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Extreme Value Theory for Random Exponentials

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Abstract. We study the limit distribution of upper extreme values of i.i.d. exponential samples $\{e^{tX_i}\}_{i=1}^N$ as $t \to \infty$, $N \to \infty$. Two cases are considered: (A) $\operatorname{ess\,sup} X = 0$ and (B) $\operatorname{ess\,sup} X = \infty$. We assume that the function $h(x) = -\log P\{X > x\}$ (case B) or $h(x) = -\log P\{X > -1/x\}$ (case A) is (normalized) regularly varying at ∞ with index $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). The growth scale of N is chosen in the form $N \sim e^{\lambda H_0(t)}$ $(0 < \lambda < \infty)$, where $H_0(t)$ is a certain asymptotic version of the function $H(t) := \log E[e^{tX}]$ (case B) or $H(t) = -\log E[e^{tX}]$ (case A). As shown earlier by Ben Arous et al. [5], there are critical points $\lambda_1 < \lambda_2$, below which the LLN and CLT, respectively, break down, whereas for $0 < \lambda < \lambda_2$ the limit laws for the sum $S_N(t) = e^{tX_1} + \cdots + e^{tX_N}$ prove to be stable, with characteristic exponent $\alpha = \alpha(\varrho, \lambda) \in (0, 2)$. In this paper, we obtain the (joint) limit distribution of the upper order statistics of the exponential sample. In particular, $M_{1,N} = \max\{e^{tX_i}\}_{i=1}^N$ has asymptotically the Fréchet distribution with parameter α. We also show that the empirical extremal measure converges (in fdd) to a Poisson random measure with intensity $d(x^{-\alpha})$. These results are complemented by explicit representations of the joint limit distribution of $S_N(t)$ and $M_{1,N}(t)$ (and in particular of their ratio) in terms of i.i.d. random variables with standard exponential distribution.

1. Introduction

In this work, we are concerned with the asymptotic analysis of random samples of the form $\{e^{tX_i}, i = 1, \dots, N\}$, where $X, X_1, X_2, \dots$ are independent identically distributed (i.i.d.) random variables and both t and N tend to infinity. More specifically, our goal is to study the limiting distribution of the upper order statistics of the sample (in particular, of the maximum $M_{1,N}(t) = \max\{e^{tX_1}, \dots, e^{tX_N}\}$) and also to explore the influence of the extreme values on the asymptotic behavior of the sums $S_N(t) = \sum_{i=1}^N e^{tX_i}$ depending on various rates of growth of the parameters t and N.

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It is natural to distinguish the following two cases:

Case A. X is *bounded* from above, $\operatorname{ess\,sup} X < \infty$,

Case B. X is *unbounded* from above, $\operatorname{ess\,sup} X = \infty$.

In case A, without loss in generality we may and will assume that the upper edge of the support of X is zero, $\operatorname{ess\,sup} X = 0$.

It is clear that the results will heavily depend on the structure of the upper tail of the distribution of X. In the present work, we focus on a fairly general class of distributions with the upper tail of the *Weibull/Fréchet* form

$$\mathrm{P}\{X > x\} \approx \begin{cases} \exp(-cx^{\varrho}) & \text{as } x \to \infty \text{ (case B)}, \\ \exp(-c(-x)^{-\varrho}) & \text{as } x \to 0- \text{ (case A)}, \end{cases} \tag{1.1}$$

where $1 < \varrho < \infty$ (case B) or $0 < \varrho < \infty$ (case A). In fact, cases B and A can be combined using the following $\pm$-convention: *in the symbols* $\pm$ *and* $\mp$*, the upper sign refers to case* B *and the lower sign to case* A; furthermore, the notation $x^{\pm}$ stands for the power $x^{\pm 1}$, and $f(x)^{\pm}$ is understood as $[f(x)]^{\pm}$. Setting

$$h(x) := -\log \mathrm{P}\{X > \pm x^{\pm}\}, \tag{1.2}$$

the precise meaning of (1.1) is then furnished by the assumption that the function h is regularly varying at infinity with index $\varrho \in (1, \infty)$ (case B) or $\varrho \in (0, \infty)$ (case A). For example, a normal distribution fits in case B with $\varrho = 2$.

Systematic study of this class of random exponentials has been initiated by Ben Arous et al. [4] (case B only), later on extended to both cases B and A in [5] (see also an earlier preprint [3]). (Let us also mention that the "limiting" value $\varrho = 0$ in case B, leading to random variables of the form X_i^t, has been recently considered by Bogachev [6].) Motivation comes from various areas in theoretical and applied probability, including the problem of a unified treatment of limit laws for sums and extreme values (see Schlather [22]), analysis of the free energy and its fluctuations in the Random Energy Model (REM) of spin glass (see Bovier et al. [7] and further references therein and in [5]), branching random walks in random environments (see [3, 5]), and risk theory (see [3, 5] and references therein).

One can expect that if the number of terms N in the sum $S_N(t)$ grows fast enough relative to the parameter t, then the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) should hold true in a conventional form. Ben Arous et al. [5] have shown that an appropriate growth scale of $N = N(t)$ is of the form

$$N \sim \mathrm{e}^{\lambda H_0(t)} \quad (t \to \infty), \tag{1.3}$$

where the rate function $H_0(t)$ is a certain asymptotic version of the cumulant generating function

$$H(t) = \pm \log \mathrm{E}[\mathrm{e}^{tX}]. \tag{1.4}$$

By a Tauberian theorem of Kasahara–de Bruijn (see Section 2 below), the function H is regularly varying with index ϱ' such that

$$\pm \frac{1}{\varrho} + \frac{1}{\varrho'} = 1. \tag{1.5}$$

The link between $h(x)$ and $H_0(t) \sim H(t)$ can be characterized explicitly, and in particular $H_0(t)$ can be found (for all t large enough) as the unique solution of the equation

$$\varrho' H_0 = \varrho h\big((\varrho' H_0/t)^{\pm}\big). \tag{1.6}$$

For example, if $h(x) = x^{\varrho}/\varrho$ $(x \ge 0)$ then the solution H_0 is given by $H_0(t) = t^{\varrho'}/\varrho'$.

As shown in [5], the critical values of the parameter λ are given by

$$\lambda_1 = \frac{\varrho'}{\varrho}, \quad \lambda_2 = 2^{\varrho'}\,\frac{\varrho'}{\varrho}, \tag{1.7}$$

so that the LLN and CLT break down if $\lambda < \lambda_1$ and $\lambda < \lambda_2$, respectively. Moreover, for $0 < \lambda < \lambda_2$ one can prove (under a slightly more restrictive condition of *normalized regular variation* of h, see Section 2 below) that the distribution of $S_N(t)$, properly centered and normalized, converges to a stable law with characteristic exponent

$$\alpha \equiv \alpha(\varrho, \lambda) := \left(\frac{\lambda\varrho}{\varrho'}\right)^{1/\varrho'} \tag{1.8}$$

(note that the critical values (1.7) correspond to the canonical values $\alpha_1 = 1$, $\alpha_2 = 2$). Let us point out that the centering constant vanishes for $0 < \lambda < \lambda_1$, while the normalization constant is given for all $0 < \lambda < \lambda_2$ by $B(t) = \mathrm{e}^{\pm t\eta_1(t)}$, where η_1 is the (unique) root of the equation $h(\eta_1^{\pm}) = \lambda H_0(t)$ (note that the right-hand side of this equation is asymptotically equivalent to $\log N$, due to the scaling condition (1.3)). A more precise statement of these results will be given below in Section 3.

In the present paper, we complement this analysis by studying the asymptotic behavior of the upper extreme values of the exponential sample $\{\mathrm{e}^{tX_i}\}_{i=1}^N$. Surprisingly enough, the limiting picture here replicates the classical results in the i.i.d. extreme value theory, known in the case of attraction to the Fréchet distribution. In particular, the maximal term $M_{1,N}(t)$ normalized by $B(t) = \mathrm{e}^{\pm t\eta_1(t)}$ (for *all* $\lambda > 0$) converges in distribution to the Fréchet law Φ_α with the distribution function

$$\Phi_\alpha(x) = \exp(-x^{-\alpha}), \quad x > 0 \tag{1.9}$$

(Theorem 4.3). The proof was first sketched in [4] (in case B) and later on extended to both cases B and A in [3]. This result should be contrasted with the limit distribution of the maximum of "plain" exponentials, $\{\mathrm{e}^{X_i}\}_{i=1}^N$, which appears to be the Gumbel (double exponential) law Λ with the distribution function

$$\Lambda(x) = \exp(-\mathrm{e}^{-x}), \quad x \in \mathbb{R} \tag{1.10}$$

(Theorem 4.1).

