozen" medium. We pursue the moment approach via studying the asymptotics of the annealed moments $\langle m_n^p \rangle$ $(p \geq 1)$, obtained by averaging over the medium realizations. Under the assumption that the random branching potential has a Weibull type upper tail, we compute the corresponding Lyapunov exponents $\lambda_{n,p}$. Our results show that the quenched moments of all orders grow in a non-regular, intermittent fashion. The proofs are based on the study of a Cauchy problem for the Anderson operator with random potential and random source. In particular, we derive the Feynman – Kac representation for the solution of the inhomogeneous problem.

*Research supported in part by grants from DFG (# 436 RUS 113/534), RFBR (# 99-01-00298), and NSF (# DMS-9971592)

Keywords: branching random walk, random branching environment, quenched moments, annealed moments, backward equations, Feynman – Kac representation, Lyapunov exponents, intermittency

AMS Subject Classification: Primary 60K37, 60H25; Secondary 60J80

1. Introduction

In this paper, we consider a continuous-time branching random walk on the lattice $\mathbf{Z}^d$, $d \geq 1$, evolving in a random branching environment constituted by random birth and death rates at the lattice sites. We assume that the branching medium is spatially homogeneous, that is, its probability distribution is translation invariant (with respect to shifts of the lattice). The motion of particles proceeds according to the law of a simple symmetric random walk.

The quantities of interest in the model are the local number of particles (at lattice sites) and the total population size. We are concerned with their long-time behavior, given that initially the population consists of a single particle placed at some point $x \in \mathbf{Z}^d$.

We pursue the moment approach to the problem. Let us consider the "quenched" statistical moments of the local and total particle numbers (of all integer orders), where expectation is taken with respect to the law of the branching random walk conditional on the branching medium. As functions in time and space, these moments satisfy an infinite chain of backward evolution equations. For example, the expected total number of particles (i.e., the quenched moment of the first order) satisfies the Cauchy problem for the celebrated Anderson operator with random potential:

$$\partial_t u(t,x) = \varkappa \Delta u(t,x) + \xi(x) u(t,x), \qquad (1.1)$$

$$u(0,x) \equiv 1. \qquad (1.2)$$

Here $\partial_t := \partial/\partial t$ stands for a partial derivative with respect to time variable t, the operator Δ is the discrete Laplacian on $\mathbf{Z}^d$ acting in variable x as

$$\Delta\psi(x) = \frac{1}{2d} \sum_{|x'-x|=1} \psi(x') - \psi(x), \qquad (1.3)$$

the coefficient $\varkappa > 0$ is a "diffusion" constant, and the potential $\xi(x) = \xi(x,\omega)$, $x \in \mathbf{Z}^d$, is a certain random function determined by the random branching medium. (Here and below, the letter ω refers to an elementary event in an appropriate probability space, corresponding to a sample realization of the medium.)

Much attention has been devoted to the study of spectral properties of the Anderson operator $\varkappa\Delta + \xi(x)$, in particular in connection with the localization problem (see, e.g., [3] and bibliography therein). These properties are closely related to the asymptotic behavior of the solution $u(t,x)$ of the problem (1.1) + (1.2) (see discussion in [6]). It has been discovered that the evolution of the field $u(t,x)$ leads to the formation of highly irregular time-space structures, characterized by the generation of rare high peaks accumulating the bulk "mass" of the evolving field (see [6, 7, 13, 20]). Such a phenomenon has received the name "intermittency".

A key device for studying intermittency first employed in the works [6,13,20], is the asymptotic analysis of the "annealed" moments obtained by averaging the powers of the quenched moment over medium's realizations. Namely, it can be shown that intermittency is displayed in an anomalous, progressive growth, as $t \to \infty$, of the annealed moments with regard to their order, in that, for instance, the second-order moment grows much faster than the squared first-order moment, etc. (see [6, 13]). In particular, important information is contained in the Lyapunov exponents characterizing the growth of the annealed moments (in an appropriate time scale). Let us remark that in [7] the next term of the annealed moment asymptotics is found, revealing in particular the role of diffusion in the development of intermittency.

Equations for the higher-order quenched moments are similar to equation (1.1), with the only (but quite essential) difference that they are inhomogeneous, the right-hand part now containing in addition a source term which depends on the medium and on the moments of lower orders. By analogy with the first-order case, we are led to the problem of studying the annealed asymptotics for the solutions of these equations. In the present paper, we restrict our considerations to the case where the potential ξ has the Weibull type upper tail, and our goal is to compute explicitly the annealed Lyapunov exponents for the moments of all orders.

2. Model and the main result

Let us describe our model in greater detail. The branching random environment is formed by a collection of pairs of non-negative random variables, $\Xi(x) := (\xi_0(x), \xi_2(x))$, $x \in \mathbf{Z}^d$, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is convenient to realize this space as the canonical one, so that elementary events $\omega \in \Omega$ represent sample realizations of the field $\Xi(\cdot)$. In particular, we can assume that $\Omega = (\mathbf{R}^2_+)^{Z^d}$. Expectation with respect to the probability measure $\mathbf{P}$ will be denoted by the angular brackets, $\langle\cdot\rangle$. As already mentioned in Introduction, we assume that the random field Ξ is spatially homogeneous, that is, its distribution $\mathbf{P}$ is invariant with respect to translations $x \mapsto x + h$, $x, h \in \mathbf{Z}^d$.

Given the medium realization ω, we consider the process Z_t of branching random walk for which the branching rates at the lattice sites are given by the field Ξ, whereas the spatial motion of particles is described by a continuous-time simple symmetric random walk on $\mathbf{Z}^d$. To be more precise, at each site $x \in \mathbf{Z}^d$ a Markov birth-and-death branching process is defined, $\xi_0(x)$ and $\xi_2(x)$ having the meaning of its transition rates of death and binary splitting, respectively.

As is well known from the standard theory of branching processes (see, e.g., [8, Chapter 5]), this implies that the random time until branching transformation has exponential distribution with the parameter $\xi_0(x)+\xi_2(x)$, and as soon as this exponential "clock" rings, the particle either dies or splits into two descendants, with probabilities proportional to $\xi_0(x)$ respectively $\xi_2(x)$.

The simple symmetric random walk is specified by its generator of the form $\varkappa\Delta$, where Δ is the lattice Laplacian defined above (see (1.3)) and $\varkappa > 0$ is the jumping rate. In particular, this implies that the random walk spends at each site an exponential random time (with parameter $\varkappa$), thereafter jumping to one of the $2d$ adjacent sites chosen with equal probability $(2d)^{-1}$.

As usual, it is assumed that the mechanisms of random walk and branching act independently of each other, which amounts to saying that exponential clocks measuring time until jumping or branching are independent. Therefore, the development of the population ensemble proceeds, in "microscopic" time scale, according to the following rule. If a particle is found at some time at point x, then during a small subsequent amount of time $h > 0$, with probability $(2d)^{-1}\varkappa h + o(h)$ it can jump to a neighboring site x' ($|x' - x| = 1$), with probability $\xi_2(x)h + o(h)$ it is replaced by two descendants at the point x, or otherwise, with probability $\xi_0(x)h + o(h)$ the particle dies without leaving offspring. Accordingly, with probability $1-\varkappa h-(\xi_2(x)+\xi_0(x))h+o(h)$ the particle does not experience any transformations during the whole time h. Newly born particles evolve by the same rule, independently of the other particles and the past history. We also impose the initial condition by assuming that at time $t = 0$ there is a single particle in the system located, say, at point $x \in \mathbf{Z}^d$.

A formal construction of such a process, including the description of an appropriate probability space, can be found, for instance, in [4, Chapter 9], [8, Chapter V, § 3], [10, Chapter III], [15]. Let us denote by $\mathsf{P}_x^{(\omega)}$ the *quenched* probability law of this process, and let $\mathsf{E}_x^{(\omega)}$ stand for the corresponding expectation. (The label ω is to refer to the fixed realization of the branching medium Ξ, and the subscript x indicates the initial position of the single original particle.) Then the *annealed* probability measure P_x defining the process Z_t in the random medium Ξ can be symbolically represented by the total probability formula

$$\mathsf{P}_x(\cdot) = \int_\Omega \mathsf{P}_x^{(\omega)}(\cdot)\, \mathbf{P}(d\omega).$$

Let us denote by $\mu_t(y)$ the number of particles at time $t \geq 0$ at point $y \in \mathbf{Z}^d$, and consider also the total population size $\mu_t := \sum_y \mu_t(y)$. The quenched generating functions associated with the random variables $\mu_t(y)$ and μ_t are defined for $z \geq 0$ by

$$\begin{aligned} F(z; t, x, y) &:= \mathsf{E}_x^{(\omega)}\,[e^{-z\mu_t(y)}], \\ F(z; t, x) &:= \mathsf{E}_x^{(\omega)}\,[e^{-z\mu_t}]. \end{aligned}$$

It is well known (see, e.g., [13,16,17]; cf. also [1]) that the functions $F(z;t,x,y)$, $F(z;t,x)$ satisfy the Skorohod equation

$$\partial_t F = \varkappa\Delta F + [\xi_2(x)F - \xi_0(x)]\cdot(F-1)$$

with the initial conditions

$$F(z;0,x,y) = e^{-z\delta_y(x)} \quad \text{resp.} \quad F(z;0,x) = e^{-z},$$

where $\delta_y(x)$ is the Kronecker symbol.

Let us now introduce the quenched integer-order moments

$$m_n(t,x,y) := \mathsf{E}_x^{(\omega)}[\mu_t^n(y)], \qquad m_n(t,x) := \mathsf{E}_x^{(\omega)}[\mu_t^n], \qquad n\in\mathbf{N}.$$

(We implicitly assume that all these moment functions are finite. Later, this will be guaranteed by the conditions imposed on the random medium Ξ.) By taking the nth derivative of the functions $F(z;t,x,y)$, $F(z;t,x)$ (which is clearly permissible, see, e.g., [5, Chapter XIII, § 2]) and letting z tend to 0, we obtain

$$m_n(t,x,y) = (-)^n \lim_{z\to 0+} \partial_z^n F(z;t,x,y), \tag{2.1}$$

$$m_n(t,x) = (-)^n \lim_{z\to 0+} \partial_z^n F(z;t,x). \tag{2.2}$$

Let us now introduce the random field

$$\xi(x) := \xi_2(x) - \xi_0(x), \qquad x\in\mathbf{Z}^d, \tag{2.3}$$

called the *potential*. As shown in [14] (see also [1, 18]), the moment functions $m_n(t,x,y)$, $m_n(t,x)$ $(n=1,2,\ldots)$ satisfy the chain of linear (non-homogeneous) differential equations

$$\partial_t m_n = \varkappa\Delta m_n + \xi(x)m_n + \xi_2(x)h_n[m_1,\ldots,m_{n-1}] \tag{2.4}$$

with the initial conditions

$$m_n(0,\cdot,y) = \delta_y(\cdot) \quad \text{resp.} \quad m_n(0,\cdot)\equiv 1, \tag{2.5}$$

where $h_1\equiv 0$ and for $n\ge 2$

$$h_n[m_1,\ldots,m_{n-1}] := \sum_{i=1}^{n-1}\binom{n}{i} m_i m_{n-i}, \tag{2.6}$$

here $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ are the binomial coefficients. In particular, the first-order moments, as mentioned in Introduction, satisfy the homogeneous equation

$$\partial_t m_1 = \varkappa\Delta m_1 + \xi(x)m_1. \tag{2.7}$$

Equations (2.4) can be obtained formally from the Skorohod equation for the generating functions via an appropriate differentiation using relations (2.1), (2.2).

We are now in a position to explain our main result on the asymptotics of the annealed moments $\langle m_n^p \rangle$, where $n \in \mathbf{N}$, $p \geq 1$. In order to make the formulations clearer, we temporarily assume that $\xi_0 \equiv 0$, that is, the particles do not die and hence branching reduces to splitting (the fission process). Therefore, the random medium is completely determined by the potential $\xi(\cdot)$ (which is in particular non-negative now). Due to our general assumption of the medium's homogeneity, the random field $\xi(\cdot)$ is homogeneous. In order to prevent the possible explosion of the moments, we need to have some control on the upper tail of the distribution of $\xi(x)$. In this work, we consider the concrete case where such tails are of the Weibull type:

$$\lim_{z \to +\infty} z^{-\alpha} \ln \mathsf{P}\{\xi(x) > z\} = -c,$$

for some $\alpha > 1$, $c > 0$.

Under these assumptions, we show the following result.

Theorem 2.1. *The annealed moment Lyapunov exponents defined for the moment functions m_n as the limits*

$$\lambda_{n,p} := \lim_{t \to \infty} \frac{\ln \langle m_n^p \rangle}{a(t)},$$

exist in the time scale $a(t) = t^{\alpha'}$, where $\alpha' := \alpha/(\alpha - 1)$. Moreover, they are explicitly given by

$$\lambda_{n,p} = \gamma(\alpha, c) \cdot (np)^{\alpha'}, \qquad n \in \mathbf{N}, \quad p \geq 1, \tag{2.8}$$

where

$$\gamma(\alpha, c) := (\alpha - 1)\alpha^{-\alpha'} c^{-1/(\alpha-1)}. \tag{2.9}$$

Note that the right-hand side of equation (2.8) does not depend on the starting point x, which is obviously due to the medium's homogeneity. More striking is the fact that the Lyapunov exponents for the local moments $m_n(t, x, y)$ coincide with those for the total moments $m_n(t, x)$ (and thus are independent of the point y as well). This shows that in the long run, all the sites $y \in \mathbf{Z}^d$, no matter how far they may be from the starting point x, make the same contribution to the growth of the annealed mean population size.

In the general case, where the possibility of death is allowed, we require the random potential $\xi(\cdot)$ to be non-percolating from below (see Sections 3 and 6 of the present paper). In words, this property asserts that the lower level set of the field $\xi(\cdot)$ does not percolate to infinity provided the level is low enough [12, Section 9] (the connectivity on the lattice $\mathbf{Z}^d$ is understood in the usual nearest-neighbor sense). We also need some technical condition on the upper tail of the splitting rate $\xi_2(x)$ (see Theorem 5.2).

The main qualitative inference from the result (2.8) is that we deal here with a strongly intermittent structure. Indeed, from (2.8) it follows that for each fixed n, the *specific* Lyapunov exponent

$$\frac{\lambda_{n,p}}{p} = \gamma(\alpha, c)\, n^{\alpha'} p^{\alpha'-1}$$

appears to be strictly increasing as a function of p, since $\alpha' > 1$. According to [6, 13], this implies the intermittency of the field m_n. Moreover, we observe that the same is valid with respect to the index n. Namely, for each fixed p

$$\frac{\lambda_{n,p}}{n} = \gamma(\alpha, c)\, p^{\alpha'} n^{\alpha'-1}$$

is a strictly increasing function in n.

It is also important to observe from (2.8) that for all $n \in \mathbf{N}$, $p \geq 1$, the following identity holds:

$$\lambda_{n,p} = \lambda_{1,np}. \tag{2.10}$$

Heuristically, relation (2.10) implies that

$$\langle m_n^p \rangle \approx \langle m_1^{np} \rangle, \qquad t \to \infty.$$

This suggests that the intermittency of the moments m_n, $n \in \mathbf{N}$, is reduced, in a sense, to that of the first-order moment m_1 (cf. [14] where this observation was first made).

It is worth noting that the distinguished role of the first-order moment is also seen from the proofs of our results (see below Sections 6 and 8). Namely, we break up the proof of relation (2.8) into two steps, for $n = 1$ (Theorem 6.1) where the exponents $\lambda_{1,p}$ are found directly, and for $n \geq 2$ (Theorem 8.1) where we prove the result by deriving two-sided estimates in terms of m_1.

Let us remark that in the case $n = 1$, the result (2.8) is well known for the total moment $m_1(t, x)$ (see [6, 13]). Our result in Theorem 6.1 related to the local moment $m_1(t, x, y)$ is a new one. In a short note [18], formulas (2.8) (for integer orders p) were announced in the case of pure-birth branching medium with i.i.d. values and Weibull tails. This case has also been considered in [14] where a sketch proof of (2.8) is given, however based on a different approach.

Let us point out that the main technical tool used in [6, 13] is the Feynman–Kac representation of the solution to the problem (1.1) + (1.2), having the form

$$u(t, x) = E_x \Big[\exp\Big(\int_0^t \xi(x_s)\, ds \Big) \Big].$$

Here $(x_t,\, t \geq 0)$ is an auxiliary simple symmetric random walk on $\mathbf{Z}^d$ with the generator $\varkappa\Delta$, and E_x is the expectation under the condition that the random walk is started at point x. In our analysis of the higher-order quenched moments, we extend this formula to the inhomogeneous case (see Theorems 4.2 and 5.2), which may be of interest in its own right.