Limit theorems for the (joint) distribution of individual order statistics confirm the Fréchet-type nature of exponential extremes (Theorems 4.8, 4.9). Of particular interest is Theorem 4.10, which reveals a Poisson limiting structure of the empirical extremal measure (in the sense of convergence of finite-dimensional distributions), with a "stable" intensity measure $d(x^{-\alpha})$. We also explore the role of the maximum in the sum. To this end, we use some representations of order statistics via uniform and exponential random variables, which allows us to work with a.s.-convergence. Using this approach, we obtain series representations for the joint limit distribution of $S_N(t)$ and $M_{1,N}(t)$.

The rest of the paper is laid out as follows. In Section 2 we specify our regularity assumptions and derive some consequences. Section 3 summarizes results about the limit laws for the sums $S_N(t)$ proved earlier. Section 4 deals with the limit (joint) laws for upper order statistics. In Section 5 we work out certain useful representations of order statistics, in order to obtain certain explicit characterizations of the limit laws of the sums $S_N(t)$. Finally, the Appendix contains a few direct proofs of a formula for the expected value of the limiting ratio $S_N(t)/M_{1,N}(t)$ in the case $0<\alpha<1$.

2. Regularity assumptions

Using the *log-tail distribution function* h defined in (1.2), the tail probability of X takes the form

$$\mathrm{P}\{X>x\}=\exp\{-h(\pm x^{\pm})\}. \tag{2.1}$$

We assume that h *is regularly varying at infinity with index* ϱ (we write $h\in \mathrm{R}_\varrho$), where $1<\varrho<\infty$ (case B) or $0<\varrho<\infty$ (case A). That is to say, for any $\kappa>0$

$$\frac{h(\kappa x)}{h(x)}\to\kappa^\varrho\quad(x\to+\infty). \tag{2.2}$$

Moreover, by the Uniform Convergence Theorem (UCT) (see [2, Theorem 1.5.2], the limit (2.2) holds uniformly in κ on each interval $(0,K]$.

The *generalized inverse* of a function f can be defined as $f^{\leftarrow}(y):=\inf\{x: f(x)\ge y\}$ [20, Section 0.2]. Note [2, Theorem 1.5.12] that if $f\in\mathrm{R}_\varrho$ with $\varrho>0$ then $f^{\leftarrow}\in\mathrm{R}_{1/\varrho}$ and

$$f^{\leftarrow}\bigl(f(x)\bigr)\sim f\bigl(f^{\leftarrow}(x)\bigr)\sim x\quad(x\to\infty). \tag{2.3}$$

Since the log-tail distribution function h is nondecreasing and right-continuous, its inverse $h^{\leftarrow}$ has the following useful property [20, eq. (0.6c)]:

$$h^{\leftarrow}(y)\le x\iff y\le h(x). \tag{2.4}$$

The asymptotic link between $h(x)$ and the cumulant generating function $H(t)$ (see (1.4)) is described by the Kasahara – de Bruijn exponential Tauberian theorem (see [2, Theorems 4.12.7 and 4.12.9]). Set

$$\varrho':=\frac{\varrho}{\varrho\mp 1}\in\begin{cases}(1,\infty) & \text{(case B)}\\ (0,1) & \text{(case A)}\end{cases}$$

(cf. (1.5)), then $h\in\mathrm{R}_\varrho$ if and only if $H\in\mathrm{R}_{\varrho'}$. More precisely, let $\varphi\in\mathrm{R}_{1/\varrho}$ and put

$$\psi(u):=u\varphi(u)^{\mp}\in\mathrm{R}_{1/\varrho'}. \tag{2.5}$$

Then

$$h(x)\sim\frac{1}{\varrho}\varphi^{\leftarrow}(x)\ (x\to\infty)\iff H(t)\sim\frac{1}{\varrho'}\psi^{\leftarrow}(t)=:H_0(t)\ (t\to\infty). \tag{2.6}$$

In addition to regular variation, $h\in\mathrm{R}_\varrho$, we will we assume throughout the paper that h is *normalized regularly varying*, $h\in\mathrm{NR}_\varrho$ [2, p. 15], that is, h is absolutely continuous (and hence a.e.-differentiable) and satisfies

$$\frac{xh'(x)}{h(x)}\to\varrho\quad(x\to\infty). \tag{2.7}$$

Equivalently, for every $\delta>0$ the function $h(x)/x^{\varrho-\delta}$ is ultimately increasing, whereas the function $h(x)/x^{\varrho+\delta}$ is ultimately decreasing [2, p. 24]. In particular, it follows that $h(x)$ itself is ultimately (strictly) increasing and therefore is invertible for x large enough.

As a benefit of the assumption $h \in \mathrm{NR}_\varrho$, the function $H_0(t) \sim H(t)$ in the Kasahara – de Bruijn theorem (see (2.6)) can be identified explicitly via equation (1.6) (see details in [5]). Furthermore, integration of (2.7) shows that h can be represented in the form

$$h(x) = h(0) + \int_0^x \frac{h(u)}{u}\big(\varrho + \varepsilon(u)\big)\,\mathrm{d}u, \tag{2.8}$$

where $\varepsilon(x) \to 0$ as $x \to \infty$.

The next lemma can be viewed as a refinement of the UCT.

Lemma 2.1. *If $h \in \mathrm{NR}_\varrho$ then, uniformly in κ on each interval $[\kappa_0, \kappa_1] \subset (0, \infty)$,*

$$\frac{h(\kappa x) - h(x)}{h(x)} = (\kappa^\varrho - 1)\big(1 + o(1)\big) \quad (x \to \infty).$$

Proof. Using representation (2.8), after the substitution $u = xy$ we have

$$\frac{h(\kappa x) - h(x)}{h(x)} = \int_1^\kappa \frac{h(xy)}{h(x)y}\big(\varrho + \varepsilon(xy)\big)\,\mathrm{d}y. \tag{2.9}$$

By the UCT, the integrand in (2.9) tends to $\varrho y^{\varrho-1}$ as $x \to \infty$, uniformly in $y \in [\kappa_0, \kappa_1]$. Hence, the integral (2.9) converges, uniformly in $\kappa \in [\kappa_0, \kappa_1]$, to $\int_1^\kappa \varrho y^{\varrho-1}\,\mathrm{d}y = \kappa^\varrho - 1$. $\square$

Let $\eta_1 \equiv \eta_1(t)$ be the unique solution of the equation

$$h(\eta_1^\pm) = \lambda H_0(t) \tag{2.10}$$

($\eta_1(t)$ is well defined for all t large enough). For $x > 0$, set

$$\eta_x \equiv \eta_x(t) := \eta_1(t) \pm \frac{\log x}{t}. \tag{2.11}$$

Combining equations (1.6) and (2.10) as $t \to \infty$ and using that $h \in \mathrm{R}_\varrho$, one can find

$$\lim_{t\to\infty} \frac{t\eta_1(t)}{H_0(t)} = \frac{\lambda\varrho}{\alpha}. \tag{2.12}$$

The next lemma plays the crucial role.

Lemma 2.2. *Uniformly in x on each interval $[x_0, x_1] \subset (0, \infty)$,*

$$\lim_{t\to\infty} \big[h(\eta_x(t)^\pm) - h(\eta_1(t)^\pm)\big] = \alpha \log x.$$

Proof. Note that

$$\kappa_x(t) := \left(\frac{\eta_x(t)}{\eta_1(t)}\right)^\pm = \left(1 \pm \frac{\log x}{t\eta_1(t)}\right)^\pm \to 1 \quad (t \to \infty),$$

uniformly in $x \in [x_0, x_1] \subset (0, \infty)$. Therefore, for all large enough t the function $\kappa_x(t)$ is uniformly bounded, $0 < \kappa_0 \le \kappa_\tau(t) \le \kappa_1 < \infty$. Applying Lemma 2.1 and using (2.10), in the limit as $t \to \infty$ we obtain, uniformly in x,

$$\begin{aligned} h(\eta_x(t)^\pm) - h(\eta_1(t)^\pm) \sim h(\eta_1(t)^\pm)(\kappa_x(t)^\varrho - 1) &= \lambda H_0(t)\left(\left(1 \pm \frac{\log x}{t\eta_1(t)}\right)^{\pm\varrho} - 1\right) \\ &\sim \lambda H_0(t)\frac{\varrho \log x}{t\eta_1(t)} \sim \alpha \log x, \end{aligned}$$

according to (2.12). $\square$

3. Limit theorems for the sums

The main results of the paper [5] can be summarized in the following two theorems. We assume throughout that

$$\lim_{t\to\infty} N\mathrm{e}^{-\lambda H_0(t)} = 1, \tag{3.1}$$

where $\lambda > 0$ is a parameter (see (1.3)). Recall that α is given by equation (1.8).

Theorem 3.1. *Set*

$$A(t) := \begin{cases} N\,\mathrm{E}[\mathrm{e}^{tX}] = N\mathrm{e}^{\pm H(t)}, & \lambda > \lambda_1 \quad (\alpha > 1),\\ N\,\mathrm{E}\bigl[\mathrm{e}^{tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\bigr], & \lambda = \lambda_1 \quad (\alpha = 1),\\ 0, & 0<\lambda<\lambda_1\ (0<\alpha<1),\end{cases} \tag{3.2}$$

$$B(t) := \begin{cases} (N\,\mathrm{E}[\mathrm{e}^{2tX}])^{1/2} = N^{1/2}\mathrm{e}^{\pm H(2t)/2}, & \lambda > \lambda_2 \quad (\alpha > 2),\\ \bigl(N\,\mathrm{E}\bigl[\mathrm{e}^{2tX}\mathbf{1}_{\{X\le\pm\eta_1\}}\bigr]\bigr)^{1/2}, & \lambda = \lambda_2 \quad (\alpha = 2),\\ \mathrm{e}^{\pm t\eta_1(t)}, & 0<\lambda<\lambda_2\ (0<\alpha<2).\end{cases} \tag{3.3}$$

Then, as $t\to\infty$,

$$\frac{S_N(t)-A(t)}{B(t)} \xrightarrow{\mathrm{d}} \begin{cases}\mathcal{N}(0,1), & \lambda\ge\lambda_2 \quad (\alpha\ge 2)\\ \mathcal{F}_\alpha, & 0<\lambda<\lambda_2\ (0<\alpha<2),\end{cases} \tag{3.4}$$

where $\mathcal{N}(0,1)$ is the standard normal distribution (with zero mean and unit variance) and $\mathcal{F}_\alpha$ is a stable law with characteristic exponent α and skewness parameter $\beta\equiv 1$. The characteristic function of $\mathcal{F}_\alpha$ is given by[1]

$$\phi_\alpha(u) = \begin{cases} \exp\Bigl\{-\Gamma(1-\alpha)|u|^\alpha \exp\Bigl(-\dfrac{i\pi\alpha}{2}\operatorname{sgn} u\Bigr)\Bigr\}, & \alpha\ne 1,\\ \exp\Bigl\{iu(1-\gamma) - \dfrac{\pi}{2}|u|\Bigl(1+i\operatorname{sgn} u\cdot\dfrac{2}{\pi}\log|u|\Bigr)\Bigr\}, & \alpha = 1,\end{cases} \tag{3.5}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is the gamma function, $\operatorname{sgn} u := u/|u|$ *and* $\gamma = 0.5772\dots$ *is the Euler constant.*[2]

With some effort (see details in [3, 5]), one can check that for all $\lambda > 0$

$$\lim_{t\to\infty}\frac{B(t)}{A(t)} = 0.$$

Therefore, Theorem 3.1 implies the following LLN.

Theorem 3.2. *If $\lambda\ge\lambda_1$ then*

$$\frac{S_N(t)}{A(t)} \xrightarrow{\mathrm{p}} 1 \quad (t\to\infty), \tag{3.6}$$

where $A(t)$ is given by (3.2).

Remark 3.3. In fact, in the regions $\lambda>\lambda_1$ and $\lambda>\lambda_2$ the LLN and CLT, respectively, hold under the plain regularity condition $h\in\mathrm{R}_\varrho$, without any extra assumptions. Moreover, the function $H(t)$ can be used here instead of $H_0(t)$, and the scaling condition (3.1) can be relaxed to $\log N \gg \lambda_1 H(t)$ and $\log N\gg\lambda_2 H(t)$, respectively (see [3, 5]).

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[1] For $1<\alpha<2$, we use an analytic continuation of the gamma function in (3.5), given by $\Gamma(1-\alpha) = \Gamma(2-\alpha)/(1-\alpha)$.

[2] See [12, #8.367].

Theorem 3.1 can be proved using the known asymptotic methods for sums of independent random variables (see, e.g., [17]). The proofs are technically quite involved since the condition of normalized regular variation guarantees only the limited smoothness of the distribution of X (i.e., ultimate monotonicity and a.e.-differentiability). However, this condition (in particular, Lemma 2.2), combined with the scaling condition (3.1) provide enough analytical control. For instance, in view of (2.1) the tail probability for e^{tX} is asymptotically given by

$$\begin{aligned}(3.7)\quad \mathrm{P}\{\mathrm{e}^{tX} > xB(t)\} = \mathrm{P}\{X > \pm\eta_x(t)\} = \mathrm{e}^{-h(\eta_x^{\pm})} &\sim N^{-1}\mathrm{e}^{h(\eta_1^{\pm})-h(\eta_x^{\pm})}\\ &\sim N^{-1}\mathrm{e}^{-\alpha\log x} = N^{-1}x^{-\alpha},\end{aligned}$$

where $B(t) = \mathrm{e}^{\pm t\eta_1(t)}$ (see (3.3)). Therefore, the Lévy–Khinchin spectral function, being the main ingredient of the limiting infinitely divisible law [17, pp. 81–82], is given for $x > 0$ by

$$L(x) := -\lim_{t\to\infty} N\,\mathrm{P}\{\mathrm{e}^{tX} > xB(t)\} = -x^{-\alpha}, \tag{3.8}$$

which indicates that α is indeed the characteristic exponent of the limit (stable) law (see [15, Theorem 2.2.1]).

Using (3.1), note that for $\lambda < \lambda_2$ we have

$$B(t) = \mathrm{e}^{\pm t\eta_1(t)} \sim N^{\pm t\eta_1(t)/(\lambda H_0(t))} \quad (t \to \infty),$$

and (2.12) implies that in case B, N is being raised to the power

$$\frac{t\eta_1(t)}{\lambda H_0(t)} \sim \frac{\varrho}{\alpha} > \frac{1}{\alpha}.$$

This should be compared to classical results in the i.i.d. case (see, e.g., [15, Theorem 2.1.1]), where the normalization is essentially of the form $N^{1/\alpha}$. As we see, in case B the sums $S_N(t)$ have a stable limit by virtue of a nonclassical (heavier) normalization. As for case A, we have $B(t) \sim N^{-t\eta_1(t)/(\lambda H_0(t))} \to 0$, which has no analogies in the classical theory.

However, another look at the tail probability reveals the mechanism of settling down to a stable law, analogous to that in the i.i.d. situation and acting in both cases A and B. Indeed, in order that i.i.d. random variables $\{Y_i\}$ belong to the domain of attraction of a stable law with characteristic exponent $\alpha > 0$, it is sufficient that for each $x > 0$

$$\mathrm{P}\{Y > N^{1/\alpha}x\} \sim N^{-1}x^{-\alpha} \quad (N \to \infty) \tag{3.9}$$

(see [15, Theorem 2.6.1]). If we set $Y_i := \mathrm{e}^{tX_i}/B(t)N^{-1/\alpha}$ then, according to (3.7),

$$\mathrm{P}\{Y > N^{1/\alpha}x\} = \mathrm{P}\{\mathrm{e}^{tX} > xB(t)\} \sim N^{-1}x^{-\alpha} \quad (t \to \infty),$$

which mimics the condition (3.9). Thus, in the normalizing function represented in the form $B(t) = B(t)N^{-1/\alpha} \cdot N^{1/\alpha}$, the factor $B(t)N^{-1/\alpha}$ is responsible for the correct behavior of the distribution tail, whereas the conventional power $N^{1/\alpha}$ performs averaging towards a stable law with characteristic exponent α.