The paper is organized as follows. We start in Section 3 with the consideration of the inhomogeneous Cauchy problem with non-random data, recalling first a few necessary facts known for the homogeneous one. In particular, sufficient conditions for the existence and uniqueness of a non-negative solution are given in Theorem 3.1. In the next Section 4, we obtain an analogue of these results for the case of random data (Theorem 4.2). In turn, the latter theorem is applied in Section 5 to obtain the Feynman – Kac representation for the higher-order moments (Theorem 5.2). We then proceed to the study of the annealed moment asymptotics, beginning with the case $n = 1$ (Section 6). A few technical lemmas have to be proved beforehand. In particular, we establish a simple scaling relation for the first-order moments (Lemma 6.3), which plays an important role in the subsequent estimates. The Lyapunov exponents $\lambda_{1,p}$ are then computed in Theorem 6.1. The proof of the second part of our main result, for $n \geq 2$, requires some preparations made in Section 7. The central result here is Lemma 7.10, which allows one to estimate a certain functional of the branching random walk by a similar functional of the ordinary random walk. Finally, Section 8 is devoted to the derivation of higher-order annealed moment Lyapunov exponents (Theorem 8.1).

In the sequel, we frequently use the following notation:

$$\mathrm{Exp}[f(\cdot)]_a^b := \exp\Big(\int_a^b f(s)\,ds\Big). \tag{2.11}$$

The distance on the lattice $\mathbf{Z}^d$ is defined as

$$|y-x| := \sum_{i=1}^{d} |y_i - x_i|, \qquad x = (x_1, \dots, x_d), \quad y = (y_1, \dots, y_d) \in \mathbf{Z}^d.$$

3. Inhomogeneous Cauchy problem with non-random data

Let us first consider the homogeneous Cauchy problem

$$\begin{aligned} \partial_t u(t,x) &= \varkappa \Delta u(t,x) + q(x) u(t,x), && (3.1)\\ u(0,x) &= u_0(x), && (3.2) \end{aligned}$$

where $q\colon \mathbf{Z}^d \to \mathbf{R}$ and $u_0\colon \mathbf{Z}^d \to \mathbf{R}_+$ are given non-random functions. We are interested in non-negative solutions $u(t,x)$.

In [6], it has been established that there is a deep connection between the problem (3.1) + (3.2) and the Feynman – Kac functional

$$\underline{u}(t,x) := E_x\big[\,\mathrm{Exp}[q(x_\cdot)]_0^t\, u_0(x_t)\big], \tag{3.3}$$

where $(x_t,\, t \ge 0)$ is the simple symmetric random walk (with continuous time) on $\mathbf{Z}^d$ with generator $\varkappa\Delta$, E_x is the corresponding expectation conditioned on $x_t|_{t=0} = x$, and the notation Exp is introduced in (2.11). For the sake of the presentation convenience, let us cite here some relevant results from [6].

Lemma 3.1. (Cf. [6, Lemma 2.2].) *If the function* $\underline{u}(t,x)$ *defined by* (3.3) *is finite for all* $t \ge 0$ *and* $x \in \mathbf{Z}^d$*, then* $\underline{u}(t,x)$ *is the minimal non-negative solution to the Cauchy problem* (3.1) + (3.2). *In the opposite case, the problem* (3.1) + (3.2) *does not have non-negative solutions.*

Definition 3.1. (Cf. [6, p. 623].) We say that the potential $q(\cdot)$ *does not percolate from below* if there exists $h \in \mathbf{R}$ such that the lower level set $A_h^- := \{x \in \mathbf{Z}^d : q(x) \le h\}$ does not percolate to infinity, that is, does not contain an infinite connected component (connectivity in $\mathbf{Z}^d$ is understood in the sense of nearest neighbors, $|x-y| = 1$).

Lemma 3.2. (Cf. [6, Lemma 2.3].) *If the potential* $q(\cdot)$ *does not percolate from below, then the Cauchy problem* (3.1) + (3.2) *has at most one non-negative solution.*

Let us now consider the non-homogeneous Cauchy problem

$$\partial_t v(t,x) = \varkappa\Delta v(t,x) + q(x)v(t,x) + g(t,x), \tag{3.4}$$

$$v(0,x) = v_0(x), \tag{3.5}$$

where $q\colon \mathbf{Z}^d \to \mathbf{R}$, $g\colon \mathbf{R}_+ \times \mathbf{Z}^d \to \mathbf{R}_+$, $v_0\colon \mathbf{Z}^d \to \mathbf{R}_+$. Let us define

$$\begin{aligned} \underline{v}(t,x) &= \underline{v}_1(t,x) + \underline{v}_2(t,x) \qquad (3.6)\\ &:= E_x\left[\operatorname{Exp}[q(x_\cdot)]_0^t v_0(x_t)\right] + E_x\left[\int_0^t \operatorname{Exp}[q(x_\cdot)]_0^s g(t-s,x_s)\,ds\right]. \end{aligned}$$

Theorem 3.1. *Assume that the function* $g(t,x)$ *is continuous in* t *at each fixed* $x \in \mathbf{Z}^d$*. Then the Cauchy problem* (3.4) + (3.5) *has at least one non-negative solution if and only if the function* $\underline{v}(t,x)$ *defined in* (3.6) *assumes finite values for all* $t \ge 0$, $x \in \mathbf{Z}^d$:

$$\underline{v}(t,x) < \infty. \tag{3.7}$$

In case condition (3.7) *is satisfied,* $\underline{v}(t,x)$ *is the minimal non-negative solution. If moreover the potential* $q(\cdot)$ *does not percolate from below in the sense of Definition* 3.1, *then* $\underline{v}(t,x)$ *is the unique non-negative solution to the problem* (3.4) + (3.5).

Remark 3.1. According to Lemma 3.1, the summand $\underline{v}_1(t,x)$ in (3.6) is a solution to the homogeneous problem with the given initial condition (3.5). Therefore, the second summand, $\underline{v}_2(t,x)$, plays the role of a particular solution of the inhomogeneous equation (3.4) with null initial condition. The latter can be verified by a formal substitution into (3.4). Let us also remark that one can arrive at the representation (3.6) by using the formula of variation of constants and taking into account Lemma 3.1 and the expression (3.3) for the solution of the homogeneous problem. However, we will not justify these calculations because it would be difficult to prove along this line the minimality property which plays a key role in the proof of uniqueness.

By analogy with the paper [6] where the homogeneous case was considered (i.e., the problem (3.1) + (3.2)), we will first study the corresponding initial boundary value problem in the strip $(t,x) \in \mathbf{R}_+ \times (Q_N \cup \partial Q_N)$, obtained by complementing equation (3.4) and the initial condition (3.5) (both considered now for $x \in Q_N$) with the boundary condition

$$v(t,x) = \psi_N(t,x), \qquad t \geq 0, \quad x \in \partial Q_N, \tag{3.8}$$

where

$$\begin{aligned} Q_N \quad &:= \quad \{x \in \mathbf{Z}^d : |x| < N\}, \qquad N \in \mathbf{N}, \\ \partial Q_N \quad &:= \quad Q_{N+1} \setminus Q_N \quad = \quad \{x \in \mathbf{Z}^d : |x| = N\}, \end{aligned}$$

and $\psi_N(t,x)$ is a function defined on $\mathbf{R}_+ \times \partial Q_N$. Denote by τ_N the hitting time of the set ∂Q_N by the random walk x_t:

$$\tau_N := \inf\{t \geq 0 : |x_t| = N\}. \tag{3.9}$$

Of course, τ_N is a stopping time with respect to the natural filtration $(\mathcal{F}_t,\, t \geq 0)$, $\mathcal{F}_t := \sigma\{x_s : s \leq t\}$. From the well-known properties of the simple symmetric random walk, it follows that $P_x\{\tau_N < \infty\} = 1$ for all $x \in Q_N \cup \partial Q_N$. Let us prove the following general lemma.

Lemma 3.3. *Assume that for a given $N \in \mathbf{N}$ the functions $g(t,x)$, $\psi_N(t,x)$ are continuous in t for each $x \in Q_N$ and each $x \in \partial Q_N$, respectively. Then the initial boundary value problem (3.4) + (3.5) + (3.8) has the unique solution $v_N(t,x)$, which can be written in the form*

$$\begin{aligned} v_N(t,x) \quad &= \quad E_x\left[\operatorname{Exp}[q(x.)]_0^{t\wedge\tau_N}\, \Psi_N(t - t\wedge\tau_N, x_{t\wedge\tau_N})\right] \\ &\qquad + E_x\Big[\int_0^{t\wedge\tau_N} \operatorname{Exp}[q(x.)]_0^s\, g(t-s, x_s)\, ds\Big], \end{aligned} \tag{3.10}$$

where $a \wedge b := \min(a,b)$ and

$$\Psi_N(s,x) := \begin{cases} \psi_N(s,x), & \text{if} \quad x \in \partial Q_N,\ s \geq 0, \\ v_0(x), & \text{if} \quad s = 0,\ x \in Q_N. \end{cases} \tag{3.11}$$

Proof. We first remark that the initial boundary value problem (3.4) + (3.5) + (3.8) in fact amounts to a Cauchy problem for a finite system of inhomogeneous ordinary differential equations with constant coefficients. Moreover, from the hypotheses of the lemma it follows that the functions determining the inhomogeneity[1] are continuous in $t \ge 0$. This immediately implies existence and uniqueness of a solution which we denote by $v_N(t,x)$. It then only remains to check that this solution can be represented in the form (3.10).

Let us continue the function $v_N(t,x)$ from the strip $\mathbf{R}_+ \times Q_{N+1}$ onto $\mathbf{R}_+ \times \mathbf{Z}^d$ by setting $v_N(t,x) \equiv 0$ for $x \in \mathbf{Z}^d \setminus Q_{N+1}$. Fix $t > 0$ and consider for $s \in [0,t]$ the random process

$$\begin{aligned} \zeta_s \quad := \quad & v_N(t-s,x_s)\,\mathrm{Exp}\big[q(x.)\big]_0^s \\ & + \int_0^s [\partial_1 v_N(t-u,x_u) - H v_N(t-u,x_u)]\,\mathrm{Exp}[q(x.)]_0^u\,du, \end{aligned} \tag{3.12}$$

where $\partial_1 v_N(\cdot,\cdot)$ is the partial derivative of the function $v_N(\cdot,\cdot)$ with respect to the first argument, and $H := \varkappa\Delta + q(\cdot)$. Let us show that $(\zeta_s,\, 0 \le s \le t)$ is a martingale with respect to the filtration $(\mathcal{F}_s,\, s \le t)$, where $\mathcal{F}_s = \sigma\{x_u : u \le s\}$. Using the Markov property of the random walk x_s, for $0 \le r \le s$ we obtain (P_x-a.s.)

$$\begin{aligned} E_x\left[\zeta_s \mid \mathcal{F}_r\right] \quad = \quad & v_N(t-r,x_r)\,\mathrm{Exp}[q(x.)]_0^r \\ & + \int_0^r [\partial_1 v_N(t-u,x_u) - H v_N(t-u,x_u)]\,\mathrm{Exp}[q(x.)]_0^u\,du \\ & + \mathrm{Exp}[q(x.)]_0^r\, E_x\Big[v_N(t-s,x_s)\,\mathrm{Exp}[q(x.)]_r^s - v_N(t-r,x_r) \\ & + \int_r^s [\partial_1 v_N(t-u,x_u) - H v_N(t-u,x_u)]\,\mathrm{Exp}[q(x.)]_r^u\,du \,\Big|\, \mathcal{F}_r\Big] \\ = \quad & \zeta_r + \mathrm{Exp}[q(x.)]_0^r\, E_z\Big[v_N(\tilde t-\tilde s,\tilde x_{\tilde s})\,\mathrm{Exp}[q(\tilde x.)]_0^{\tilde s} - v_N(\tilde t,z) \\ & + \int_0^{\tilde s} [\partial_1 v_N(\tilde t-u,\tilde x_u) - H v_N(\tilde t-u,\tilde x_u)]\,\mathrm{Exp}[q(\tilde x.)]_0^u\,du\Big]_{z=x_r}, \end{aligned}$$

where $\tilde s := s - r$, $\tilde t := t - r$, and $\tilde x_u := z + x_{r+u} - x_r$ $(u \ge 0)$ is a "shifted" random walk started at point z. Comparing this expression with formula (3.12), we see that in order to prove the required equality

$$E_x\left[\zeta_s \mid \mathcal{F}_r\right] = \zeta_r \qquad (P_x\text{-a.s.}) \tag{3.13}$$

it suffices to check that $E_z[\zeta_s] = E_z[\zeta_0]$, that is,

$$\begin{aligned} & E_z\left[v_N(t-s,x_s)\,\mathrm{Exp}[q(x.)]_0^s - v_N(t,z)\right] \\ & \qquad + E_z\Big[\int_0^s [\partial_1 v_N(t-u,x_u) - H v_N(t-u,x_u)]\,\mathrm{Exp}[q(x.)]_0^u\,du\Big] \equiv 0. \end{aligned} \tag{3.14}$$

[1] The contribution to the inhomogeneity is brought both by the function $g(t,x)$ and (for "near-boundary" points x) by the boundary function $\psi_N(t,x)$.

To this end, let us represent the first expectation in (3.14) as

$$\begin{aligned} & E_z\left[v_N(t-s,x_s)\,\mathrm{Exp}[q(x.)]_0^s - v_N(t,z)\right] \\ = \quad & \int_0^s \partial_u E_z\left[v_N(t-u,x_u)\,\mathrm{Exp}[q(x.)]_0^u\right]du \\ = \quad & E_z\Big[\int_0^s \big[-\partial_1 v_N(t-u,x_u) + \varkappa\Delta v_N(t-u,x_u) \\ & \qquad + q(x_u)v_N(t-u,x_u)\big]\,\mathrm{Exp}[q(x.)]_0^u\,du\Big]. \end{aligned}$$

Substitution of the obtained expression into (3.14) immediately shows that the left-hand side of (3.14) identically vanishes, which proves (3.13).

Let us now use an appropriate version of the optional stopping theorem which guarantees that the martingale property (3.13) is preserved for stopped martingales with right-continuous paths and bounded stopping times (see, e.g., [19, Chapter 2, § 8, Theorem 8.10]). In particular, for the stopping time $t\wedge\tau_N$ (where τ_N is defined in (3.9)) we have

$$E_x\,[\zeta_{t\wedge\tau_N}] = E_x\,[\zeta_0]. \tag{3.15}$$

By the definition of the process ζ (see (3.12)), the right-hand side of (3.15) is equal to $E_x[v_N(t,x_0)] = v_N(t,x)$, whereas the left-hand side is rewritten as

$$\begin{aligned} & E_x\left[v_N(t-t\wedge\tau_N, x_{t\wedge\tau_N})\,\mathrm{Exp}[q(x.)]_0^{t\wedge\tau_N}\right] \\ & \quad + E_x\Big[\int_0^{t\wedge\tau_N}[\partial_1 v_N(t-u,x_u) - Hv_N(t-u,x_u)]\,\mathrm{Exp}[q(x.)]_0^u\,du\Big]. \end{aligned} \tag{3.16}$$

In view of the initial-boundary conditions (3.5), (3.8) and the definition of Ψ_N (see (3.11)), the random function $v_N(t-t\wedge\tau_N, x_{t\wedge\tau_N})$ P_x-a.s. coincides with $\Psi_N(t-t\wedge\tau_N, x_{t\wedge\tau_N})$. On the other hand, the function $v_N(\cdot,z)$ satisfies equation (3.4) for all $z\in Q_N$ and hence P_x-a.s.

$$\partial_1 v_N(t-u,x_u) - Hv_N(t-u,x_u) = g(t-u,x_u), \qquad 0\le u< t\wedge\tau_N.$$

As a result, expression (3.16) amounts to the right-hand side of (3.10), and the proof is complete. □

Lemma 3.4. *Let the condition* (3.7) *be fulfilled, that is, the function* $\underline{v}(t,x)$ *defined by* (3.6) *be finite for all* $t\ge 0$, $x\in\mathbf{Z}^d$. *Then for each* $N\in\mathbf{N}$, *the function* $\underline{v}(t,x)$ *in the strip* $(t,x)\in\mathbf{R}_+\times Q_{N+1}$ *admits the following representation:*

$$\begin{aligned} \underline{v}(t,x) \quad = \quad & E_x\left[\mathrm{Exp}[q(x.)]_0^{t\wedge\tau_N}\cdot\Psi_{\underline{v},N}(t-t\wedge\tau_N, x_{t\wedge\tau_N})\right] \\ & + E_x\Big[\int_0^{t\wedge\tau_N}\mathrm{Exp}[q(x.)]_0^s g(t-s,x_s)\,ds\Big], \end{aligned} \tag{3.17}$$

where the function $\Psi_{\underline{v},N}(s,x)$ *is defined on the boundary of the above mentioned strip by*

$$\Psi_{\underline{v},N}(s,x) := \begin{cases} \underline{v}(s,x), & \text{if } x \in \partial Q_N, \quad s \geq 0, \\ v_0(x), & \text{if } s = 0, \quad x \in Q_N. \end{cases} \tag{3.18}$$

Proof. Distinguishing between the cases $\tau_N > t$ and $\tau_N \leq t$, let us write the first summand in (3.6) in the form

$$\begin{aligned} \underline{v}_1(t,x) \quad &= \quad E_x\left[\operatorname{Exp}[q(x.)]_0^t\, v_0(x_t)\, I\{\tau_N > t\}\right] \\ &\quad + E_x\left[\operatorname{Exp}[q(x.)]_0^{\tau_N}\, I\{\tau_N \leq t\} \cdot E_x\left\{\operatorname{Exp}[q(x.)]_{\tau_N}^t v_0(x_t) \,\middle|\, \tau_N\right\}\right], \end{aligned} \tag{3.19}$$

where $I(A)$ stands for the indicator of the event A. Thanks to the strong Markov property of the random walk x_t applied to the stopping time τ_N, the conditional expectation in equation (3.19) P_x-a.s. equals

$$\begin{aligned} &E_x\left[\operatorname{Exp}\big[q(x_{\tau_N} + \cdot)\big]_0^{t-\tau_N}\, v_0(x_{\tau_N+(t-\tau_N)}) \,\middle|\, \tau_N\right] \\ &\quad = \quad E_z\left[\operatorname{Exp}[q(\tilde{x}.)]_0^T\, v_0(\tilde{x}_T)\right]_{z=x_{\tau_N},\, T=t-\tau_N} \quad = \quad \underline{v}_1(t-\tau_N, x_{\tau_N}), \end{aligned} \tag{3.20}$$

where $\tilde{x}_u := z + x_{\tau_N+u} - x_{\tau_N}$, $u \geq 0$, is a shifted random walk started at point z.