This observation helps explain heuristically the many similarities between the limit behavior of random exponentials e^{tX_i} and that of the usual i.i.d. random variables — from convergence to a stable law (Theorem 3.1) to the asymptotic properties of extreme values (Sections 4, 5 below).

We conclude this section with the following LLN for $\log S_N(t)$: as $t \to \infty$,

$$\frac{\log S_N(t)}{t\eta_1(t)} \xrightarrow{\mathrm{P}} \begin{cases} \dfrac{\alpha(\lambda \pm 1)}{\lambda\varrho}, & \lambda \geq \lambda_1, \\ \pm 1, & 0 < \lambda \leq \lambda_1. \end{cases} \tag{3.10}$$

For $\lambda < \lambda_1$ this follows directly from Theorem 3.1. For $\lambda \geq \lambda_1$, Theorem 3.2 implies

$$\log S_N(t) = \log A(t) \cdot \theta(t), \tag{3.11}$$

where $\theta(t) \xrightarrow{\mathrm{P}} 1$ as $t \to \infty$. If $\lambda > \lambda_1$, then from (3.1) and (3.2)

$$\log A(t) = \log N \pm H(t) \sim (\lambda \pm 1) H_0(t),$$

and so (3.10) follows by (2.12). If $\lambda = \lambda_1 = \varrho'/\varrho$ (i.e., $\alpha = \alpha_1 = 1$), then $A(t) \leq N\mathrm{e}^{\pm H(t)}$ (see (3.2)), so from (3.11) we get, according to (2.12) and (3.1),

$$\frac{\log S_N(t)}{t\eta_1(t)} \leq \frac{\log N \pm H(t)}{t\eta_1(t)} \cdot \theta(t) \xrightarrow{\mathrm{P}} \frac{(\lambda_1 \pm 1)}{\lambda_1 \varrho} = \pm 1 \tag{3.12}$$

(cf. (1.5)). On the other hand, $S_N(t) \geq M_{1,N}(t) := \max\{\mathrm{e}^{tX_i}, i = 1, \dots, N\}$, hence

$$\frac{\log S_N(t)}{t\eta_1(t)} \geq \frac{\log M_{1,N}(t)}{t\eta_1(t)} \xrightarrow{\mathrm{P}} \pm 1 \quad (t \to \infty), \tag{3.13}$$

as we will show below (see (4.14)). Now, combination of (3.12) and (3.13) yields (3.10).

Remark 3.4. The limit (3.10) has the meaning of the *free energy* in the REM (see [3, Section 9; 5; 7] and further references therein). It is easy to check that (3.10), as a function of λ, is continuous and continuously differentiable everywhere including the critical point $\lambda = \lambda_1$, but its second derivative has a jump at this point. This corresponds to a "third order" phase transition [7].

4. Asymptotic behavior of extreme values

4.1. The plain exponential maximum. Let us first obtain the distribution of the maximum for plain exponentials, e^{X_i} (i.e., without the large scaling parameter t in the exponent). Namely, we will show that under our conditions on X_i, the random variables e^{X_i} belong to the domain of attraction of the Gumbel (double exponential) distribution Λ (see (1.10)).

Theorem 4.1. *Denote* $Y_i := \mathrm{e}^{X_i}$, $Y_{1,N} := \max\{Y_1, \dots, Y_N\}$, *and set*

$$a_N := \exp\bigl(\pm h^{\leftarrow}(\log N)^{\pm}\bigr), \quad b_N := \pm \frac{a_N \log a_N}{\varrho \log N}. \tag{4.1}$$

Then

$$\lim_{N \to \infty} \mathrm{P}\left\{\frac{Y_{1,N} - a_N}{b_N} \leq x\right\} = \exp(-\mathrm{e}^{-x}), \quad x \in \mathbb{R}. \tag{4.2}$$

Proof. It is not difficult to verify available sufficient conditions for convergence of the maximum's distribution to Λ (see, e.g., [11, Theorem 2.1.3]). However, it appears even simpler to prove (4.2) directly. Using (2.1), we can write

$$\mathrm{P}\{Y_{1,N} \leq a_N + x b_N\} = \bigl(\mathrm{P}\{X \leq \log(a_N + x b_N)\}\bigr)^N = \bigl(1 - \mathrm{e}^{-h(L_N(x)^{\pm})}\bigr)^N, \tag{4.3}$$

where

$$L_N(x) := \pm \log(a_N + x b_N).$$

Note that, according to (4.1),

$$L_N(0)^{\pm} = \pm(\log a_N)^{\pm} = h^{\leftarrow}(\log N), \tag{4.4}$$

and since $h \in \mathrm{NR}_\varrho$, equation (4.4) can be inverted for all large enough N to yield

$$h(L_N(0)^{\pm}) = \log N. \tag{4.5}$$

Observe that, as $N \to \infty$,

$$\frac{b_N}{a_N \log a_N} = \pm\frac{1}{\varrho \log N} \to 0 \tag{4.6}$$

and, moreover,

$$\frac{b_N}{a_N} = \pm\frac{\log a_N}{\varrho \log N} = \frac{h^{\leftarrow}(\log N)^{\pm}}{\varrho \log N} \to 0, \tag{4.7}$$

since $h^{\leftarrow}(x) \in \mathrm{R}_{1/\varrho}$ and $h^{\leftarrow}(x)^{\pm}/x \in \mathrm{R}_{\pm 1/\varrho - 1}$ with $\pm 1/\varrho - 1 = -1/\varrho' < 0$. In view of (4.6) and (4.7), we obtain

$$\kappa_N(x) := \frac{L_N(x)^{\pm}}{L_N(0)^{\pm}} = \left(1 + \frac{\log(1 + x b_N/a_N)}{\log a_N}\right)^{\pm} \to 1 \quad (N \to \infty), \tag{4.8}$$

and hence $L_N(x)^{\pm} \sim L_N(0)^{\pm} \to +\infty$ (see (4.4)). Using (4.3) and (4.5), it follows that the limit (4.2) is reduced to showing that

$$\lim_{N\to\infty} \left[h(L_N(x)^{\pm}) - h(L_N(0)^{\pm})\right] = x.$$

To this end, we apply Lemma 2.1 and use (4.5), (4.8) to get

$$\begin{aligned} h(L_N(x)^{\pm}) - h(L_N(0)^{\pm}) &\sim h(L_N(0)^{\pm})(\kappa_N(x)^{\varrho} - 1) \\ &= \log N \left(\left(1 + \frac{\log(1 + x b_N/a_N)}{\log a_N}\right)^{\pm\varrho} - 1\right) \\ &\sim \log N \cdot \frac{(\pm\varrho) x b_N}{a_N \log a_N} = x, \end{aligned}$$

according to (4.6). □

Passing to logarithms, it is easy to show that the background random variables $X_i = \log \mathrm{e}^{X_i}$ also belong to the domain of attraction of the Gumbel distribution (see also [6, Proposition 10.1]).

Corollary 4.2. *Let* $X_{1,N} := \max\{X_1, \dots, X_N\}$ *and set*

$$\tilde{a}_N := \pm h^{\leftarrow}(\log N)^{\pm}, \quad \tilde{b}_N := \frac{h^{\leftarrow}(\log N)^{\pm}}{\varrho \log N}. \tag{4.9}$$

Then

$$\lim_{N\to\infty} \mathrm{P}\left\{\frac{X_{1,N} - \tilde{a}_N}{\tilde{b}_N} \le x\right\} = \exp(-\mathrm{e}^{-x}), \quad x \in \mathbb{R}. \tag{4.10}$$

Since $\tilde{b}_N/\tilde{a}_N \to 0$, the limit (4.10) readily implies the following LLN:

$$\frac{X_{1,N}}{h^{\leftarrow}(\log N)^{\pm}} \xrightarrow{\mathrm{p}} \pm 1 \quad (N \to \infty). \tag{4.11}$$

4.2. Limit distribution of the exponential maximum. We shall now see that when a nonlinear (power) scaling is switched on, $\mathrm{e}^X \mapsto \mathrm{e}^{tX}$, the limit law for the maximum changes dramatically. Denote by $M_{k,N} \equiv M_{k,N}(t) := \mathrm{e}^{tX_{k,N}}$ the nonincreasing order statistics of the sample $\{\mathrm{e}^{tX_i}\}_{i=1}^N$. In particular, $M_{1,N}(t) = \max\{\mathrm{e}^{tX_i}, i = 1, \dots, N\}$. For all $\lambda \in (0, \infty)$ denote

$$B(t) := \mathrm{e}^{\pm t\eta_1(t)}, \tag{4.12}$$

where $\eta_1(t)$ is defined by equation (2.10) (cf. (3.3)).

In this section, we obtain the limit distribution of $M_{1,N}(t)$.

Theorem 4.3. *For all $\lambda > 0$, as $t \to \infty$, $M_{1,N}(t)/B(t)$ converges in distribution to the Fréchet law Φ_α (see* (1.9)*).*

Proof. Using (2.1), (2.11), and (4.12), for $x > 0$ we have

$$\begin{aligned} \mathrm{P}\{M_{1,N}(t) \le xB(t)\} &= \mathrm{P}\left\{X_{1,N} \le \frac{\log x}{t} \pm t\eta_1(t)\right\} \\ &= \mathrm{P}\{X_{1,N} \le \pm\eta_x\} = (\mathrm{P}\{X \le \pm\eta_x\})^N \\ &= \bigl(1 - \mathrm{e}^{-h(\eta_x^\pm)}\bigr)^N = \exp\Bigl(-N\mathrm{e}^{-h(\eta_x^\pm)}\bigl(1 + o(1)\bigr)\Bigr). \end{aligned} \tag{4.13}$$

Furthermore, recalling (3.1) and (2.10) we obtain, as $t \to \infty$,

$$N\mathrm{e}^{-h(\eta_x^\pm)} \sim \mathrm{e}^{\lambda H_0(t) - h(\eta_x^\pm)} = \mathrm{e}^{h(\eta_1^\pm) - h(\eta_x^\pm)} \to \mathrm{e}^{-\alpha \log x} = x^{-\alpha},$$

according to Lemma 2.2. Returning to (4.13) we get

$$\lim_{t\to\infty} \mathrm{P}\{M_{1,N}(t) \le xB(t)\} = \exp(-x^{-\alpha}),$$

and the theorem is proved. $\square$

Remark 4.4. The probability law Φ_α, known as the Fréchet distribution, represents one of the three types of possible weak limits for maxima of i.i.d. random variables (see [11, Theorem 2.4.1]). However, the known general theorems about convergence to Φ_α are not directly applicable in our case.

In view of (4.12), Theorem 4.3 implies the following LLN for $\log M_{1,N}$:

$$\frac{\log M_{1,N}(t)}{t\eta_1(t)} \xrightarrow{\mathrm{P}} \pm 1 \qquad (t \to \infty). \tag{4.14}$$

Remark 4.5. In contrast with a similar log-LLN for $S_N(t)$ (see (3.10)), the limit (4.14) does not involve any "phase transitions."

Remark 4.6. Comparing (4.14) and (3.10), we note that in the case $0 < \lambda \le \lambda_1$

$$\frac{\log M_{1,N}(t)}{\log S_N(t)} \xrightarrow{\mathrm{P}} 1 \quad (t \to \infty), \tag{4.15}$$

which indicates that the contribution of the maximal term $M_{1,N}(t)$ to the sum $S_N(t)$ is logarithmically equivalent to the whole sum. In the opposite case where $\lambda > \lambda_1$, the limit of the left-hand side in (4.15) can be shown to be strictly less than 1, so that $M_{1,N}(t)$ is negligible as compared to $S_N(t)$. This observation is supported by the LLN being valid for $\lambda \ge \lambda_1$ (see Theorem 3.2), and is further evidenced by Theorem 5.11 characterizing the limit distribution of the ratio $S_N(t)/M_{1,N}(t)$ in the case $\lambda < \lambda_1$.

Remark 4.7. Note that $\log M_{1,N} = tX_{1,N}$, so (4.14) amounts to

$$\frac{X_{1,N}(t)}{\eta_1(t)} \xrightarrow{\mathrm{P}} \pm 1 \quad (t \to \infty), \tag{4.16}$$

which is of course consistent with (4.11); indeed, by (2.3), (2.10), (3.1) and the UCT we have

$$\eta_1(t)^{\pm} \sim h^{\leftarrow}(h(\eta_1(t)^{\pm})) = h^{\leftarrow}\bigl(\lambda H_0(t)\bigr) \sim h^{\leftarrow}(\log N),$$

and so $\eta_1(t) \sim h^{\leftarrow}(\log N)^{\pm}$ (cf. (4.11) and (4.16)).

4.3. Joint asymptotic behavior of exponential order statistics. Recall that $M_{k,N} = \mathrm{e}^{tX_{k,N}}$ denotes the kth order statistic (arranged in the nonincreasing order) of the exponential sample $\{\mathrm{e}^{tX_i}\}_{i=1}^{N}$.

Theorem 4.8. *For all $\lambda > 0$ and each $k \in \mathbb{N}$,*

$$\lim_{t\to\infty} \mathrm{P}\biggl\{\frac{M_{k,N}(t)}{B(t)} \le x\biggr\} = \mathrm{e}^{-x^{-\alpha}} \sum_{j=0}^{k-1} \frac{x^{-\alpha j}}{j!} \quad (x > 0). \tag{4.17}$$

Proof. We have

$$\mathrm{P}\{M_{k,N} \le xB(t)\} = \mathrm{P}\{X_{k,N} \le \pm\eta_x\} \tag{4.18}$$

(cf. (4.13)). The distribution of the kth order statistic $X_{k,N}$ is given by the following known formula (see [11, Section 2.8, eq. (135)]):

$$\mathrm{P}\{X_{k,N} \le u\} = \sum_{j=0}^{k-1} \binom{N}{j} [1 - F(u)]^j [F(u)]^{N-j},$$

where $F(u)$ is the common distribution function of the random variables X_i:

$$F(u) := \mathrm{P}\{X \le u\} = 1 - \mathrm{e}^{-h(\pm u^{\pm})}.$$

Setting $u = \pm\eta_x(t)$, similarly to the proof of Theorem 4.3 we obtain

$$\begin{aligned}
\mathrm{P}\{X_{k,N} \le \pm\eta_x(t)\} &= \sum_{j=0}^{k-1} \binom{N}{j} \mathrm{e}^{-jh(\eta_x^{\pm})}\bigl(1 - \mathrm{e}^{-h(\eta_x^{\pm})}\bigr)^{N-j} \\
&\sim \bigl(1 - \mathrm{e}^{-h(\eta_x^{\pm})}\bigr)^{N} \sum_{j=0}^{k-1} \frac{N^j}{j!} \mathrm{e}^{-jh(\eta_x^{\pm})} \\
&\sim \exp\bigl(-\mathrm{e}^{h(\eta_1^{\pm}) - h(\eta_x^{\pm})}\bigr) \sum_{j=0}^{k-1} \frac{1}{j!} \mathrm{e}^{j(h(\eta_1^{\pm}) - h(\eta_x^{\pm}))},
\end{aligned}$$

and (4.17) follows by Lemma 2.2. □

One can also derive the limiting form of the joint distribution for a finite number of upper extremes. For instance, let us prove the following assertion.

Theorem 4.9. *For any integers $r < s$ and all real $x, y > 0$,*

$$\lim_{t\to\infty} \mathrm{P}\left\{\frac{M_{r,N}}{B(t)} \le x, \frac{M_{s,N}}{B(t)} \le y)\right\} \tag{4.19}$$

$$= \begin{cases} \mathrm{e}^{-y^{-\alpha}} \displaystyle\sum_{j=0}^{r-1}\sum_{k=j}^{s-1} \frac{x^{-\alpha j}(y^{-\alpha}-x^{-\alpha})^{k-j}}{j!(k-j)!} & \text{if } x > y, \\ \mathrm{e}^{-x^{-\alpha}} \displaystyle\sum_{j=0}^{r-1} \frac{x^{-\alpha j}}{j!} & \text{if } x \le y. \end{cases}$$

Proof. For $x \le y$ the inequality $M_{r,N} \le x$ implies $M_{s,N} \le y$, so that

$$\mathrm{P}\{M_{r,N} \le xB(t), M_{s,N} \le yB(t)\} = \mathrm{P}\{M_{r,N} \le xB(t)\},$$

and (4.19) follows by Theorem 4.17. For $x > y$, similarly to (4.18) we obtain

$$\mathrm{P}\{M_{r,N} \le xB(t), M_{s,N} \le yB(t)\} = \mathrm{P}\{X_{r,N} \le \pm\eta_x(t), X_{s,N} \le \pm\eta_y(t)\}.$$