Similarly,

$$\begin{aligned} \underline{v}_2(t,x) \quad &= \quad E_x\left[\int_0^t \operatorname{Exp}[q(x.)]_0^s\, g(t-s, x_s)\, ds \cdot I\{\tau_N > t\}\right] \\ &\quad + E_x\left[\int_0^{\tau_N} \operatorname{Exp}[q(x.)]_0^s\, g(t-s, x_s)\, ds \cdot I\{\tau_N \leq t\}\right] \\ &\quad + E_x\left[\int_{\tau_N}^t \operatorname{Exp}[q(x.)]_0^s\, g(t-s, x_s)\, ds \cdot I\{\tau_N \leq t\}\right] \\ &= \quad E_x\left[\int_0^{t\wedge\tau_N} \operatorname{Exp}[q(x.)]_0^s\, g(t-s, x_s)\, ds\right] \\ &\quad + E_x\Big[\operatorname{Exp}[q(x.)]_0^{\tau_N}\, I\{\tau_N \leq t\} \\ &\quad \times E_x\left\{\int_{\tau_N}^t \operatorname{Exp}[q(x.)]_{\tau_N}^s\, g(t-s, x_s)\, ds \,\middle|\, \tau_N\right\}\Big]. \end{aligned} \tag{3.21}$$

The conditional expectation in (3.21) P_x-a.s. can be written in the form (cf. (3.20))

$$E_z\left[\int_0^T \operatorname{Exp}[q(\tilde{x}.)]_0^u\, g(T-u, \tilde{x}_u)\, du\right]_{z=x_{\tau_N},\, T=t-\tau_N} = \underline{v}_2(t-\tau_N, x_{\tau_N}). \tag{3.22}$$

Thus, substituting expressions (3.20), (3.22) into formulas (3.19), (3.21) respectively, by virtue of (3.6) we obtain

$$\begin{aligned}\underline{v}(t,x) \;&=\; E_x\Big[\operatorname{Exp}[q(x_\cdot)]_0^{t\wedge\tau_N}\\ &\qquad\times\big(v_0(x_t)\,I\{\tau_N>t\}+\underline{v}(t-\tau_N,x_{\tau_N})\,I\{\tau_N\le t\}\big)\Big]\\ &\qquad+E_x\Big[\int_0^{t\wedge\tau_N}\operatorname{Exp}[q(x_\cdot)]_0^s\,g(t-s,x_s)\,ds\Big].\end{aligned}\tag{3.23}$$

Since $x_{\tau_N}\in Q_N$, by the definition of $\Psi_{\underline{v},N}$ (see (3.18)) we have

$$\Psi_{\underline{v},N}(t-t\wedge\tau_N,x_{t\wedge\tau_N})=\begin{cases}\underline{v}(t-\tau_N,x_{\tau_N}), & \text{if}\quad \tau_N\le t,\\ v_0(x_t), & \text{if}\quad \tau_N>t.\end{cases}$$

Therefore, expression (3.23) amounts to (3.17). □

Lemma 3.5. *Under the condition* (3.7) *the function* $\underline{v}(t,x)$ *defined by formula* (3.6) *is continuous in* $t\in[0,\infty)$ *for each* $x\in\mathbf{Z}^d$.

Proof. Without loss of generality, one can assume that $x=0$. Let us apply Lemma 3.4 with $x=0$ and $N=1$. Since the distribution of the random time τ_1 with respect to P_0 is exponential with parameter $\varkappa$, from the representation (3.17) we obtain

$$\begin{aligned}\underline{v}(t,0)\;&=\;P_0\{\tau_1>t\}\Big[e^{tq(0)}\,v_0(0)+\int_0^t e^{sq(0)}\,g(t-s,0)\,ds\Big]\\ &\qquad+\frac{1}{2d}\sum_{|x|=1}E_0\Big[I\{\tau_1\le t\}\Big(e^{\tau_1 q(0)}\,\underline{v}(t-\tau_1,x)\\ &\qquad+\int_0^{\tau_1}e^{uq(0)}\,g(t-u,0)\,du\Big)\Big]\\ &=\;e^{-\varkappa t}\Big[e^{tq(0)}\,v_0(0)+\int_0^t e^{sq(0)}\,g(t-s,0)\,ds\Big]\\ &\qquad+\frac{1}{2d}\sum_{|x|=1}\int_0^t\varkappa e^{-\varkappa t_1}\Big[e^{t_1q(0)}\underline{v}(t-t_1,x)\\ &\qquad+\int_0^{t_1}e^{uq(0)}\,g(t-u,0)\,du\Big]dt_1.\end{aligned}\tag{3.24}$$

By means of the substitutions $s \mapsto t - s,\ t_1 \mapsto t - t_1,\ u \mapsto t - u$, expression (3.24) is rewritten as

$$\begin{aligned} \underline{v}(t,0) &= e^{-\varkappa t}\Big[e^{tq(0)}\, v_0(0) + \int_0^t e^{(t-s)q(0)}\, g(s,0)\, ds\Big] \\ &\quad + \frac{1}{2d}\sum_{|x|=1}\int_0^t \varkappa e^{-\varkappa(t-t_1)}\Big[e^{(t-t_1)q(0)}\, \underline{v}(t_1,x) \\ &\quad + \int_{t_1}^t e^{(t-u)q(0)}\, g(u,0)\, du\Big] dt_1, \end{aligned}$$

whence it is obvious that $\underline{v}(\cdot,0)$ is continuous. □

Now, we are in a position to establish Theorem 3.1.

Proof of Theorem 3.1. Note that by Lemmas 3.3, 3.4 and 3.5, the function $\underline{v}(t,x)$ for each $N \in \mathbf{N}$ is the (unique) solution to the initial boundary value problem (3.4) + (3.5) + (3.8), in which the boundary condition (3.8) is induced by the function $\underline{v}(t,x)$ itself:

$$\psi_N \equiv \underline{v}|_{x\in\partial Q_N}.$$

Since N is arbitrary, it follows that $\underline{v}(t,x)$ satisfies equation (3.4) and the initial condition (3.5) for all $x \in \mathbf{Z}^d$, $t \ge 0$.

Suppose now that $v(t,x)$ is a non-negative solution of the problem (3.4) + (3.5). Clearly, for each $N \in \mathbf{N}$ the function $v(t,x)$ solves the initial boundary value problem (3.4) + (3.5) + (3.8) with the natural boundary condition induced by $v(t,x)$, that is, determined by the function

$$\psi_N \equiv v|_{x\in\partial Q_N}.$$

Therefore, by Lemma 3.3 the representation (3.10) holds in the strip $(t,x) \in \mathbf{R}_+ \times Q_{N+1}$, where

$$\Psi_N(s,x) = \Psi_{v,N}(s,x) = \begin{cases} v(s,x), & \text{if } \ x \in \partial Q_N, \quad s \ge 0, \\ v_0(x), & \text{if } \ s = 0, \quad x \in Q_N. \end{cases}$$

Hence, for $(t,x) \in \mathbf{R}_+ \times Q_N$

$$\begin{aligned} v(t,x) &\ge E_x\left[I\{\tau_N > t\}\,\mathrm{Exp}[q(x.)]_0^t\, v_0(x_t)\right] \\ &\quad + E_x\Big[I\{\tau_N > t\}\int_0^t \mathrm{Exp}[q(x.)]_0^s\, g(t-s,x_s)\, ds\Big]. \end{aligned}$$

Passing to the limit on the right-hand side as $N \to \infty$ (by the monotone convergence theorem), we arrive at the inequality

$$v(t,x) \ge \underline{v}(t,x), \qquad (t,x) \in \mathbf{R}_+ \times \mathbf{Z}^d,$$

and therefore $\underline{v}$ is the minimal solution. Our argument also shows that (3.7) is a necessary condition for the solvability of the problem $(3.4)+(3.5)$.

To prove uniqueness (under the assumption of non-percolation from below of the potential $q(\cdot)$), we consider the difference of two solutions, $v_*(t,x) := v(t,x) - \underline{v}(t,x)$. Obviously, $v_*(t,x)$ satisfies the homogeneous Cauchy problem $(3.1)+(3.2)$ with null initial condition: $v_*(0,x) \equiv 0$, and moreover $v_*(t,x) \ge 0$, thanks to the minimality of the solution $\underline{v}(t,x)$. By Lemma 3.2, it then follows that $v_*(t,x) \equiv 0$, that is, $v(t,x) \equiv \underline{v}(t,x)$, and the proof is complete. $\square$

4. Inhomogeneous Cauchy problem with random data

Let us first formulate the basic results [6] for the homogeneous Cauchy problem

$$\begin{aligned} \partial_t u(t,x) &= \varkappa\Delta u(t,x) + \xi(x)u(t,x), && (4.1)\\ u(0,x) &= u_0(x), && (4.2)\end{aligned}$$

where $\xi(x) = \xi(x,\omega)$, $u_0(x) = u_0(x,\omega) \ge 0$ are random fields on $\mathbf{Z}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Let us introduce the notation

$$\ln_+ x := \begin{cases} \ln x, & \text{if } \ x > e,\\ 1, & \text{if } \ x \le e. \end{cases} \tag{4.3}$$

For $\rho \in \mathbf{R}$, denote by $\varPhi_\rho$ the class of non-negative functions $\varphi(x),\ x \in \mathbf{Z}^d$, such that

$$\limsup_{|x|\to\infty} \frac{\ln_+ \varphi(x)}{|x| \ln |x|} < \rho. \tag{4.4}$$

Clearly, $\varPhi_\rho \subset \varPhi_{\rho'}$ for $\rho < \rho'$.

Theorem 4.1. (See [6, Theorem 2.1].) *Suppose that the random field $\{\xi(x), x \in \mathbf{Z}^d\}$ is homogeneous, and let the following condition be satisfied:*

$$\left\langle \Bigl(\frac{\xi^+(0)}{\ln_+ \xi^+(0)} \Bigr)^d \right\rangle < \infty, \tag{4.5}$$

where $\xi^+ := \max(\xi, 0)$. Furthermore, assume that the field $\xi(\cdot)$ is $\mathbf{P}$-a.s. non-percolating from below (see Definition 3.1). Finally, let $u_0 \in \varPhi_1$ ($\mathbf{P}$-a.s.).

Then the problem $(4.1)+(4.2)$ $\mathbf{P}$-a.s. has a unique non-negative solution, which admits the Feynman–Kac representation

$$u(t,x) = E_x \bigl[\operatorname{Exp}[\xi(x_\cdot)]_0^t \, u_0(x_t) \bigr]. \tag{4.6}$$

Remark 4.1. A sufficient condition for the random field to be $\mathbf{P}$-a.s. non-percolating from below is provided by the so-called PF-estimate (see [12, Section 9, Theorem 9.1′]), asserting that for arbitrary $x_1, \dots, x_n \in \mathbf{Z}^d$

$$\mathbf{P}\{\xi(x_1) \le h, \dots, \xi(x_n) \le h\} \le CF^n(h), \qquad h \in \mathbf{R}, \tag{4.7}$$

where $F : \mathbf{R} \to \mathbf{R}_+$ is an appropriate non-decreasing function satisfying $F(h) \to 0$ as $h \to -\infty$, and $C > 0$ is a constant. In the simplest case, where the field $\xi(\cdot)$ has independent identically distributed values, the PF-estimate is obviously valid with $F(h) = \mathbf{P}\{\xi(0) \le h\}$ and $C = 1$. Moreover, the condition (4.7) is satisfied if the field $\xi(\cdot)$ has a finite correlation radius (see [12, Section 9]). Other examples and further references can be found in [12].

For the sake of further presentation, let us recall the scheme of the theorem's proof (see [6, Section 2.3]). By Lemma 3.1, in order to prove that (4.6) is a solution, it suffices to check $\mathbf{P}$-a.s. finiteness of the right-hand side of (4.6) for all $(t, x) \in \mathbf{R}_+ \times \mathbf{Z}^d$. From (4.6) we have the estimate

$$u(t, x) \le \sum_{n=0}^{\infty} P_x \big\{ \max_{s\in[0,t]} |x_s| = n \big\} \exp \big(t \max_{|y|\le n} \xi(y) + \max_{|y|\le n} \ln_+ u_0(y) \big). \tag{4.8}$$

Lemma 4.1. (See [6, Lemma 2.4].) *For each $x \in \mathbf{Z}^d$ and arbitrary $t > 0$*

$$\limsup_{n\to\infty} \frac{\ln P_x \{\max_{s\in[0,t]} |x_s| \ge n\}}{n \ln n} \le -1. \tag{4.9}$$

Lemma 4.2. (See [6, Lemma 2.5].) *Let the random variables $\{\xi(x),\, x \in \mathbf{Z}^d\}$ be identically distributed. If the condition* (4.5) *is satisfied then with probability* 1

$$\limsup_{|x|\to\infty} \frac{\xi(x)}{|x| \ln |x|} \le 0. \tag{4.10}$$

Remark 4.2. If in addition the random variables $\{\xi(x),\, x \in \mathbf{Z}^d\}$ are independent, then condition (4.5) is necessary for (4.10); moreover, its violation implies that with probability 1 the upper limit in (4.10) equals $+\infty$ (see [6, Lemma 2.5]).

By virtue of (4.9), (4.10), and the condition $u_0 \in \Phi_1$ (see (4.4)), we get that the common term of the series in (4.8) is dominated, with probability 1, by $O(n^{-\varepsilon n})$, where $\varepsilon = \varepsilon(\omega) > 0$. This implies that the right-hand side of (4.8) is $\mathbf{P}$-a.s. finite as claimed.

As for uniqueness of the solution (4.6), it is automatically guaranteed by Lemma 3.2, since the field $\xi(\cdot)$ is assumed to non-percolate from below ($\mathbf{P}$-a.s.).

Let us now proceed to the non-homogeneous Cauchy problem with random data:

$$\begin{aligned} \partial_t v(t, x) &= \varkappa \Delta v(t, x) + \xi(x) v(t, x) + \eta(t, x), && (4.11) \\ v(0, x) &= v_0(x), && (4.12) \end{aligned}$$

where $\xi(x) = \xi(x,\omega)$, $\eta(t,x) = \eta(t,x,\omega) \ge 0$, $v_0(x) = v_0(x,\omega) \ge 0$ are random fields on $\mathbf{Z}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Theorem 4.2. *Let all the hypotheses of Theorem* 4.1 *on the random field* $\{\xi(x),\ x \in \mathbf{Z}^d\}$ *be satisfied. Furthermore, assume that* $\mathbf{P}$*-a.s. the following conditions hold:*

(*a*) $v_0 \in \Phi_1$*;*

(*b*) *for each fixed* x*, the function* $\eta(t,x)$ *is continuous in* t*;*

(*c*) $Y(t,\cdot) := \max_{0\le s\le t} \eta(s,\cdot) \in \Phi_1$ *for each fixed* t*.*

Then with probability 1 *the Cauchy problem* (4.11) + (4.12) *has the unique non-negative solution*[2]

$$v(t,x) = E_x\left[\operatorname{Exp}[\xi(x_\cdot)]_0^t\, v_0(x_t)\right] + E_x\left[\int_0^t \operatorname{Exp}[\xi(x_\cdot)]_0^s\, \eta(t-s,x_s)\,ds\right]. \quad (4.13)$$

Proof. According to the first part of Theorem 3.1, in order to show that formula (4.13) provides a solution to the problem (4.11) + (4.12), it suffices to check that (4.13) is finite, with probability 1, for all $t \ge 0$, $x \in \mathbf{Z}^d$. But the conditions of our theorem imply that Theorem 4.1 is applicable, and hence Lemma 3.1 shows that the first summand in (4.13) is finite ($\mathbf{P}$-a.s.).

The second summand in (4.13) is estimated from above by

$$\begin{aligned} &E_x\left[\int_0^t \exp\Bigl(s \max_{u\in[0,t]} \xi(x_u)\Bigr) Y(t,x_s)\,ds\right] \\ &\quad\le t\sum_{n=0}^{\infty} P_x\Bigl\{\max_{s\in[0,t]} |x_s| = n\Bigr\} \exp\bigl(t\max_{|y|\le n} \xi^+(y) + \max_{|y|\le n} \ln_+ Y(t,y)\bigr), \quad (4.14)\end{aligned}$$

similarly to (4.8). Now, the same argument as that used in the proof of Theorem 4.1 justifies $\mathbf{P}$-a.s. finiteness of the expression (4.14).

Finally, the uniqueness of the solution (4.13) follows from the second part of Theorem 3.1 and the assumption of the $\mathbf{P}$-a.s. non-percolation from below. □

5. Feynman – Kac representation of the quenched moments

In this section, we obtain integral representations of the Feynman – Kac type for the moment functions $m_n(t,x,y)$, $m_n(t,x)$, which will be the basic tool for deriving the long-time asymptotics.

In the case $n = 1$, such a representation is well known (see [6, 13]).

[2] Cf. (4.13) with formula (3.6) for the solution of a homogeneous Cauchy problem (3.4) + (3.5).

Theorem 5.1. *Under the hypotheses of Theorem* 4.1 *on the random field* $\{\xi(x),\, x \in \mathbf{Z}^d\}$*, the quenched moments of the first order* **P**-a.s. *admit Feynman – Kac representation:*

$$m_1(t,x,y) = E_x \big[\operatorname{Exp}[\xi(x.)]_0^t \, \delta_y(x_t)\big], \tag{5.1}$$

$$m_1(t,x) = E_x \big[\operatorname{Exp}[\xi(x.)]_0^t\big]. \tag{5.2}$$

Proof. Clearly, the functions $\delta_y(\cdot)$, $\mathbf{1}(\cdot)$ determining the initial conditions (2.5) in equation (2.7) for $m_1(t,x,y)$, $m_1(t,x)$, respectively, belong to the class Φ_1. Applying Theorem 4.1, we obtain (5.1) and (5.2). □

Consider now the case $n \geq 2$. In order to apply Theorem 4.2, we need a lemma allowing one to control the growth of the solution $v(t,x)$ of the Cauchy problem (4.11) + (4.12) as $|x| \to \infty$. We set

$$\overline{\Phi}_0 := \bigcap_{\rho>0} \Phi_\rho. \tag{5.3}$$

Thus, the class $\overline{\Phi}_0$ consists of functions for which the upper limit in (4.4) equals 0.

Lemma 5.1. *Let the conditions of Theorem* 4.2 *be satisfied and, moreover,* **P**-a.s. $v_0(\cdot), Y(t,\cdot) \in \overline{\Phi}_0$ *(for all* $t \geq 0$*). Then with probability* 1 *the solution* $v(t,x)$ *defined in* (4.13) *also belongs to the class* $\overline{\Phi}_0$ *for all* $t \geq 0$*.*

Proof. In view of the elementary inequality

$$\ln_+(a+b) \leq \ln_+ a + \ln_+ b, \qquad a,b \geq 0, \tag{5.4}$$

which can be easily verified via the definition (4.3), it is sufficient to show that in the representation (4.13), each of the two summands v_1, v_2 is contained in the class $\overline{\Phi}_0$.

Similarly to (4.8), we estimate v_1:

$$\begin{aligned} v_1(t,x) &= E_0 \big[\operatorname{Exp}[\xi(x+x.)]_0^t \, v_0(x+x_t)\big] \\ &\leq \sum_{n=0}^{\infty} P_0 \big\{ \max_{s\in[0,t]} |x_s| = n \big\} \\ &\qquad \times \exp\big(t \max_{|y|\leq|x|+n} \xi^+(y) + \max_{|y|\leq|x|+n} \ln_+ v_0(y)\big). \end{aligned} \tag{5.5}$$

By Lemma 4.1, for each $\varepsilon > 0$ there exists $N \geq 1$ such that for all $n > N$

$$P_0 \big\{ \max_{s\in[0,t]} |x_s| = n \big\} \leq e^{-(1-\varepsilon)\, n \ln n}.$$

On the other hand, according to Lemma 4.2 and the condition $v_0 \in \overline{\Phi}_0$, with probability 1 there exists $r_0 = r_0(\omega)$ such that

$$\begin{aligned} \xi^+(x) &\le \varepsilon |x| \ln |x|, \\ \ln_+ v_0(x) &\le \varepsilon |x| \ln |x|, \end{aligned}$$

whenever $|x| > r_0$. For such x, from (5.5) we get **P**-a.s. the estimate

$$\begin{aligned} v_1(t,x) &\le \sum_{n=0}^{\infty} P_0 \Big\{ \max_{s\in[0,t]} |x_s| = n \Big\} \cdot e^{K+\varepsilon\,(t+1)\,(|x|+n)\,\ln\,(|x|+n)} \\ &\le e^{K+\varepsilon\,(t+1)\,(|x|+N)\,\ln\,(|x|+N)} \\ &\quad + \sum_{n=N+1}^{\infty} e^{K-(1-\varepsilon)\,n\,\ln\,n+\varepsilon\,(t+1)\,(|x|+n)\,\ln\,(|x|+n)}, \end{aligned} \tag{5.6}$$

where

$$K = K(r_0) := \max_{|y|\le r_0} \{ t\xi^+(y) + \ln_+ v_0(y) \}.$$

Let us estimate the sum on the right-hand side of (5.6) as a function of $|x|$. Let $\varepsilon_0 \in (0,1)$. The part of the sum in (5.6) extended over $n \le |x|^{\varepsilon_0}$ is bounded from above by

$$O(1)\, e^{\varepsilon\,(t+1)\,(|x|+|x|^{\varepsilon_0})\,\ln\,(|x|+|x|^{\varepsilon_0})} = O(1)\, e^{\varepsilon\,(t+1)\,|x|\,\ln\,|x|\,(1+o(1))}. \tag{5.7}$$

To estimate the remaining sum in (5.6) (i.e., over $n > \max\{|x|^{\varepsilon_0}, N+1\}$), let us first show that the function

$$f(u) := -(1-2\varepsilon)\,u \ln u + \varepsilon\,(t+1)\,(|x|+u) \ln\,(|x|+u)$$

is monotone decreasing on the set $u \ge |x|^{\varepsilon_0}$, provided $|x|$ is large enough. Indeed, the condition $f'(u) < 0$ is equivalent to the inequality

$$\frac{u}{(|x|+u)^{\varepsilon_1}} > e^{-(1-\varepsilon_1)}, \tag{5.8}$$

where

$$\varepsilon_1 := \frac{\varepsilon(t+1)}{1-2\varepsilon}.$$

By taking ε sufficiently small, we may and will suppose that $0 < \varepsilon_1 < 1$. Since the left-hand side of (5.8) is increasing in $u \ge 0$, inequality (5.8) will follow if we check, putting $u = |x|^{\varepsilon_0}$, that

$$\frac{|x|^{\varepsilon_0}}{(|x|+|x|^{\varepsilon_0})^{\varepsilon_1}} > 1. \tag{5.9}$$

But the inequality (5.9) is obviously fulfilled for $|x|$ large enough, once ε_0 has been chosen such that

$$0 < \varepsilon_1 < \varepsilon_0 < 1.$$

Coming back to (5.6), by the monotonicity of $f(u)$ we have

$$\begin{aligned}\sum_{n>|x|^{\varepsilon_0}} e^{K-\varepsilon n\ln n}\cdot e^{f(n)} &\leq e^{f(|x|^{\varepsilon_0})}\sum_{n=1}^{\infty} e^{K-\varepsilon n\ln n}\\ &\leq O(1)\, e^{\varepsilon(t+1)|x|\ln|x|\,(1+o(1))}. \qquad (5.10)\end{aligned}$$

Therefore, according to (5.6), (5.7), and (5.10) we have

$$\limsup_{|x|\to\infty}\frac{\ln_+ v_1(t,x)}{|x|\ln|x|}\leq \varepsilon(t+1).$$

Passing now to the limit as $\varepsilon\to 0$ yields $v_1(t,\cdot)\in\overline{\Phi}_0$ as required.

The summand v_2 in (4.13) is considered similarly, via the inequality

$$v_2(t,x)\leq t\sum_{n=0}^{\infty}P_0\bigl\{\max_{s\in[0,t]}|x_s|=n\bigr\}\exp\bigl(t\max_{|y|\leq|x|+n}\xi^+(y)+\max_{|y|\leq|x|+n}\ln_+ Y(t,y)\bigr)$$

(cf. (4.14), (5.5)) and using the condition $Y(t,x)\in\overline{\Phi}_0$. □

Theorem 5.2. *Let the hypotheses of Theorem* 4.1 *on the random potential* $\xi(\cdot)$ *be fulfilled. Assume moreover that the random field* $\{(\xi_0(x),\xi_2(x)),\ x\in\mathbf{Z}^d\}$ *determining the branching environment is homogeneous, and let the distribution of* ξ_2 *satisfy the condition*

$$\Bigl\langle\Bigl(\frac{\ln_+\xi_2(0)}{\ln_+\ln_+\xi_2(0)}\Bigr)^d\Bigr\rangle<\infty. \qquad (5.11)$$

Then for each $n\in\mathbf{N}$ *with probability* 1 *the following representations are valid:*

$$m_n(t,x,y) = m_1(t,x,y)+\widetilde{m}_n(t,x,y), \qquad (5.12)$$
$$m_n(t,x) = m_1(t,x)+\widetilde{m}_n(t,x), \qquad (5.13)$$

where

$$\widetilde{m}_n(t,x,y) := E_x\Bigl[\int_0^t \mathrm{Exp}[\xi(x_\cdot)]_0^s\,\xi_2(x_s)\,h_n[m_1,\dots,m_{n-1}](t-s,x_s,y)\,ds\Bigr], \qquad (5.14)$$

$$\widetilde{m}_n(t,x) := E_x\Bigl[\int_0^t \mathrm{Exp}[\xi(x_\cdot)]_0^s\,\xi_2(x_s)\,h_n[m_1,\dots,m_{n-1}](t-s,x_s)\,ds\Bigr], \qquad (5.15)$$

and $h_n[m_1,\dots,m_{n-1}]$ *is defined in* (2.6).

Proof. Along with the moments $m_n(t,x,y)$, $m_n(t,x)$, $n\in\mathbf{N}$, satisfying equations (2.4) with respective initial conditions (2.5), let us consider the sequence $\{m_n^{(+)}(t,x)\}$, defined recursively

as the solution of the same chain of equations (2.4) with unit initial conditions (see (2.5)), but with potential ξ replaced by ξ^+:

$$\partial_t m_n^{(+)} = \varkappa\Delta m_n^{(+)} + \xi^+(x) m_n^{(+)} + \xi_2(x) h_n[m_1^{(+)},\dots,m_{n-1}^{(+)}]. \tag{5.16}$$

We now prove by induction that *for all* $n\in\mathbf{N}$ *with probability* 1 *the following hypothesis* H_n *is true:*

(a) $m_n(t,x,y)$, $m_n(t,x)$ *satisfy equations* (5.12), (5.13), *respectively;*

(b) $m_n^{(+)}(t,x)$ *has a similar representation*

$$m_n^{(+)}(t,x) = m_1^{(+)}(t,x) + \widetilde{m}_n^{(+)}(t,x), \tag{5.17}$$

where $\widetilde{m}_n^{(+)}(t,x)$ *is defined as in* (5.15) *but with* ξ *replaced by* ξ^+ *and* $h_n[m_1,\dots,m_{n-1}]$ *by* $h_n[m_1^{(+)},\dots,m_{n-1}^{(+)}]$*;*

(c) *for all* $t\ge 0$, $x,y\in\mathbf{Z}^d$

$$m_n(t,x,y)\le m_n(t,x)\le m_n^{(+)}(t,x)\,;$$

(d) $m_n^{(+)}(t,x)$ *is monotone increasing in* t *for each* $x\in\mathbf{Z}^d$*;*

(e) $m_n^{(+)}(t,\cdot)\in\overline{\varPhi}_0$ *for all* $t\ge 0$.

Note that H_n(a) is just the statement of the theorem.

For $n=1$, the statements (a) and (b) are obviously true, because we have formally set $h_1\equiv 0$ (see (2.6)). Using for $m_1(t,x,y)$, $m_1(t,x)$, $m_1^{(+)}(t,x)$ the representation of Theorem 5.1, it is easy to see that the statements H_1(c), H_1(d) are also valid. Next, an application of Lemma 5.1 proves H_1(e).

Assume that the hypotheses $\mathrm{H}_1,\dots,\mathrm{H}_{n-1}$ have already been proved. In order to apply Theorem 4.2 to the equations of the form (2.4) for $m_n(t,x,y)$, $m_n(t,x)$, $m_n^{(+)}(t,x)$, let us check that the functions

$$\begin{aligned} Y_n(t,x,y) &:= \xi_2(x)\max_{s\in[0,t]} h_n[m_1,\dots,m_{n-1}](s,x,y),\\ Y_n(t,x) &:= \xi_2(x)\max_{s\in[0,t]} h_n[m_1,\dots,m_{n-1}](s,x),\\ Y_n^{(+)}(t,x) &:= \xi_2(x)\max_{s\in[0,t]} h_n[m_1^{(+)},\dots,m_{n-1}^{(+)}](s,x) \end{aligned} \tag{5.18}$$

with probability 1 belong to the class $\overline{\varPhi}_0\subset\varPhi_1$ for all $t\ge 0$. Using the inequalities H_i(c) ($i=1,\dots,n-1$) and taking into account the increase monotonicity of the polynomial $h_n[m_1,\dots,m_{n-1}]$ in each of its arguments (see (2.6)), we obtain for all $t\ge 0$, $x,y\in\mathbf{Z}^d$

$$Y_n(t,x,y)\le Y_n(t,x)\le Y_n^{(+)}(t,x).$$

Therefore, it suffices to prove that

$$Y_n^{(+)}(t,\cdot)\in\overline{\Phi}_0. \tag{5.19}$$

To this end, let us observe that by the monotonicity of the functions $m_i^{(+)}(t,x)$ in t (according to the conditions H_i(d), $i=1,\dots,n-1$), the maximum in (5.18) is attained for $s=t$:

$$\begin{aligned} Y_n^{(+)}(t,x) &\equiv \xi_2(x)\,h_n[m_1^{(+)},\dots,m_{n-1}^{(+)}](t,x)\\ &= \xi_2(x)\sum_{i=1}^{n-1}\binom{n}{i}m_i^{(+)}(t,x)m_{n-i}^{(+)}(t,x). \end{aligned}$$

By Lemma 4.2 applied to $\ln_+\xi_2(\cdot)$, from (5.11) it follows that $\mathbf{P}$-a.s. $\xi_2(\cdot)\in\overline{\Phi}_0$. According to the induction hypotheses H_i(e), the functions $m_i^{(+)}(t,\cdot)$ are also in the class $\overline{\Phi}_0$ for $i=1,\dots,n-1$. Hence, in view of (5.4) and by the obvious inequality

$$\ln_+(ab)\le\ln_+a+\ln_+b,\qquad a,b\ge 0,$$

the inclusion (5.19) follows. As a result, we may apply Theorem 4.2, which proves H_n(b) and hence H_n(a).

Next, H_n(c) easily follows from the representations (5.12), (5.13) and (5.17), and the form of the functions $\widetilde{m}_n^{(+)}(t,x)$, $\widetilde{m}_n^{(+)}(t,x)$ and $\widetilde{m}_n^{(+)}(t,x)$, by using the properties H_i(c) ($i=1,\dots,n-1$). In a similar way, H_n(d) follows from H_i(d) ($i=1,\dots,n-1$). Finally, using (5.19) and applying Lemma 5.1, we conclude that H_n(e) is true as well.