The joint distribution of the order statistics $X_{r,N}$ and $X_{s,N}$ (with $r < s$, $u > v$) is given by (cf. [9, Section 2.2])

$$\mathrm{P}\{X_{r,N} \le u, X_{r,N} \le v\} = \sum_{j=0}^{r-1}\sum_{k=j}^{s-1} \frac{N!}{j!(k-j)!(N-j)!}[1-F(v)]^j[F(u)-F(v)]^{k-j}[F(u)]^{N-k}.$$

Taking $u = \pm\eta_x(t)$, $v = \pm\eta_y(t)$, similarly as above we arrive at (4.19). □

This theorem can be extended to the case of any given number of upper extremes. But it is more instructive to characterize the limit distribution of extreme values in a different way. Let us consider the random measure μ_N on $(0,\infty)$ (corresponding to the *empirical extremal process*) which counts the order statistics $M_{k,N}$ "from the right":

$$\mu_N(x,\infty) := \sum_{k=1}^{N} \mathbf{1}_{\{M_{k,N} > xB(t)\}}, \qquad x > 0. \tag{4.20}$$

The following theorem reveals a Poisson asymptotic structure of μ_N as $t \to \infty$.

Theorem 4.10. *In the sense of convergence of all finite-dimensional distributions, the measure μ_N converges to a Poisson random measure on $(0,\infty)$ with the mean measure $\nu(x,\infty) = x^{-\alpha}$. That is, for any $x_1 > \dots > x_n > 0$ and any integers $m_1,\dots,m_n \ge 0$*

$$\lim_{t\to\infty} \mathrm{P}\{\mu_N(\Delta_i) = m_i, i = 1,\dots,n\} = \prod_{i=1}^{n} \frac{[\nu(\Delta_i)]^{m_i}}{m_i!}\mathrm{e}^{-\nu(\Delta_i)}, \tag{4.21}$$

where $\Delta_i := (x_i, x_{i-1}]$, $x_0 := \infty$, and $\nu(\Delta_i) = x_i^{-\alpha} - x_{i-1}^{-\alpha}$.

Remark 4.11. The same answer was established by Weissman [23] for the counting measure of extremes in the case of i.i.d. random variables in the domain of attraction of the Fréchet distribution Φ_α. Analogous results were also obtained in [23] for the cases of attraction to the Weibull distribution Ψ_α and the Gumbel distribution Λ (see also [9, Section 9.4]).

Proof of Theorem 4.10. By (4.20), the condition $\mu_N(\Delta_i) = m_i$ means that the interval $(\pm\eta_{x_i}, \pm\eta_{x_{i-1}}]$ contains exactly m_i terms from the sample $\{X_1, \dots, X_N\}$. Hence, putting $m_{n+1} := N - m_1 - \dots - m_n$, by the multinomial formula we obtain

$$\begin{aligned}
\mathrm{P}\bigl\{\mu_N(\Delta_i) = m_i, i = 1, \dots, n\bigr\}&\\
&= N! \prod_{i=1}^{n} \frac{[F(\pm\eta_{x_{i-1}}) - F(\pm\eta_{x_i})]^{m_i}}{m_i!} \cdot \frac{[F(\pm\eta_{x_n})]^{m_{n+1}}}{m_{n+1}!}\\
&= N! \prod_{i=1}^{n} \frac{\bigl[\mathrm{e}^{-h(\eta^{\pm}_{x_i})} - \mathrm{e}^{-h(\eta^{\pm}_{x_{i-1}})}\bigr]^{m_i}}{m_i!} \cdot \frac{\bigl(1 - \mathrm{e}^{-h(\eta^{\pm}_{x_n})}\bigr)^{m_{n+1}}}{m_{n+1}!}\\
&\sim N^{m_1+\dots+m_n} \prod_{i=1}^{n} \frac{\bigl[\mathrm{e}^{-h(\eta^{\pm}_{x_i})} - \mathrm{e}^{-h(\eta^{\pm}_{x_{i-1}})}\bigr]^{m_i}}{m_i!} \cdot \bigl(1 - \mathrm{e}^{-h(\eta^{\pm}_{x_n})}\bigr)^{N}\\
&\sim \prod_{i=1}^{n} \frac{\bigl[\mathrm{e}^{h(\eta^{\pm}_1)-h(\eta^{\pm}_{x_i})} - \mathrm{e}^{h(\eta^{\pm}_1)-h(\eta^{\pm}_{x_{i-1}})}\bigr]^{m_i}}{m_i!} \cdot \exp\bigl\{\mathrm{e}^{h(\eta^{\pm}_1)-h(\eta^{\pm}_{x_n})}\bigr\}.
\end{aligned}$$

Using that $h(\eta^{\pm}_1) - h(\eta^{\pm}_{x_i}) \to -\alpha \log x_i$ (see Lemma 2.2), we arrive at (4.21). $\square$

5. Sums of random exponentials via order statistics

5.1. Some representations of extreme values. In this section, we record a few (basically well known) representations in distribution for extreme values $M_{k,N}$ and hence for the sum $S_N(t)$. These representations are expressed in terms of auxiliary sequences of i.i.d. random variables with either exponential or uniform distribution. In particular, they will be used to study the asymptotic behavior of $S_N(t)$ in the case $0 < \lambda < \lambda_1$. The advantage of such an approach (usually called the "method of common probability space") is that the random variables of interest will have a limit *with probability one*, rather than just in distribution.

Consider the random variable $\Xi := \pm h^{\leftarrow}(\xi)^{\pm}$, where ξ has the unit exponential distribution, that is, $\mathrm{P}\{\xi > x\} = \mathrm{e}^{-x}$ $(x > 0)$. A key observation is that $\Xi \stackrel{\mathrm{d}}{=} X$. Indeed, since h is right-continuous, we can use the property (2.4) to obtain

$$\begin{aligned}
\mathrm{P}\{\Xi \le x\} = \mathrm{P}\{\pm h^{\leftarrow}(\xi)^{\pm} \le x\} &= \mathrm{P}\{h^{\leftarrow}(\xi) \le \pm x^{\pm}\}\\
&= \mathrm{P}\{\xi \le h(\pm x^{\pm})\} = 1 - \mathrm{e}^{-h(\pm x^{\pm})} = \mathrm{P}\{X \le x\},
\end{aligned}$$

according to (2.1). Furthermore, if $\xi_1, \dots, \xi_N$ are i.i.d. random variables with unit exponential distribution, then the random variables $\Xi_i := \pm h^{\leftarrow}(\xi_i)^{\pm}$ are also independent and hence the random vector $(\Xi_i)_{i=1}^N$ has the same distribution as $(X_i)_{i=1}^N$. In particular, the joint distribution of the order statistics $\Xi_{1,N} \ge \dots \ge \Xi_{N,N}$ coincides with that of the order statistics $X_{1,N} \ge \dots \ge X_{N,N}$. We also note that since the function $\pm h^{\leftarrow}(x)^{\pm}$ is nondecreasing in its domain, the order statistics $\Xi_{k,N}$ can be represented through the underlying exponential order statistics $\xi_{1,N} \ge \dots \ge \xi_{N,N}$ as $\Xi_{k,N} = \pm h^{\leftarrow}(\xi_{k,N})^{\pm}$ $(k = 1, \dots, N)$.

Furthermore, let us recall the following representation in distribution of the exponential order statistics (see [9, Section 2.7]).

Lemma 5.1. *For each $k = 1, \dots, N$, the distribution of the kth order statistic $\xi_{k,N}$ coincides with the distribution of the random variable*

$$T_{k,N} := \sum_{i=k}^{N} \frac{\zeta_i}{i}, \tag{5.1}$$

where (ζ_i) is an auxiliary sequence of i.i.d. random variables, each having the unit exponential distribution. Moreover, $(\xi_{1,N}, \dots, \xi_{N,N}) \stackrel{\mathrm{d}}{=} (T_{1,N}, \dots, T_{N,N})$.