Thus, the induction step is complete and the theorem is proved. □

6. Asymptotics of the annealed moments $\langle m_1^p\rangle$

Before turning to technical preparations for the proof of our first theorem, we record a lemma about the scaling property of the first-order quenched moments. This simple but important relation plays a crucial role in our arguments and will be exploited a few times in the sequel. Let us introduce in the notation the parameters of the branching random walk under consideration, viz., the diffusion constant $\varkappa$ and the environment realization ω, by writing explicitly $m_1^{(\varkappa,\omega)}$.

Lemma 6.1. *Let the conditions of Theorem* 4.1 *on the random potential* $\xi(\cdot)$ *be fulfilled. Then for any* $\beta > 0$, *the first-order moments* **P**-a.s. *satisfy the following scaling identities:*

$$m_1^{(\varkappa,\beta\omega)}(t,x,y) \;=\; m_1^{(\varkappa/\beta,\,\omega)}(\beta t,x,y), \tag{6.1}$$

$$m_1^{(\varkappa,\beta\omega)}(t,x) \;=\; m_1^{(\varkappa/\beta,\,\omega)}(\beta t,x). \tag{6.2}$$

Proof. According to Section 2 (see equation (2.7)), the function $u(t,x) := m_1^{(\varkappa/\beta,\,\omega)}(t,x)$ is a solution to the Cauchy problem

$$\begin{aligned} \partial_t u(t,x) &= \frac{\varkappa}{\beta}\Delta u(t,x) + \xi(x)u(t,x),\\ u(0,x) &\equiv 1. \end{aligned}$$

Therefore, the function $w(t,x) := m_1^{(\varkappa/\beta,\,\omega)}(\beta t,x)$ solves the Cauchy problem

$$\begin{aligned} \partial_t w(t,x) &= \varkappa\Delta w(t,x) + \beta\xi(x)w(t,x),\\ w(0,x) &\equiv 1. \end{aligned}$$

On the other hand, $m_1^{(\varkappa,\beta\omega)}(t,x)$ also satisfies the latter problem, and due to uniqueness, these two solutions coincide **P**-a.s. and (6.2) follows. The other relation, (6.1), is proved in exactly the same way with an obvious change in the initial condition. □

Remark 6.1. The proof of this lemma, as it stands, is a formal verification of the identities (6.1), (6.2), giving no idea of their origin. However, in fact these relations have a very clear meaning. Indeed, let Z_t denote a generic realization of the branching random walk under consideration (in a frozen branching environment ω), governed by the quenched law $\mathsf{P}_x^{(\varkappa,\omega)}$ (we use the same notation as in Lemma 6.1). Along with Z_t, let us consider the time-scaled process

$$\tilde{Z}_t := Z_{\beta t}, \qquad t \ge 0, \tag{6.3}$$

where $\beta > 0$ is a scaling parameter. That is to say, the process $\tilde{Z}_t$ has the same realizations as the former process, but the time now goes β times slower. Let $\tilde{\mathsf{P}}_x^{(\varkappa,\omega)}$ stand for the law of the new process. We claim that

$$\tilde{\mathsf{P}}_x^{(\varkappa,\omega)} = \mathsf{P}_x^{(\beta\varkappa,\beta\omega)}. \tag{6.4}$$

Heuristically, relation (6.4) can be seen by recalling that the process evolution is governed by independent exponential times, or "clocks", responsible for either jumping or branching of particles. (It is just the parameters of those clocks, i.e., the jumping and branching rates, which now explicitly appear in the notation.) But for exponential random variables, it is straightforward to check the above scaling property. For example, if τ and $\tilde{\tau}$ are the amounts of time spent by the processes Z_t and $\tilde{Z}_t$, respectively, at the initial state (i.e., until the first transition, either by jumping or by branching), then (6.3) implies that $\tau = \beta\tilde{\tau}$. Therefore,

$$\begin{aligned}\tilde{\mathsf{P}}_x^{(\varkappa,\omega)}\{\tilde{\tau} > t\} &= \mathsf{P}_x^{(\varkappa,\omega)}\{\tau > \beta t\} = e^{-(\varkappa+\xi_0(x)+\xi_2(x))\beta t} \\ &= e^{-(\beta\varkappa+\beta\xi_0(x)+\beta\xi_2(x))t} = \mathsf{P}_x^{(\beta\varkappa,\beta\omega)}\{\tau > t\},\end{aligned}$$

which is in agreement with (6.4).

This argument can be made rigorous (and even more simple!) by using the semigroup approach. Namely, the transition semigroup $Q_t^{(\varkappa,\omega)}$ associated with the Markov process Z_t with the law $\mathsf{P}_x^{(\varkappa,\omega)}$ acts on the space of continuous bounded functions defined on the Polish state space of the process Z_t. The semigroup $Q_t^{(\varkappa,\omega)}$ is then strongly continuous, as can be checked by methods like, e.g., in [4, Chapter 4] or [11, Chapter I]. Let A be its generator, so that $Q_t^{(\varkappa,\omega)} = e^{tA}$. Under the scaling $t \mapsto \beta t$, the semigroup $Q_t^{(\varkappa,\omega)}$ is transformed as follows:

$$Q_t^{(\varkappa,\omega)} = e^{tA} \mapsto e^{(\beta t)A} = e^{t(\beta A)} = Q_t^{(\beta\varkappa,\beta\omega)}.$$

Hence, the new semigroup $\tilde{Q}_t^{(\varkappa,\omega)}$ corresponding to the scaled process coincides with $Q_t^{(\beta\varkappa,\beta\omega)}$, which is equivalent to (6.4).

Now, as an immediate consequence of (6.4), we obtain the following identities for the average numbers of particles:

$$\begin{aligned}m_1^{(\varkappa,\omega)}(\beta t, x, y) &= m_1^{(\beta\varkappa,\beta\omega)}(t, x, y), \\ m_1^{(\varkappa,\omega)}(\beta t, x) &= m_1^{(\beta\varkappa,\beta\omega)}(t, x).\end{aligned}$$

Replacing here $\varkappa$ by $\varkappa/\beta$, we get the statement of Lemma 6.1.

To formulate the next lemma, let us introduce the *cumulant generating function* of the random potential $\xi(\cdot)$ (having, by our assumptions, identically distributed values):

$$G_\xi(t) \equiv G(t) := \ln\langle e^{t\xi(0)}\rangle \equiv \ln\langle e^{t\xi(x)}\rangle. \tag{6.5}$$

Lemma 6.2. *Let $F(z) := \mathbf{P}\{\xi(0) \le z\}$ denote the distribution function of the potential $\xi(\cdot)$. Assume that its upper tail has a Weibull type asymptotics:*

$$\lim_{z\to+\infty} z^{-\alpha} \ln(1 - F(z)) = -c, \tag{6.6}$$

for some $\alpha > 1$, $c > 0$. Set $\alpha' := \alpha/(\alpha-1) > 1$. Then

$$\lim_{t\to\infty} t^{-\alpha'} G(t) = (\alpha-1)\alpha^{-\alpha'} c^{-1/(\alpha-1)} =: \gamma(\alpha, c). \tag{6.7}$$

Proof. Passing from ξ to $c^{1/\alpha}\xi$, it is easy to see that it suffices to prove the lemma in the case $c = 1$. According to (6.5),

$$e^{G(t)} = \langle e^{t\xi(0)}\rangle = \int_{-\infty}^{\infty} e^{tz}\, dF(z) = t\int_{-\infty}^{\infty} (1 - F(z))\, e^{tz}\, dz.$$

By condition (6.6) (with $c = 1$), for any $0 < \varepsilon < 1$ there exists $z_0 = z_0(\varepsilon) > 0$ such that for all $z \geq z_0$

$$(1 - \varepsilon) z^{\alpha} \leq -\ln(1 - F(z)) \leq (1 + \varepsilon) z^{\alpha}.$$

Therefore,

$$\delta_{\varepsilon}^{(-)}(t) + J_{\varepsilon}^{(-)}(t) \geq e^{G(t)} \geq \delta_{\varepsilon}^{(+)}(t) + J_{\varepsilon}^{(+)}(t), \tag{6.8}$$

where

$$\begin{aligned} \delta_{\varepsilon}^{(\pm)}(t) \;&:=\; t\int_{-\infty}^{z_0} e^{tz}\,(1 - F(z))\, dz - t\int_{0}^{z_0} e^{tz-(1\pm\varepsilon)z^{\alpha}}\, dz, \\ J_{\varepsilon}^{(\pm)}(t) \;&:=\; t\int_{0}^{\infty} e^{tz-(1\pm\varepsilon)z^{\alpha}}\, dz. \end{aligned} \tag{6.9}$$

The first term is easily estimated:

$$|\delta_{\varepsilon}^{(\pm)}(t)| \leq 2t\int_{-\infty}^{z_0} e^{tz}\, dz = 2e^{tz_0}. \tag{6.10}$$

Further, by the substitution $z \mapsto t^{1/(\alpha-1)}z$ the expression (6.9) is reduced to the form

$$J_{\varepsilon}^{(\pm)}(t) = t^{\alpha'}\int_{0}^{\infty} e^{t^{\alpha'}(z-(1\pm\varepsilon)z^{\alpha})}\, dz.$$

The maximum of the integrand is attained at $z = z_* := \bigl(\alpha(1 \pm \varepsilon)\bigr)^{-1/(\alpha-1)}$. Applying the Laplace method (see, e.g., [2, Chapter 4]), we easily find the rough asymptotics of the above integral:

$$\begin{aligned} \lim_{t\to\infty} t^{-\alpha'} \ln J_{\varepsilon}^{(\pm)}(t) \;&=\; z_* - (1 \pm \varepsilon) z_*^{\alpha} \;=\; z_*(1 - (1 \pm \varepsilon) z_*^{\alpha-1}) \\ &=\; z_*(1 - \alpha^{-1}) \;=\; (1 \pm \varepsilon)^{-1/(\alpha-1)}(\alpha - 1)\,\alpha^{\alpha'} \\ &=\; (1 \pm \varepsilon)^{-1/(\alpha-1)}\gamma(\alpha, 1), \end{aligned} \tag{6.11}$$

in accordance with (6.7).

Comparing (6.10) and (6.11) and taking into account that $\alpha' > 1$, from (6.8) we obtain

$$\begin{aligned} (1 + \varepsilon)^{-1/(\alpha-1)}\gamma(\alpha, 1) \;&\leq\; \liminf_{t\to\infty} t^{-\alpha'} G(t) \\ &\leq\; \limsup_{t\to\infty} t^{-\alpha'} G(t) \;\leq\; (1 - \varepsilon)^{-1/(\alpha-1)}\gamma(\alpha, 1), \end{aligned}$$

whence by passing to the limit $\varepsilon \to 0$ it follows

$$\lim_{t\to\infty} t^{-\alpha'} G(t) = \gamma(\alpha, 1),$$

as required. □

Lemma 6.3. *Suppose that $f_1(t)$, $f_2(t)$ $(t \geq 0)$ are continuous positive functions, such that*

$$\lim_{t\to\infty} t^{-a} \ln f_i(t) = C_i > 0, \qquad i = 1, 2,$$

where $a > 1$. Then their convolution

$$W(t) := f_1 * f_2(t) = \int_0^t f_1(s)\, f_2(t-s)\, ds$$

satisfies

$$\lim_{t\to\infty} t^{-a} \ln W(t) = \max(C_1, C_2). \tag{6.12}$$

Proof. Let us write down the functions $\ln f_i(t)$ in the form

$$\ln f_i(t) = C_i t^a + \varphi_i(t), \qquad i = 1, 2.$$

By the assumptions of the lemma, for any $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that for all $t \geq 0$

$$|\varphi_i(t)| \leq K + \varepsilon t^a, \qquad i = 1, 2.$$

Consequently,

$$W_0(t)\, e^{-2K-2\varepsilon t^a} \leq W(t) \leq W_0(t)\, e^{2K+2\varepsilon t^a}, \tag{6.13}$$

where

$$W_0(t) := \int_0^t e^{C_1 s^a + C_2 (t-s)^a}\, ds.$$

By the substitution $s \mapsto ts$ this integral is reduced to the form

$$W_0(t) = t \int_0^1 e^{t^a [C_1 s^a + C_2 (1-s)^a]}\, ds.$$

The maximum of the integrand is attained at an end-point of the segment $[0, 1]$ and is equal to $\max(C_1, C_2) =: C$ (in case $C_1 = C_2$, the values at $s = 0$ and $s = 1$ are equal to each other). Hence, the Laplace method [2, Chapter 4] yields

$$\lim_{t\to\infty} t^{-a} \ln W_0(t) = C. \tag{6.14}$$

Then, by (6.13) and (6.14),

$$\begin{aligned} -2\varepsilon + C \quad &\leq \quad \liminf_{t\to\infty} t^{-a} W(t) \\ &\leq \quad \limsup_{t\to\infty} t^{-a} \ln W(t) \quad \leq \quad 2\varepsilon + C. \end{aligned}$$

Letting ε tend to zero, we obtain (6.12). □

In a similar way, one can prove a "multivariate" version of the previous lemma.

Lemma 6.4. *Under the conditions of Lemma* 6.3, *the integral*

$$W^{(p)}(t) := \int_0^t \cdots \int_0^t f_1(s_1 + \cdots + s_p)\, f_2(pt - s_1 - \cdots - s_p)\, ds_1 \cdots ds_p$$

has the following asymptotics:

$$\lim_{t\to\infty} t^{-a} \ln W^{(p)}(t) = Cp^a, \tag{6.15}$$

where $C := \max(C_1, C_2)$.

Proof. As in the proof of Lemma 6.3, it is sufficient to establish (6.15) for integrals of the form

$$\begin{aligned} W_0^{(p)}(t) \quad &:= \quad \int_0^t \cdots \int_0^t e^{C_1(s_1+\cdots+s_p)^a + C_2(pt-s_1-\cdots-s_p)^a}\, ds_1 \cdots ds_p \qquad (6.16)\\ &= \quad t^p \int_0^1 \cdots \int_0^1 e^{t^a[C_1(s_1+\cdots+s_p)^a + C_2(p-s_1-\cdots-s_p)^a]}\, ds_1 \cdots ds_p. \end{aligned}$$

It is not difficult to verify that the maximum of the function $C_1 s^a + C_2 (p-s)^a$, considered on the segment $[0, p]$, is attained at an end-point of the segment and is equal to Cp^a, where $C = \max(C_1, C_2)$. Applying to (6.16) the Laplace method for multiple integrals (see, e.g., [2, Chapter 4, Section 4.6]), we obtain asymptotics (6.15). □

Theorem 6.1. *Assume that the random potential* $\{\xi(x),\, x \in \mathbf{Z}^d\}$ *is homogeneous and all the conditions of Theorem* 5.1 *are satisfied, so that the first-order moments* $m_1(t, x, y)$, $m_1(t, x)$ *are* $\mathbf{P}$-a.s. *represented via the Feynman – Kac formula* (*see* (5.1), (5.2)). *Suppose also that there exist constants* $\alpha > 1$ *and* $c > 0$ *such that*

$$\lim_{z\to+\infty} z^{-\alpha} \ln(1 - F(z)) = -c, \tag{6.17}$$

where $F(z) := \mathbf{P}\{\xi(x) \le z\} \equiv \mathbf{P}\{\xi(0) \le z\}$ *is the distribution function of the potential* $\xi(\cdot)$.[3] *Then for the annealed moments* $\langle m_1^p(t, x)\rangle$, $\langle m_1^p(t, x, y)\rangle$, *for any real* $p \ge 1$ *the limiting relation holds*

$$\lim_{t\to\infty} t^{-\alpha'} \ln \langle m_1^p\rangle = \gamma(\alpha, c)\, p^{\alpha'}, \tag{6.18}$$

where $\alpha' := \alpha/(\alpha - 1) > 1$ *and* $\gamma(\alpha, c)$ *is defined in* (6.7).

[3] It is easy to see that the condition (4.5) required by the hypotheses of Theorem 5.1 is automatically satisfied due to the limiting relation (6.17).

Proof. ¿From the very definition of moments, it is clear that $0 \le m_1(t,x,y) \le m_1(t,x)$ (this also follows from the representations (5.1), (5.2)). Therefore, for the proof of the theorem it suffices to find an appropriate upper estimate for $\langle m_1^p(t,x)\rangle$ and a respective lower estimate for $\langle m_1^p(t,x,y)\rangle$.