As a result, the joint distribution of the order statistics $M_{k,N} = \mathrm{e}^{tX_{k,N}}$ coincides with the joint distribution of the random variables $\mathrm{e}^{\pm th^{\leftarrow}(T_{k,N})^{\pm}}$. Hence, the sum $S_N(t) = \sum_{i=1}^{N} \mathrm{e}^{tX_i} = \sum_{k=1}^{N} \mathrm{e}^{tX_{k,N}}$ has the same distribution as the sum

$$\sum_{k=1}^{N} \mathrm{e}^{\pm th^{\leftarrow}(T_{k,N})^{\pm}} \tag{5.2}$$

with $T_{k,N}$ given by (5.1). It is convenient to rewrite $T_{k,N}$ as

$$T_{k,N} = \sum_{i=k}^{N} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) + \log \frac{N+1}{k}. \tag{5.3}$$

Lemma 5.2. *The series*

$$\sum_{i=1}^{\infty} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) \tag{5.4}$$

converges with probability one.

Proof. Let us represent the series (5.4) as

$$\sum_{i=1}^{\infty} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) = \sum_{i=1}^{\infty} \frac{\zeta_i - 1}{i} + \sum_{i=1}^{\infty} \left(\frac{1}{i} - \log \frac{i+1}{i} \right) \tag{5.5}$$

and show that both series on the right are convergent. Recalling that ζ_i are exponentially distributed with mean 1, we have $\mathrm{E}[(\zeta_i - 1)/i] = 0$ and

$$\sum_{i=1}^{\infty} \mathrm{Var}\left(\frac{\zeta_i - 1}{i} \right) = \sum_{i=1}^{\infty} \frac{\mathrm{Var}[\zeta_i]}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

Since (ζ_i) are independent, this implies a.s.-convergence of the series $\sum_i (\zeta_i - 1)/i$ (see [17, Chapter IX, Section 2, Lemma 8]). Further, note that

$$\frac{1}{i} - \log\left(1 + \frac{1}{i}\right) = O\big(i^{-2}\big) \quad (i \to \infty),$$

hence the last series in (5.5) converges. □

Remark 5.3. A similar approach was used by Hall [14] to obtain a canonical representation for limiting extreme values in the i.i.d. scheme, following the idea suggested by Rényi [18] (see also [19, Chapter VIII, Section 9]]). LePage et al. [16] have used the same approach to study convergence to a stable law in the classical situation of i.i.d. random variables.

Lemma 5.2 implies that for each $k \geq 1$ the sum of the series

$$Z_k := \sum_{i=k}^{\infty} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) \tag{5.6}$$

is finite with probability one. It is not difficult to find the (joint) distribution of Z_k.

Lemma 5.4. *Let us set $\tau_k := k\mathrm{e}^{-Z_k}$ $(k = 1, 2, \dots)$. Then $(\tau_k) \stackrel{\mathrm{d}}{=} (\sigma_k)$, where $\sigma_k := \zeta'_1 + \cdots + \zeta'_k$ and (ζ'_i) is a sequence of independent exponential random variables with mean 1. In particular, τ_k has the gamma distribution with mean k.*

Remark 5.5. The random variables (τ_k) are distributed as a sequence of arrival times of a Poisson process with unit rate.

Proof of Lemma 5.4. According to the definitions (5.6) and (5.1), we have

$$Z_k = \lim_{N \to \infty} \sum_{i=k}^{N} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) = \lim_{N \to \infty} (T_{k,N} - \log N) + \log k.$$

Hence,

$$\tau_k = \mathrm{e}^{-Z_k + \log k} = \lim_{N \to \infty} \left(N \mathrm{e}^{-T_{k,N}} \right). \tag{5.7}$$

Recall that by Lemma 5.1, the random variable $T_{k,N}$ has the same distribution as the kth (decreasing) order statistic $\xi_{k,N}$ of N independent exponential random variables $(\xi_i)_{i=1}^N$ (with mean 1). From this, one could derive the distribution of τ_k using the known limit results for the exponential order statistics (see [11, Example 1.3.1 for $k = 1$ and Theorem 2.8.1 for $k \geq 1$]).

However, a more neat proof is possible that requires almost no calculations and simultaneously allows one to establish independence of the successive differences $\tau_{k+1} - \tau_k$. Namely, observe that the random variables $U_i := \mathrm{e}^{-\xi_i}$ $(i = 1, \dots, N)$ are independent and uniformly distributed in $[0, 1]$. Therefore, $(\mathrm{e}^{-\xi_{k,N}})_{k=1}^N \stackrel{\mathrm{d}}{=} (U_{k,N})_{k=1}^N$, where $U_{1,N} \leq \cdots \leq U_{N,N}$ are the order statistics of $(U_i)_{i=1}^N$. In turn, it is well known (see [10, Chapter III, §3]) that

$$(U_{1,N}, \dots, U_{N,N}) \stackrel{\mathrm{d}}{=} \left(\frac{\sigma_1}{\sigma_{N+1}}, \dots, \frac{\sigma_N}{\sigma_{N+1}} \right), \tag{5.8}$$

where σ_k are described in the lemma. Returning to (5.7), the distribution of the vector $(\tau_1, \dots, \tau_k)$ for each $k \geq 1$ can be computed as the weak limit

$$\left(\frac{N\sigma_k}{\sigma_{N+1}}, \dots, \frac{N\sigma_k}{\sigma_{N+1}} \right) \xrightarrow{\mathrm{d}} (\sigma_1, \dots, \sigma_k) \quad (N \to \infty),$$

where we used that, due to the Law of Large Numbers, $\sigma_{N+1}/N \xrightarrow{\mathrm{p}} 1$. $\square$

5.2. Preparatory estimates. The main goal of this section is to establish a suitable uniform upper bound (with probability one) for the terms of the sum (5.2)) (see Lemma 5.7 below), which will allow us to pass to the limit in (5.2) as $t \to \infty$.

Lemma 5.6. *For each fixed $k \geq 1$, with probability one,*

$$\lim_{t \to \infty} \frac{\exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}}{B(t)} = k^{-1/\alpha} \mathrm{e}^{Z_k/\alpha}, \tag{5.9}$$

where $B(t)$ is given by (3.3) *and Z_k is defined in* (5.6)*.*

Proof. Note that a number sequence (a_n) has a limit $a \in \mathbb{R}$ if and only if for each $c \neq a$,

$$\lim_{n\to\infty} \mathbf{1}_{\le c}(a_n) = \mathbf{1}_{\le c}(a), \tag{5.10}$$

where $\mathbf{1}_{\le c}(\cdot)$ denotes the indicator function of the interval $(-\infty, c]$. Let us fix $c > 0$ and consider the "level" inequality

$$\frac{\exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}}{B(t)} \le c, \tag{5.11}$$

which, in view of notations (2.10), (3.3) amounts to $h^{\leftarrow}(T_{k,N}) \le \eta_c(t)^{\pm}$. Furthermore, due to the property (2.4) the last inequality can be rewritten as

$$T_{k,N} \le h(\eta_c(t)^{\pm}). \tag{5.12}$$

Our next step is to use equation (2.10) and represent inequality (5.12) as

$$T_{k,N} - \log N \le h(\eta_c(t)^{\pm}) - h(\eta_1(t)^{\pm}) + \lambda H_0(t) - \log N. \tag{5.13}$$

According to (5.3) and (5.6), the left-hand side of (5.13) amounts to

$$\sum_{i=k}^{N}\left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) + \log\frac{N+1}{N} - \log k \to Z_k - \log k \quad (N\to\infty), \tag{5.14}$$

while on the right, using Lemma 2.2 and the scaling relation (3.1), we have

$$h(\eta_c(t)^{\pm}) - h(\eta_1(t)^{\pm}) + \lambda H_0(t) - \log N \to \alpha \log c \quad (t\to\infty).$$

Therefore, in the limit $t\to\infty$, $N\to\infty$, inequality (5.13) takes the form $Z_k - \log k \le \alpha \log c$, or equivalently, $k^{-1/\alpha}\mathrm{e}^{Z_k/\alpha} \le c$. Comparing this inequality with (5.11) and applying (5.10), we obtain (5.9). □

Let us note that if $\sum_i a_i$ is a convergent series, then its partial sums $\sum_{i=k}^n a_i$ are uniformly bounded. Indeed, set $s_n := \sum_{i=1}^n a_i$, $s_0 := 0$, then $s^* := \sup_n s_n < +\infty$, $s_* := \inf_n s_n > -\infty$ and $\left|\sum_{i=k}^n a_i\right| = |s_n - s_{k-1}| \le s^* - s_* < \infty$ for all n and $k \le n$.