(i) Upper estimate (see [6, page 637]). Applying Lyapunov's and Jensen's inequalities (the latter in its integral form) and using Fubini's theorem, we obtain

$$\begin{aligned}
\langle m_1^p(t,x)\rangle &= \left\langle\left(E_x\left[\exp\left\{\int_0^t \xi(x_s)\,ds\right\}\right]\right)^p\right\rangle \\
&\le \left\langle E_x\left[\exp\left\{p\int_0^t \xi(x_s)\,ds\right\}\right]\right\rangle = E_x\left\langle \exp\left\{\frac{1}{t}\int_0^t pt\xi(x_s)\,ds\right\}\right\rangle \\
&\le E_x\left\langle \frac{1}{t}\int_0^t e^{pt\xi(x_s)}\,ds\right\rangle = \frac{1}{t}\int_0^t E_x\,\langle e^{pt\xi(x_s)}\rangle ds.
\end{aligned} \tag{6.19}$$

Since the potential $\xi(\cdot)$ has identically distributed values, the inner expectation under the integral sign in (6.19) does not depend on s, being thus equal to $\langle e^{pt\xi(0)}\rangle = e^{G(pt)}$. Hence, from (6.19) we get

$$\langle m_1^p(t,x)\rangle \le e^{G(pt)},$$

and by Lemma 6.2

$$\limsup_{t\to\infty} t^{-\alpha'} \ln\langle m_1^p(t,x)\rangle \le \gamma(\alpha,c)\,p^{\alpha'}. \tag{6.20}$$

(ii) Lower estimate. Let us show that

$$\liminf_{t\to\infty} t^{-\alpha'} \ln\langle m_1^p(t,x,y)\rangle \ge \gamma(\alpha,c)\,p^{\alpha'}. \tag{6.21}$$

If $y = x$, the required estimate is easily obtained by taking into account in the Feynman−Kac formula (5.1) only those paths of the random walk x_s which stay at the initial point x during the whole time interval $[0,t]$. Indeed, let τ denote the time spent by the random walk in the initial state until exit. Since τ is exponentially distributed with parameter $\varkappa$, we have

$$\begin{aligned}
m_1(t,x,x) &\ge E_x\left[I\{\tau > t\}\,\exp\Big(\int_0^t \xi(x_s)\,ds\Big)\delta_x(x_t)\right] \\
&= P_x\{\tau > t\}\cdot e^{\xi(x)t} = e^{-\varkappa t}\cdot e^{\xi(x)t}.
\end{aligned}$$

Therefore,

$$\langle m_1^p(t,x,x)\rangle \ge e^{-p\varkappa t}\langle e^{pt\xi(x)}\rangle = e^{-p\varkappa t}\cdot e^{G(pt)},$$

whence by Lemma 6.2, (6.21) follows.

If $y \neq x$, we consider a path connecting the points x, y and passing through pairwise different lattice sites, $x_0 = x$, x_1, ..., $x_n = y$, so that $|x_i - x_{i-1}| = 1$, $i = 1, \dots, n$, and $n = |y - x|$. By the strong Markov property applied to the Feynman–Kac formula (5.1) with the exit time τ as a stopping time (cf. above), we can write

$$\begin{aligned} m_1(t,x,y) &\geq E_x \Big[I\{\tau \leq t\}\, E_x \Big\{ \exp\Big(\int_0^t \xi(x_u)\, du \Big) \delta_y(x_t) \,\Big|\, \tau \Big\} \Big] \\ &= E_x \Big[I\{\tau \leq t\} \exp\Big(\int_0^\tau \xi(x_u)\, du \Big) \\ &\qquad \times E_z \Big[\exp\Big(\int_0^{\tilde t} \xi(\tilde x_u)\, du \Big) \delta_y(\tilde x_{\tilde t}) \Big]_{z = x_\tau,\, \tilde t = t - \tau} \Big] \\ &= E_x \,[I\{\tau \leq t\}\, e^{\tau \xi(x)}\, m_1(t-\tau, x_\tau, y)], \end{aligned} \tag{6.22}$$

where $\tilde x_u := z + x_{\tau+u} - x_\tau$, $u \geq 0$, is a shifted random walk started at point z. Recalling that τ is exponentially distributed and restricting ourselves to the case $x_\tau = x_1$ (which occurs with probability $1/(2d)$), from (6.22) we have

$$m_1(t,x,y) \geq \frac{\varkappa}{2d} \int_0^t e^{-\varkappa s}\, e^{s\xi(x)}\, m_1(t-s, x_1, y)\, ds.$$

Iterating this relation by taking into account successive jumps from $x = x_0$ to x_1, x_2, ..., $x_n = y$, yields

$$\begin{aligned} & m_1(t, x_0, x_n) \\ &\geq \Big(\frac{\varkappa}{2d}\Big)^n \int_0^t e^{-\varkappa s_1} e^{s_1 \xi(x_0)}\, ds_1 \int_0^{t-s_1} e^{-\varkappa s_2} e^{s_2\xi(x_1)}\, ds_2 \\ &\qquad \cdots \int_0^{t-s_1-\dots-s_{n-1}} e^{-\varkappa s_n} e^{s_n \xi(x_{n-1})} e^{-\varkappa(t-s_1-\dots-s_n)} e^{(t-s_1-\dots-s_n)\xi(x_n)}\, ds_n \\ &= \Big(\frac{\varkappa}{2d}\Big)^n e^{-\varkappa t} \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1-\dots-s_{n-1}} ds_n \\ &\qquad \times e^{s_1\xi(x_0)}\, e^{s_2\xi(x_1)} \dots e^{s_n\xi(x_{n-1})}\, e^{(t-s_1-\dots-s_n)\xi(x_n)}. \end{aligned} \tag{6.23}$$

Consider first the case $p = 1$. Since each factor under the integral in (6.23), being of the form $\exp\{s\xi(x)\}$, is an increasing function of the field $\xi(\cdot)$, we can apply the FKG inequality (see, e.g., [9, Chapter 2, § 2.2]) to conclude that

$$\begin{aligned} & \big\langle e^{s_1\xi(x_0)}\, e^{s_2\xi(x_1)} \dots e^{s_n\xi(x_{n-1})}\, e^{(t-s_1-\dots-s_n)\xi(x_n)} \big\rangle \\ &\geq \big\langle e^{s_1\xi(x_0)} \big\rangle \cdot \big\langle e^{s_2\xi(x_1)} \big\rangle \cdots \big\langle e^{s_n\xi(x_{n-1})} \big\rangle \cdot \big\langle e^{(t-s_1-\dots-s_n)\xi(x_n)} \big\rangle \\ &= e^{G(s_1)} \cdot e^{G(s_2)} \cdots e^{G(s_n)} \cdot e^{G(t-s_1-\dots-s_n)}. \end{aligned}$$

Therefore, taking expectation $\langle\cdot\rangle$ of (6.23), we have

$$\begin{aligned}\langle m_1(t,x_0,x_n)\rangle &\ge \Big(\frac{\varkappa}{2d}\Big)^n e^{-\varkappa t}\int_0^t ds_1\int_0^{t-s_1} ds_2\cdots\int_0^{t-s_1-\cdots-s_{n-1}} ds_n \\ &\quad\times e^{G(s_1)}\,e^{G(s_2)}\cdots e^{G(s_n)}\,e^{G(t-s_1-\cdots-s_n)} \\ &= \Big(\frac{\varkappa}{2d}\Big)^n e^{-\varkappa t}\underbrace{f*\cdots*f}_{n}(t), \qquad (6.24)\end{aligned}$$

where $f(t):=e^{G(t)}$. Using Lemma 6.2 and n times applying Lemma 6.3 with $a=\alpha'$, from (6.24) we get

$$\liminf_{t\to\infty} t^{-\alpha'}\ln\langle m_1(t,x,y)\rangle \ge \gamma(\alpha,c).$$

Thus, for $p=1$ inequality (6.21) is proved.

For integer $p\ge 2$, we take the pth power of the estimate (6.23) and rewrite the product of integrals as a multiple integral. Collecting terms with $\xi(x_0)$, $\xi(x_1)$, ..., $\xi(x_n)$ and again applying the FKG inequality, similarly as in (6.24) we obtain

$$\begin{aligned}&\langle m_1^p(t,x_0,x_n)\rangle \\ &\ge \Big(\frac{\varkappa}{2d}\Big)^{pn} e^{-\varkappa pt}\int_0^t\cdots\int_0^t f(s_{11}+\cdots+s_{1p})\,ds_{11}\cdots ds_{1p} \\ &\quad\times\int_0^{t-s_{11}}\cdots\int_0^{t-s_{1p}} f(s_{21}+\cdots+s_{2p})\,ds_{21}\cdots ds_{2p} \\ &\quad\cdots\int_0^{t-s_{11}-\cdots-s_{n-1,1}}\cdots\int_0^{t-s_{1p}-\cdots-s_{n-1,p}} f(s_{n1}+\cdots+s_{np}) \\ &\quad\times f\Big(pt-\textstyle\sum_{i=1}^n\sum_{j=1}^p s_{ij}\Big)\,ds_{n1}\cdots ds_{np}. \qquad (6.25)\end{aligned}$$

Again using Lemma 6.2 and n times applying Lemma 6.4, we arrive at (6.21).

Now, let $p\ge 1$ be a real number. We can represent it as $p=k/r$ with $k\in\mathbf{N}$, $r>1$. (For instance, one can set $k=[p]+1$, where $[p]$ is the integer part of the number p; then $r=([p]+1)/p>1$.) Since $1/r<1$, Lyapunov's inequality implies that

$$\begin{aligned}m_1^{1/r}(t,x,y) &= \Big(E_x\Big[\exp\Big\{\int_0^t\xi(x_s)\,ds\Big\}\delta_y(x_t)\Big]\Big)^{1/r} \\ &\ge E_x\Big[\exp\Big\{\frac{1}{r}\int_0^t\xi(x_s)\,ds\Big\}\delta_y(x_t)\Big]. \qquad (6.26)\end{aligned}$$

According to the Feynman–Kac representation (5.1), the right-hand side of (6.26) amounts to $m_1^{(\varkappa,\omega/r)}(t,x,y)$. Due to Lemma 6.1, this can be rewritten as $m_1^{(r\varkappa,\,\omega)}(t/r,x,y)$. As a result, the estimate (6.26) takes the form

$$m_1^{1/r}(t,x,y) \ge m_1^{(r\varkappa,\,\omega)}(t/r,x,y). \tag{6.27}$$

Taking the kth power of (6.27) and applying to the right-hand side the estimate (6.21) already proved for integer powers, we get

$$\begin{aligned}\liminf_{t\to\infty} t^{-\alpha'} \ln\langle m_1^{k/r}(t,x,y)\rangle \quad &\ge \quad r^{-\alpha'} \liminf_{t\to\infty} \frac{\ln\bigl\langle\bigl(m_1^{(r\varkappa,\,\omega)}(t/r,x,y)\bigr)^k\bigr\rangle}{(t/r)^{\alpha'}}\\ &\ge \quad r^{-\alpha'}\cdot\gamma(\alpha,c)\,k^{\alpha'}\\ &= \quad \gamma(\alpha,c)\,(k/r)^{\alpha'} \quad = \quad \gamma(\alpha,c)\,p^{\alpha'},\end{aligned}$$

and therefore (6.21) is proved in full generality.

It remains to notice that combining estimates (6.20) and (6.21) yields the statement of the theorem. □

7. Some auxiliary propositions

Before proceeding to the proof of our main theorem on the higher-order moments (see Section 8 below), we prove a few lemmas which will be instrumental for obtaining the upper estimate in the theorem.

Lemma 7.1. *For a positive random variable X, the ratio of its consecutive integer moments $m_i := E[X^i]$ $(i = 1, 2, \ldots)$ is monotone decreasing:*

$$\frac{m_i}{m_{i+1}} \ge \frac{m_{i+1}}{m_{i+2}}\,, \qquad i \in \mathbf{N}. \tag{7.1}$$

Proof. Using the Cauchy – Schwarz inequality, we have

$$m_{i+1}^2 = (E[X^{i+1}])^2 = (E[X^{i/2}\cdot X^{(i+2)/2}])^2 \le E[X^i]\cdot E[X^{i+2}] = m_i\cdot m_{i+2},$$

whence our claim follows. □

Lemma 7.2. *For any integer $n \ge 2$ and all $i = 1, \ldots, n-1$, the moments $m_i = E[X^i]$ satisfy the inequalities*

$$m_i\, m_{n-i} \le m_1\, m_{n-1}. \tag{7.2}$$

Proof. By Lemma 7.1, for each natural j

$$\frac{m_j}{m_{j+1}} \ge \frac{m_{n-i+j-1}}{m_{n-i+j}}. \tag{7.3}$$

Multiplying inequalities (7.3) over $j = 1, \dots, i-1$, we obtain

$$\frac{m_1}{m_i} = \prod_{j=1}^{i-1} \frac{m_j}{m_{j+1}} \ge \prod_{j=1}^{i-1} \frac{m_{n-i+j-1}}{m_{n-i+j}} = \frac{m_{n-i}}{m_{n-1}},$$

and (7.2) is proved. □

Lemma 7.3. *Under the assumptions of Theorem* 5.2, *for any* $n \ge 2$

$$m_n(t,x) \le m_1(t,x) + 2^n \int_0^t E_x\{\mathrm{Exp}[\xi(x.)]_0^s\, \xi_2(x_s)(m_1 \cdot m_{n-1})(t-s, x_s)\}ds. \tag{7.4}$$

Proof. Applying Lemma 7.2 to the function $h_n[m_1, \dots, m_{n-1}]$ defined in (2.6), we get

$$h_n[m_1, \dots, m_{n-1}] = \sum_{i=1}^{n-1} \binom{n}{i} m_i m_{n-i} \le \sum_{i=1}^{n-1} \binom{n}{i} \cdot m_1 m_{n-1} < 2^n m_1 m_{n-1}.$$

Substituting this into (5.15) and using equation (5.13), we arrive at (7.4). □

For the simple symmetric random walk $(x_t)_{t\ge 0}$ with generator $\varkappa\Delta$, consider the random variables

$$\tau(y;t) := \int_0^t I\{x_s = y\}\, ds = \mathrm{mes}\,\{s \in [0,t] : x_s = y\}, \qquad y \in \mathbf{Z}^d, \quad t \ge 0, \tag{7.5}$$

where $\mathrm{mes}\,(\cdot)$ is the Lebesgue measure on the line. In other words, $\tau(y;t)$ is the *local time*, that is the total time spent by the random walk $x.$ at point y until time moment t. Obviously, in terms of the local times, the Feynman – Kac representation (5.2) for the first-order moment $m_1(t,x)$ takes the form

$$m_1(t,x) = E_x[e^{\sum_y \xi(y)\tau(y;t)}] = E_0[e^{\sum_y \xi(y)\tau(y-x;t)}], \tag{7.6}$$

where E_0 is the expectation with respect to the distribution of the random walk $x.$ started from the origin. Our nearest goal is to find a similar representation (to be more accurate, an upper estimate) for the higher-order moments.

Let us introduce an auxiliary family of simple symmetric random walks on $\mathbf{Z}^d$ (with generator $\varkappa\Delta$):

$$\{x_t^{(k)},\, \bar{x}_t^{(k)},\, t \ge 0\}_{k=1}^{\infty},$$

all starting at zero and mutually independent. Expectation with respect to their (joint) distribution will be denoted by E. Introduce the corresponding local times (cf. (7.5))

$$\tau_k(y;t) := \int_0^t I\{x_s^{(k)} = y\}\, ds, \qquad \bar{\tau}_k(y;t) := \int_0^t I\{\bar{x}_s^{(k)} = y\}\, ds. \tag{7.7}$$

For $t \geq 0$, denote by $S_n(t)$ the (n-dimensional) simplex

$$S_n(t) := \{(s_0, \ldots, s_n) : s_i \geq 0,\ s_0 + \cdots + s_n = t\} \qquad (n \geq 0).$$

The null-dimensional simplex $S_0(t)$ consists of a single point t. The points of $S_n(t)$ will also be denoted by $\sigma_n := (s_0, \ldots, s_n)$. We shall write integrals over a simplex in the form

$$\int_{S_n(t)} \varphi(\sigma_n)\, d\sigma_n = \int_{S_n(t)} \varphi(s_0, \ldots, s_n)\, d\sigma_n, \tag{7.8}$$

setting for $n = 0$

$$\int_{S_0(t)} \varphi(\sigma_0)\, d\sigma_0 := \varphi(t).$$

Integral (7.8) reduces to the iterated integral

$$\begin{aligned} \int_{S_n(t)} \varphi(\sigma_n)\, d\sigma_n &= \int_0^t ds_n \int_{S_{n-1}(t-s_n)} \varphi(\sigma_{n-1}, s_n)\, d\sigma_{n-1} = \cdots \\ &= \int_0^t ds_n \int_0^{t-s_n} ds_{n-1} \\ &\quad \cdots \int_0^{t-s_n-\cdots-s_2} ds_1\, \varphi(t - s_1 - \cdots - s_n, s_1, \ldots, s_n). \end{aligned} \tag{7.9}$$

For $y \in \mathbf{Z}^d$ and $\sigma_{n-1} \in S_{n-1}(t)$ $(n \in \mathbf{N})$, we define the random variables $T_n(y; \sigma_{n-1})$ recursively by setting

$$T_1(y; \sigma_0) := \tau_1(y; t), \tag{7.10}$$

$$\begin{aligned} T_{n+1}(y; \sigma_n) &:= \tau_{n+1}(y; s_n) + \bar{\tau}_n(y - x_{s_n}^{(n+1)}; t - s_n) \\ &\quad + T_n(y - x_{s_n}^{(n+1)}; \sigma_{n-1}) \qquad (n \geq 1). \end{aligned} \tag{7.11}$$

Remark 7.1. It is easy to understand that the quantity $T_n(y; \sigma_{n-1})$ has the meaning of the *branching local time*, that is the total time spent at point y by a certain branching random walk whose branching epochs (arranged in the reverse order) are determined by the vector $\sigma_{n-1} = (s_0, \ldots, s_{n-1})$ and such that after each branching only one of the two descendants retains the reproductive ability (namely, the one which moves according to a random walk denoted *without* the bar).

Let us also introduce the notation $V_n(x; \sigma_{n-1})$ by the recursion

$$V_1(x; \sigma_0) := 1, \tag{7.12}$$

$$V_{n+1}(x; \sigma_n) := \xi_2(x + x_{s_n}^{(n+1)}) \cdot V_n(x + x_{s_n}^{(n+1)}; \sigma_{n-1}) \quad (n \geq 1). \tag{7.13}$$

Finally, we put

$$\Theta_n(x; \sigma_{n-1}) := V_n(x; \sigma_{n-1})\, e^{\sum_y \xi(y)\, T_n(y - x; \sigma_{n-1})}. \tag{7.14}$$

Lemma 7.4. *Let the the conditions of Theorem* 5.2 *be fulfilled. Then for each* $n \ge 1$ *the following estimate holds:*

$$m_n(t,x) \le \sum_{k=1}^{n} c_{kn} \int_{S_{k-1}(t)} E\left[\Theta_k(x;\sigma_{k-1})\right] d\sigma_{k-1}, \tag{7.15}$$

where $\{c_{kn}\}_{k=1}^{n}$ *are certain numerical coefficients.*

Proof. Let us proceed by induction in n. For $n = 1$, (7.15) is valid due to the Feynman – Kac representation (7.6):

$$m_1(t,x) = E_0\, [e^{\sum_y \xi(y)\tau(y-x,t)}] = \int_{S_0(t)} E\,[\Theta_1(x;\sigma_0)] d\sigma_0 \tag{7.16}$$

(so that one can put $c_{11} = 1$).

Assume now that (7.15) holds for all $k \le n$. Using Lemma 7.3 and inequality (7.15), we obtain

$$\begin{aligned}
&m_{n+1}(t,x)\\
&\quad\le\; m_1(t,x) + 2^{n+1}\int_0^t E_x\,[\xi_2(x_s) e^{\sum_y \xi(y)\tau(y;s)} (m_1\cdot m_n)(t-s,x_s)]ds\\
&\quad\le\; m_1(t,x) + 2^{n+1}\sum_{k=1}^{n} c_{kn}\int_0^t ds E_x\left[\xi_2(x_s) e^{\sum_y \xi(y)\tau(y;s)} m_1(t-s,x_s)\right.\\
&\qquad\quad\left.\times\int_{S_{k-1}(t-s)} d\sigma_{k-1} E\,[\Theta_k(z;\sigma_{k-1})]_{z=x_s}\right].
\end{aligned} \tag{7.17}$$

Let us take $x_{\cdot} \equiv x + x_{\cdot}^{(k+1)}$ as the random walk $x_{\cdot}$ implied in the expectation E_x, and represent the factor $m_1(t-s,x_s)$ by the Feynman – Kac formula (7.6) via the random walk $\bar{x}_{\cdot}^{(k)}$ (shifted at $x + x_s^{(k+1)}$). For the notational convenience, for a fixed k we replace the integration variable s in (7.17) by s_k. Then the integral term on the right-hand side of (7.17) is rewritten as

$$\begin{aligned}
&\int_0^t ds_k\, E\left[\xi_2(x+x_{s_k}^{(k+1)})\, e^{\sum_y \xi(y)\tau_{k+1}(y-x;s_k)}\, E\left[e^{\sum_y \xi(y)\bar{\tau}_k(y-x-x_{s_k}^{(k+1)})}\,\middle|\, x_{s_k}^{(k+1)}\right]\right.\\
&\qquad\left.\times\int_{S_{k-1}(t-s_k)} E\,[\Theta_k(x+x_{s_k}^{(k+1)};\sigma_{k-1}) \mid x_{s_k}^{(k+1)}] d\sigma_{k-1}\right]\\
&\quad=\;\int_0^t ds_k \int_{S_{k-1}(t-s_k)} d\sigma_{k-1} E\left[\xi_2(x+x_{s_k}^{(k+1)})\, e^{\sum_y \xi(y)\tau_{k+1}(y-x;s_k)}\right.\\
&\qquad\left.\times E\left[e^{\sum_y \xi(y)\bar{\tau}_k(y-x-x_{s_k}^{(k+1)})}\,\Theta_k(x+x_{s_k}^{(k+1)};\sigma_{k-1})\,\middle|\, x_{s_k}^{(k+1)}\right]\right]\\
&\quad=\;\int_{S_k(t)} E\left[\xi_2(x+x_{s_k}^{(k+1)})\, e^{\sum_y \xi(y)[\tau_{k+1}(y-x;s_k)+\bar{\tau}_k(y-x-x_{s_k}^{(k+1)})]}\right.\\
&\qquad\left.\times\,\Theta_k(x+x_{s_k}^{(k+1)};\sigma_{k-1})\right] d\sigma_k,
\end{aligned} \tag{7.18}$$

where we used formula (7.9). Taking into account the definition of Θ_k (see (7.14)) and also the recurrence relations (7.11), (7.13), expression (7.18) takes the form

$$\int_{S_k(t)} E\,[\Theta_{k+1}(x;\sigma_k)]\,d\sigma_k. \tag{7.19}$$

Substituting (7.19) into the right-hand side of (7.17) and using (7.16), we obtain

$$\begin{aligned} m_{n+1}(t,x) &\le m_1(t,x) + 2^{n+1}\sum_{k=1}^{n} c_{kn}\int_{S_k(t)} E\,[\Theta_{k+1}(x;\sigma_k)]d\sigma_k \\ &= \sum_{k=1}^{n+1} c_{k,n+1}\int_{S_{k-1}(t)} E\,[\Theta_k(x;\sigma_{k-1})]d\sigma_{k-1}, \end{aligned} \tag{7.20}$$

where we put

$$c_{k,n+1} := \begin{cases} 1, & k=1, \\ 2^{n+1}c_{k-1,n}, & k\ge 2. \end{cases}$$

Thus, the induction step is completed and the lemma is proved. □

Let us now obtain a few lemmas related to the averaging with respect to the distribution of the branching medium.

Lemma 7.5. *Under the assumptions of Theorem* 5.2*, for all $x \in \mathbf{Z}^d$, $n \ge 1$, and any $t \ge 0$, $\sigma_{n-1} \in S_{n-1}(t)$, the moments of all orders $\beta > 0$ of the random variable V_n defined in* (7.12), (7.13)*, are bounded:*

$$\langle V_n^\beta(x;\sigma_{n-1})\rangle \le C_{n,\beta} := \langle \xi_2^{\beta(n-1)}(0)\rangle < \infty. \tag{7.21}$$

Let us emphasize that the constant $C_{n,\beta}$ does not depend on the random walk x_t involved in the definition (7.13).

Proof. First of all, note that the condition (5.11) of Theorem 5.2 implies that the expectation on the right-hand side of (7.21) is indeed finite.

For $n = 1$, the inequality (7.21) is clearly true (see (7.12)). Suppose that (7.21) is valid for some $n \ge 1$. Then, applying to the recurrence relation (7.13) Hölder's inequality with exponents $1/n$, $1/n'$, where $n' := n/(n-1)$, we get

$$\begin{aligned} \langle V_{n+1}^\beta(x;\sigma_n)\rangle &= \langle \xi_2(x+x_{s_n}^{(n+1)})\cdot V_n(x+x_{s_n}^{(n+1)};\sigma_{n-1})\rangle \\ &\le \langle \xi_2^{\beta n}(x+x_{s_n}^{(n+1)})\rangle^{1/n}\cdot\langle V_n^{\beta n'}(x+x_{s_n}^{(n+1)};\sigma_{n-1})\rangle^{1/n'} \\ &\le \langle \xi_2^{\beta n}(0)\rangle^{1/n}\cdot\langle \xi_2^{\beta n'(n-1)}(0)\rangle^{1/n'} \;=\; \langle \xi_2^{\beta n}(0)\rangle \;=\; C_{n+1,\beta}, \end{aligned}$$

using that the values of the random field $\xi_2(\cdot)$ are identically distributed. Thus, the lemma is proved. □

Given the random field $\{\xi(x),\, x\in\mathbf{Z}^d\}$, let us introduce its *cumulant generating functional* $\mathfrak{G}_\xi[\cdot]\equiv\mathfrak{G}[\cdot]$ defined on functions $\varphi\colon\mathbf{Z}^d\to\mathbf{R}$ by

$$\mathfrak{G}[\varphi]:=\ln\langle e^{\sum_y \xi(y)\varphi(y)}\rangle. \tag{7.22}$$

The functional $\mathfrak{G}$ is well defined at least on finitely supported functions φ (that is, such that $\#\{y:\varphi(y)\neq 0\}<\infty$). The cumulant generating function $G(t)$ introduced above (see (6.5)) can obviously be expressed through $\mathfrak{G}$ as follows:

$$G_{\xi(x)}(t)=\mathfrak{G}_\xi[t\delta_x(\cdot)],\qquad x\in\mathbf{Z}^d.$$

The next lemma extends the convexity property well-known for ordinary cumulant generating functions (see, e.g., [5, Chapter V, § 8(c)]), to cumulant generating functionals.

Lemma 7.6. *The functional $\mathfrak{G}[\cdot]$ is convex, that is, for each $\alpha\in(0,1)$ and any functions φ_1, φ_2 in the domain of $\mathfrak{G}$, one has*

$$\mathfrak{G}[\alpha\varphi_1+(1-\alpha)\varphi_2]\le\alpha\mathfrak{G}[\varphi_1]+(1-\alpha)\mathfrak{G}[\varphi_2]. \tag{7.23}$$

Proof. By Hölder's inequality,

$$\begin{aligned} e^{\mathfrak{G}[\alpha\varphi_1+(1-\alpha)\varphi_2]} &= \langle e^{\alpha\sum_y\xi(y)\varphi_1(y)}\cdot e^{(1-\alpha)\sum_y\xi(y)\varphi_2(y)}\rangle\\ &\le \langle e^{\sum_y\xi(y)\varphi_1(y)}\rangle^{\alpha}\cdot\langle e^{\sum_y\xi(y)\varphi_2(y)}\rangle^{1-\alpha}\\ &= e^{\alpha\mathfrak{G}[\varphi_1]}\cdot e^{(1-\alpha)\mathfrak{G}[\varphi_2]}, \end{aligned}$$

whence by passing to logarithms we get (7.23). □

Lemma 7.7. *If the random field $\{\xi(x),\, x\in\mathbf{Z}^d\}$ is homogeneous, then its cumulant generating functional $\mathfrak{G}$ is translation invariant, in the sense that for any $x\in\mathbf{Z}^d$ and all functions φ in the domain of $\mathfrak{G}$*

$$\mathfrak{G}[\varphi(\cdot-x)]=\mathfrak{G}[\varphi(\cdot)].$$

Proof. Using the definition of $\mathfrak{G}$ (see (7.22)), we have

$$\begin{aligned} \mathfrak{G}[\varphi(\cdot-x)] &= \ln\langle e^{\sum_y\xi(y)\varphi(y-x)}\rangle = \ln\langle e^{\sum_y\xi(y+x)\varphi(y)}\rangle\\ &= \ln\langle e^{\sum_y\xi(y)\varphi(y)}\rangle = \mathfrak{G}[\varphi(\cdot)], \end{aligned}$$

due to the homogeneity of $\xi(\cdot)$. □

In the next lemma, we remark that the cumulant generating functional $\mathfrak{G}$ has a simple scaling property which is a direct consequence of the definition (7.22).

Lemma 7.8. *For any $\beta\in\mathbf{R}$*

$$\mathfrak{G}_\xi[\beta\varphi]=\mathfrak{G}_{\beta\xi}[\varphi]. \tag{7.24}$$

The following lemma gives an important representation of the annealed first-order moment, $\langle m_1(t,x)\rangle$, in terms of the cumulant generating functional $\mathfrak{G} \equiv \mathfrak{G}_\xi$ applied to the local-times function of the auxiliary random walk x_t.

Lemma 7.9. *Let the conditions of Theorem* 5.1 *be fulfilled. Then*

$$\langle m_1(t,x)\rangle = E_0\,[e^{\mathfrak{G}[\tau(\cdot\,;\,t)]}]. \tag{7.25}$$

Proof. First of all, we note that $\tau(y;\,t)$ as a function of $y \in \mathbf{Z}^d$ is P_x-a.s. finitely supported (for any $x \in \mathbf{Z}^d$), because the random walk x_t with P_x-probability 1 visits only a finite number of lattice sites at a finite time. Therefore, the right-hand side of (7.25) is well defined.

Now, from representation (7.6), by using Fubini's theorem and the definition of $\mathfrak{G}$, it follows

$$\begin{aligned}\langle m_1(t,x)\rangle &= \left\langle E_0\left[e^{\sum_y \xi(y)\tau(y-x;t)}\right]\right\rangle = E_0\left\langle e^{\sum_y \xi(y)\tau(y-x;t)}\right\rangle\\ &= E_0\,[e^{\mathfrak{G}[\tau(\cdot-x;t)]}] = E_0\,[e^{\mathfrak{G}[\tau(\cdot\,;\,t)]}],\end{aligned}$$

where in the last step we used the translation invariance of $\mathfrak{G}$ (see Lemma 7.7). □

Lemma 7.10. *Let $t \ge 0$ and $\sigma_{n-1} = (s_0,\dots,s_{n-1}) \in S_{n-1}(t)$ $(n \in \mathbf{N})$. Then for all $\lambda, \beta \ge 0$*

$$E\,[e^{\lambda\,\mathfrak{G}[\beta T_n(\cdot\,;\sigma_{n-1})]}] \le E_0\,[e^{\lambda\,\mathfrak{G}[\beta n\tau(\cdot\,;t)]}], \tag{7.26}$$

where $T_n(\cdot;\sigma_{n-1})$ is the branching local time defined in (7.10), (7.11) *and $\tau(\cdot;t)$ is the local time corresponding to the random walk x_t started at zero.*

Proof. Let us use induction. For $n = 1$, inequality (7.26) is obviously satisfied. Assume that the statement of the lemma is valid for all $k \le n$. By the recurrence relation (7.11), we have for $\sigma_n \in S_n(t)$:

$$\begin{aligned}T_{n+1}(y;\sigma_n) &= \frac{1}{n+1}\cdot(n+1)\,[\tau_{n+1}(y;s_n) + \bar{\tau}_n(y - x^{(n+1)}_{s_n}; t - s_n)]\\ &\quad + \frac{n}{n+1}\cdot\frac{n+1}{n}\,T_n(y - x^{(n+1)}_{s_n};\sigma_{n-1}).\end{aligned} \tag{7.27}$$

Let us multiply equation (7.27) by β and apply the functional $\mathfrak{G}$. Then, using the convexity of $\mathfrak{G}$ (Lemma 7.6) and its translation invariance (Lemma 7.7), we get

$$
\begin{aligned}
&\mathfrak{G}[\beta T_{n+1}(\cdot;\sigma_n)] \\
&\quad\le \frac{1}{n+1}\,\mathfrak{G}[\beta(n+1)\{\tau_{n+1}(\cdot;s_n)+\bar{\tau}_n(\cdot-x^{(n+1)}_{s_n};t-s_n)\}] \\
&\qquad + \frac{n}{n+1}\,\mathfrak{G}\Big[\frac{\beta(n+1)}{n}\,T_n(\cdot-x^{(n+1)}_{s_n};\sigma_{n-1})\Big] \\
&\quad= \frac{1}{n+1}\,\mathfrak{G}[\beta(n+1)\{\tau_{n+1}(\cdot+x^{(n+1)}_{s_n};s_n)+\bar{\tau}_n(\cdot;t-s_n)\}] \qquad (7.28) \\
&\qquad + \frac{n}{n+1}\,\mathfrak{G}\Big[\frac{\beta(n+1)}{n}\,T_n(\cdot;\sigma_{n-1})\Big]. \qquad (7.29)
\end{aligned}
$$

Note that the random variable T_n contained in (7.29) is built from the random walks $\{x^{(k)}_\cdot,\ k=1,\dots,n\}$, $\big\{\bar{x}^{(k)}_\cdot,\ k=1,\dots,n-1\big\}$ (see (7.11)), whereas expression (7.28) involves only the walks $x^{(n+1)}_\cdot$, $\bar{x}^{(n)}_\cdot$. This implies that the random variables (7.28) and (7.29) are independent, and hence

$$
\begin{aligned}
&E\left[\,\exp\{\lambda\mathfrak{G}[\beta T_{n+1}(\cdot;\sigma_n)]\}\right] \\
&\quad\le E\left[\exp\Big\{\frac{\lambda}{n+1}\mathfrak{G}[\beta(n+1)\{\tau_{n+1}(\cdot;s_n)+\bar{\tau}_n(\cdot-x^{(n+1)}_{s_n};t-s_n)\}]\Big\}\right] \\
&\qquad\times E\left[\exp\Big\{\frac{n\lambda}{n+1}\mathfrak{G}\Big[\frac{\beta(n+1)}{n}\,T_n(\cdot;\sigma_{n-1})\Big]\Big\}\right] \qquad (7.30)
\end{aligned}
$$

(under the first expectation on the right-hand side of (7.30), we made a reverse shift by $-x^{(n+1)}_{s_n}$ and again used translation invariance of $\mathfrak{G}$).

Let us continue the random walk $x^{(n+1)}_\cdot$ beyond the point s_n via the random walk $\bar{x}^{(n)}_\cdot$, independent of the former one. In other words, we put

$$
x_s := \begin{cases} x^{(n+1)}_s, & s\le s_n, \\ x^{(n+1)}_{s_n}+\bar{x}^{(n)}_{s-s_n}, & s>s_n. \end{cases} \qquad (7.31)
$$

Obviously, the "glued" process (7.31) is again a random walk (started from zero), and the sum $\tau_{n+1}(y;s_n)+\bar{\tau}_n(y-x^{(n+1)}_{s_n};t-s_n)$ is nothing else but its local time $\tau(y;t)$. Hence, the first factor on the right-hand side of (7.30) takes the form

$$
E_0\left[\exp\Big\{\frac{\lambda}{n+1}\mathfrak{G}[\beta(n+1)\tau(\cdot;t)]\Big\}\right].
$$

Applying the induction hypothesis to the second factor in (7.30) and using Lyapunov's inequality, we finally obtain from (7.30)

$$
\begin{aligned}
&E\left[\exp\{\lambda\mathfrak{G}[\beta T_{n+1}(\cdot;\sigma_n)]\}\right]\\
&\quad\le\ E_0\left[\exp\Big\{\frac{\lambda}{n+1}\mathfrak{G}[\beta(n+1)\tau(\cdot;t)]\Big\}\right]\cdot E_0\left[\exp\Big\{\frac{n\lambda}{n+1}\mathfrak{G}[\beta(n+1)\tau(\cdot;t)]\Big\}\right]\\
&\quad\le\ \big(E_0\left[\exp\{\lambda\mathfrak{G}[\beta(n+1)\tau(\cdot;t)]\}\right]\big)^{1/(n+1)}\\
&\qquad\times\big(E_0\left[\exp\{\lambda\mathfrak{G}[\beta(n+1)\tau(\cdot;t)]\}\right]\big)^{n/(n+1)}\\
&\quad=\ E_0\left[e^{\lambda\mathfrak{G}[\beta(n+1)\tau(\cdot;t)]}\right].
\end{aligned}
$$

This completes the induction step, and therefore the lemma is proved. □

8. Asymptotics of the annealed moments $\langle m_n^p\rangle$

We are now in a position to give the proof of the second part of our main result dealing with the case $n\ge 2$.

Theorem 8.1. *Assume that the hypotheses of Theorem* 6.1 *are fulfilled. In addition, let the condition* (5.11) *be satisfied. Then for all integers $n\ge\mathbf{N}$ and for each real $p\ge 1$, the annealed moments $\langle m_n^p(t,x)\rangle$, $\langle m_n^p(t,x,y)\rangle$ have the following asymptotics:*

$$\lim_{t\to\infty} t^{-\alpha'}\ln\langle m_n^p\rangle=\gamma(\alpha,c)\,(np)^{\alpha'},\tag{8.1}$$

where $\alpha':=\alpha/(\alpha-1)$ and $\gamma(\alpha,c)$ is defined in (6.7).

Proof. As in the proof of Theorem 6.1, we have to obtain an appropriate upper estimate for $\langle m_n^p(t,x)\rangle$ and a corresponding lower estimate for $\langle m_n^p(t,x,y)\rangle$.

(i) Upper estimate. Lemma 7.4 implies

$$\langle m_n^p(t,x)\rangle\le\Big\langle\Big(\sum_{k=1}^n c_{kn}\int_{S_{k-1}(t)}E\left[\Theta_k(x;\sigma_{k-1})\right]d\sigma_{k-1}\Big)^p\Big\rangle.\tag{8.2}$$

Using the elementary inequality

$$\Big\langle\Big(\sum_{k=1}^n X_k\Big)^p\Big\rangle\le n^{p-1}\sum_{k=1}^n\langle X_k^p\rangle$$

(with $X_1,\dots,X_n\ge 0$ and $p\ge 1$), which easily follows by Jensen's inequality, from (8.2) we obtain

$$\langle m_n^p(t,x)\rangle\le n^{p-1}\sum_{k=1}^n c_{kn}^p\Big\langle\Big(\int_{S_{k-1}(t)}E\left[\Theta_k(x;\sigma_{k-1})\right]d\sigma_{k-1}\Big)^p\Big\rangle.\tag{8.3}$$

Twice applying Lyapunov's inequality yields

$$\begin{aligned}
&\Big(\int_{S_{k-1}(t)} E\,[\Theta_k(x;\sigma_{k-1})]d\sigma_{k-1}\Big)^p\\
&\quad\le\ |S_{k-1}(t)|^{p-1}\int_{S_{k-1}(t)} \big(E\,[\Theta_k(x;\sigma_{k-1})]\big)^p d\sigma_{k-1}\\
&\quad\le\ |S_{k-1}(t)|^{p-1}\int_{S_{k-1}(t)} E\,[\Theta_k^p(x;\sigma_{k-1})]d\sigma_{k-1}, \qquad (8.4)
\end{aligned}$$

where

$$|S_{k-1}(t)| := \int_{S_{k-1}(t)} d\sigma_{k-1} = \frac{t^{k-1}}{(k-1)!} \qquad (8.5)$$

is the $(k-1)$-dimensional volume of the simplex $S_{k-1}(t)$.

Therefore, substituting estimate (8.4) into (8.3) and using Fubini's theorem we obtain

$$\langle m_n^p(t,x)\rangle \le n^{p-1}\sum_{k=1}^{n} c_{kn}^p |S_{k-1}(t)|^{p-1}\int_{S_{k-1}(t)} E\,[\langle\Theta_k^p(x;\sigma_{k-1})\rangle]\,d\sigma_{k-1}. \qquad (8.6)$$

Let us apply Hölder's inequality to the equation defining the quantity Θ_k (see (7.14) with $n=k$). Then

$$\langle\Theta_k^p(x;\sigma_{k-1})\rangle \le \langle V_k^{pq'}(x;\sigma_{k-1})\rangle^{1/q'}\cdot\Big\langle e^{pq\sum_y \xi(y)\,T_k(y-x;\sigma_{k-1})}\Big\rangle^{1/q}, \qquad (8.7)$$

where $q>1$ is so far arbitrary and q' is its conjugate, $q' := q/(q-1)$. According to Lemma 7.5, the first factor on the right-hand side of (8.7) is bounded:

$$\langle V_k^{pq'}(x;\sigma_{k-1})\rangle \le C_{k,pq'} < \infty. \qquad (8.8)$$

Furthermore, by the definition of $\mathfrak{G}$ and by Lemma 7.7

$$\Big\langle e^{pq\sum_y \xi(y)\,T_k(y-x;\sigma_{k-1})}\Big\rangle = e^{\mathfrak{G}[pqT_k(\cdot-x;\sigma_{k-1})]} = e^{\mathfrak{G}[pqT_k(\cdot;\sigma_{k-1})]}. \qquad (8.9)$$

Substituting (8.8) and (8.9) into (8.7), we apply Lemma 7.10 to get

$$\begin{aligned}
E\,[\langle\Theta_k^p(x;\sigma_{k-1})\rangle] &\le C_{k,pq'}^{1/q'}\cdot E\,[e^{(1/q)\,\mathfrak{G}[pqT_k(\cdot;\sigma_{k-1})]}]\\
&\le C_{k,pq'}^{1/q'}\cdot E_0\,[e^{(1/q)\,\mathfrak{G}[kpq\tau(\cdot;t)]}]\\
&\le C_{k,pq'}^{1/q'}\cdot (E_0\,[e^{\mathfrak{G}[kpq\tau(\cdot;t)]}])^{1/q}, \qquad (8.10)
\end{aligned}$$

again using Lyapunov's inequality.

Observe now that by combining Lemmas 7.8 and 7.9 and recalling the notation introduced in Lemma 6.1, we can represent the expectation in (8.10) in the form

$$E_0\,[e^{\mathfrak{G}[kpq\tau(\cdot;t)]}] = \langle m_1^{(\varkappa,kpq\omega)}(t,0)\rangle.$$

Rewriting this once more by Lemma 6.1 and substituting the result into (8.10) yields

$$E\left[\langle\Theta_k^p(x;\sigma_{k-1})\rangle\right] \le C_{k,pq'}^{1/q'}\cdot\langle m_1^{(\varkappa/kpq,\,\omega)}(kpqt,0)\rangle^{1/q}. \tag{8.11}$$

Note that the right-hand part of inequality (8.11) does not depend on σ_{k-1}. Then, coming back to (8.6), we eventually obtain

$$\begin{aligned}
&\langle m_n^p(t,x)\rangle\\
&\quad\le\ n^{p-1}\sum_{k=1}^{n} c_{kn}^p |S_{k-1}(t)|^{p-1}\, C_{k,pq'}^{1/q'}\,\langle m_1^{(\varkappa/kpq,\,\omega)}(kpqt,0)\rangle^{1/q}\int_{S_{k-1}(t)} d\sigma_{k-1}\\
&\quad=\ n^{p-1}\sum_{k=1}^{n} c_{kn}^p |S_{k-1}(t)|^{p}\, C_{k,pq'}^{1/q'}\,\langle m_1^{(\varkappa/kpq,\,\omega)}(kpqt,0)\rangle^{1/q}.
\end{aligned}\tag{8.12}$$

Theorem 6.1 implies that the main contribution to the long-time asymptotics of the right-hand side of (8.12) is brought by the summand with $k=n$. According to (8.5), the factor $|S_{n-1}(t)|^p$ grows in a powerlike manner in t and hence can be neglected. Thus, from (8.12) using (6.18) we get

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\ln\langle m_n^p(t,x)\rangle}{t^{\alpha'}} &\le \limsup_{t\to\infty}\frac{\ln\langle m_1^{(\varkappa/npq,\,\omega)}(npqt,0)\rangle}{qt^{\alpha'}}\\
&= \frac{\gamma(\alpha,c)(npq)^{\alpha'}}{q}.
\end{aligned}\tag{8.13}$$

Finally, letting the parameter $q>1$ tend to 1, from (8.13) we arrive at the required upper estimate

$$\limsup_{t\to\infty}\frac{\ln\langle m_n^p(t,x)\rangle}{t^{\alpha'}} \le \gamma(\alpha,c)(np)^{\alpha'}. \tag{8.14}$$

(ii) *Lower estimate.* Recalling that m_n and m_1 are the moments (of orders n respectively 1) of the random variable $\mu_t(y)$ with respect to the probability measure $\mathsf{P}_x^{(\omega)}$, by Lyapunov's inequality we have

$$m_n(t,x,y)\ge m_1^n(t,x,y).$$

Therefore, using estimate (6.21), for all $p\ge 1$ we obtain

$$\liminf_{t\to\infty}\frac{\ln\langle m_n^p(t,x,y)\rangle}{t^{\alpha'}} \ge \liminf_{t\to\infty}\frac{\ln\langle m_1^{np}(t,x,y)\rangle}{t^{\alpha'}} \ge \gamma(\alpha,c)\,(np)^{\alpha'}. \tag{8.15}$$

As a result, from the two-sided estimates (8.14) and (8.15) equation (8.1) follows, and thus our theorem is proved. □

References

[1] S. Albeverio and L.V. Bogachev (2000) Branching random walk in a catalytic medium. I. Basic equations. *Positivity* **4**, 41–100.

[2] N.G. de Bruijn (1970) *Asymptotic Methods in Analysis.* Bibliotheca Mathematica; Vol. IV. North-Holland Publishing, Amsterdam, and Walters-Noordhoff, Groningen.

[3] R. Carmona and J. Lacroix (1990) *Spectral Theory of Random Schroedinger Operators.* Probability and Its Applications. Birkhäuser Verlag, Basel.

[4] S.N. Ethier and T.G. Kurtz. (1986) *Markov Processes. Characterization and Convergence.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York.

[5] W. Feller (1971) *An Introduction to Probability Theory and its Applications*, Vol. 2. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York.

[6] J. Gärtner and S.A. Molchanov (1990) Parabolic problems for the Anderson model. I: Intermittency and related topics. *Commun. Math. Phys.* **132**, 613–655.

[7] J. Gärtner and S.A. Molchanov (1998) Parabolic problems for the Anderson model. II: Second-order asymptotics and structure of high picks. *Probab. Theory and Relat. Fields* **111**, 17–55.

[8] I.I. Gihman and A.V. Skorohod (1973) *The Theory of Stochastic Processes*, Vol. II. Nauka, Moscow (in Russian). English translation (1975): *The Theory of Stochastic Processes* II. Grundlehren der mathematischen Wissenschaften; 218. Springer-Verlag, Berlin, Heidelberg, New York.

[9] G. Grimmett (1999) *Percolation.* Grundlehren der mathematischen Wissenschaften; 321, 2nd ed. Springer-Verlag, Berlin, Heidelberg, New York.

[10] T.E. Harris (1963) *The Theory of Branching Processes.* Grundlehren der mathematischen Wissenschaften; 119. Springer-Verlag, Berlin, Göttingen, Heidelberg.

[11] T.M. Liggett (1985) *Interacting Particle Systems.* Grundlehren der mathematischen Wissenschaften; 276. Springer-Verlag, New York, Berlin, Heidelberg.

[12] M.V. Men'shikov, S.A. Molchanov, and A.F. Sidorenko (1986) Percolation theory and some applications. *Itogi Nauki Tekh., Ser. Teor. Veroyatn., Mat. Stat., Teor. Kibern.* **24**, 53–110 (in Russian). English translation (1988): *J. Sov. Math.* **42**, 1766–1810.

[13] S.A. Molchanov (1994) Lectures on random media. *Lect. Notes Math.* **1581**, 242–411.

[14] S.A. Molchanov (1996) Reaction-diffusion equations in the random media: Localization and intermittency. In: *Nonlinear Stochastic PDEs. Hydrodynamic Limit and Burgers' Turbulence,* T. Funaki et al., (eds.) IMA, Math. Appl. **77**, 81–109. Springer-Verlag, Berlin, Heidelberg, New York.

[15] J.E. Moyal (1962) The general theory of stochastic population processes. *Acta Math.* **108**, 1–31.

[16] B.A. Sevastyanov (1958) Branching stochastic processes for particles diffusing in a restricted domain with absorbing boundaries. *Teor. Veroyatn. Primen.* **3**,

121–136 (in Russian). English translation (1958): *Theory Probab. and Appl.* **3**, 111–126.

[17] A.V. Skorohod (1964) Branching diffusion processes. *Teor. Veroyatn. Primen.* **9**, 492–497 (in Russian). English translation (1964): *Theor. Probab. Appl.* **9**, 445–449.

[18] E.B. Yarovaya (1990) Intermittency of leading moments in the model of branching process with diffusion in a random medium. *Vestn. Mosk. Univ., Ser. I: Math. Mech.* (4), 79–82 (in Russian). English translation (1990): *Mosc. Univ. Math. Bull.* **45** (4), 49–51.

[19] J. Yeh (1995) *Martingales and Stochastic Analysis.* Series on Multivariate Analysis; Vol. 1. World Scientific, Singapore.

[20] Ya.B. Zeldovich, S.A. Molchanov, A.A. Ruzmaikin and D.D. Sokoloff (1988) Intermittency, diffusion and generation in a nonstationary random medium. *Sov. Sci. Rev., Sect. C: Math. Phys. Rev.* **7**, 1–110.