Therefore, since by Lemma 5.2 the series (5.4)) is convergent, there exists a proper random variable Z^* such that, with probability one,

$$\sum_{i=k}^{N}\left(\frac{\zeta_i}{i} - \log\frac{i+1}{i}\right) \le Z^* \quad \text{for all } N \text{ and } 1 \le k \le N. \tag{5.15}$$

Using (5.3) it follows that with probability one for all N and $1 \le k \le N$

$$T_{k,N} \le Z^* + \log\frac{N+1}{k}. \tag{5.16}$$

Lemma 5.7. *Let $B(t)$ be given by* (3.3)*. Then for any $\tilde\alpha > \alpha$ and each $\varepsilon > 0$, with probability one for all large enough t and uniformly in $k \le N$*

$$\frac{\exp(\pm t h^{\leftarrow}(T_{k,N})^{\pm})}{B(t)} \le c_k := k^{-1/\tilde\alpha}\exp\left(\frac{Z^*+\varepsilon}{\alpha}\right). \tag{5.17}$$

PROOF. Similarly to (5.11), we can rewrite (5.17) in the form $T_{k,N} \le h(\eta_{c_k}(t)^{\pm})$ (cf. (5.12)). Furthermore, on account of (5.16) it suffices to check that

$$Z^* - \log k + \log(N+1) \le h(\eta_{c_k}(t)^{\pm}), \quad k = 1, \dots, N. \tag{5.18}$$

Note that with probability one, the ratio

$$\varkappa_k(t) := \frac{\eta_{c_k}(t)}{\eta_1(t)} = 1 \pm \frac{Z^* + \varepsilon}{\alpha t \eta_1(t)} \mp \frac{\log k}{\tilde{\alpha} t \eta_1(t)} \tag{5.19}$$

is ultimately bounded above and separated from 0, uniformly in $k \le N$. Indeed, $\varkappa_k(t)^{\pm} \le \varkappa_1(t)^{\pm} \to 1$ as $t \to \infty$. On the other hand, using (2.12) and (3.1), we get

$$\varkappa_k(t)^{\pm} \ge \varkappa_N(t)^{\pm} \to \left(1 \mp \frac{\alpha}{\tilde{\alpha}\varrho}\right)^{\pm} > \left(1 \mp \frac{1}{\varrho}\right)^{\pm} > 0,$$

since $\tilde{\alpha} > \alpha$. Hence, we can apply Lemma 2.1 and, using the elementary inequality

$$x^p - 1 \ge p(x-1) \quad (p > 1 \text{ or } p < 0)$$

(see [13, Theorem 41]) and also relations (2.10), (2.12) and (5.19), obtain

$$\begin{aligned} h(\eta_{c_k}^{\pm}) - h(\eta_1^{\pm}) &\sim h(\eta_1^{\pm})(\varkappa_k^{\pm\varrho} - 1) \ge \pm\varrho h(\eta_1^{\pm})(\varkappa_k - 1) \\ &= \frac{\varrho h(\eta_1^{\pm})}{\alpha t \eta_1}\left(Z^* + \varepsilon - \frac{\alpha \log k}{\tilde{\alpha}}\right) \sim Z^* + \varepsilon - \frac{\alpha \log k}{\tilde{\alpha}}, \end{aligned} \tag{5.20}$$

uniformly in $k \le N$. We also note that by (2.10) and (3.1),

$$\log(N+1) = h(\eta_1^{\pm}) + o(1) \quad (t \to \infty). \tag{5.21}$$

Estimates (5.20), (5.21) imply that inequality (5.18) will be proved once we check that

$$Z^* - \log k + o(1) \le \left(Z^* + \varepsilon - \frac{\alpha \log k}{\tilde{\alpha}}\right)\bigl(1 + o(1)\bigr),$$

where all $o(1)$ are uniform in $k \le N$. Rearranging, this is reduced to the inequality

$$0 \le \varepsilon + \left(1 - \frac{\alpha}{\tilde{\alpha}} + o(1)\right) \log k + o(1),$$

which holds as $t \to \infty$, since $\varepsilon > 0$, $1 - \alpha/\tilde{\alpha} > 0$ and $\log k \ge 0$. ☐

5.3. Convergence of the representations of sums. We are now in a position to prove the following result.

Theorem 5.8. *Assume that $0 < \alpha < 1$. Then, as $t \to \infty$,*

$$\frac{1}{B(t)} \sum_{k=1}^{N} \mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} \xrightarrow{\text{a.s.}} V_\alpha := \sum_{k=1}^{\infty} k^{-1/\alpha} \mathrm{e}^{Z_k/\alpha}, \tag{5.22}$$

where the last series converges with probability one.

PROOF. Denote $v_k(t) := (1/B(t))\mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}}$ for $k \le N(t)$ and $v_k(t) := 0$ otherwise. By Lemma 5.6, $v_k(t) \xrightarrow{\text{a.s.}} k^{-1/\alpha}\mathrm{e}^{Z_k/\alpha}$ as $t \to \infty$. Let us pick some $\varepsilon > 0$ and for a given $\alpha \in (0,1)$ choose a number $\tilde{\alpha}$ such that $\alpha < \tilde{\alpha} < 1$. Then, according to Lemma 5.7 (see (5.17)), with probability one for all t large enough we have $v_k(t) \le v_k^* := k^{-1/\tilde\alpha} \mathrm{e}^{(Z^*+\varepsilon)/\alpha}$, $k = 1, 2, \dots$ Since $1/\tilde\alpha > 1$, the series $\sum_k v_k^*$ is convergent, so the Lebesgue dominated convergence theorem yields

$$\frac{1}{B(t)} \sum_{k=1}^{N} \mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} = \sum_{k=1}^{\infty} v_k(t) \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} k^{-1/\alpha} \mathrm{e}^{Z_k/\alpha} \equiv V_\alpha \quad (t \to \infty),$$

also implying a.s.-convergence of the limiting series. □

Remark 5.9. The representation of the limit (5.22) can be rewritten as

$$V_\alpha = \sum_{k=1}^{\infty} \tau_k^{-1/\alpha}, \tag{5.23}$$

where (τ_k) are defined in Lemma 5.4. Note that convergence of the series (5.23) is obvious, since a.s. $\tau_k \sim k$ as $k \to \infty$ (by the strong LLN) and $\sum_k k^{-1/\alpha} < \infty$.

Recalling that $S_N(t)$ has the same distribution as the sum (5.2), from Theorem 5.8 it follows that the distribution of $S_N(t)/B(t)$ weakly converges, as $t \to \infty$, to the distribution of the random variable V_α. Comparing this result with Theorem 3.1, we arrive at the following assertion.

Theorem 5.10. *For $0 < \alpha < 1$, the random variable V_α defined in* (5.22) *has the stable distribution $\mathcal{F}_\alpha$ with characteristic function ϕ_α given by formula* (3.5)*.*

This theorem can be viewed as a series representation of the stable distribution $\mathcal{F}_\alpha$ (with $0 < \alpha < 1$ and $\beta = 1$). However, being rewritten in the form (5.23) this representation amounts to one of the known formulas (cf. [21, Theorem 1.4.5]).

Theorem 5.11. *For $\alpha \in (0, 1)$, the ratio $S_N(t)/M_{1,N}(t)$ has a proper limit distribution, which can be represented via the random variable*

$$W_\alpha := \mathrm{e}^{-Z_1/\alpha} V_\alpha \equiv 1 + \sum_{k=1}^{\infty} \exp\biggl(-\frac{1}{\alpha} \sum_{i=1}^{k} \frac{\zeta_i}{i}\biggr), \tag{5.24}$$

where (ζ_i) is a sequence of independent exponential random variables with mean 1 *involved in the representation* (5.6)*.*

Proof. As stated after Lemma 5.1, the joint distribution of $M_{1,N}(t) = \mathrm{e}^{tX_{1,N}}$ and $S_N(t) = \sum_{k=1}^{N} \mathrm{e}^{tX_{k,N}}$ coincides with that of the pair

$$\exp\{\pm t h^{\leftarrow}(T_{1,N})^{\pm}\}, \quad \sum_{k=1}^{N} \exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}.$$

In particular,

$$\frac{S_N(t)}{M_{1,N}(t)} \stackrel{\mathrm{d}}{=} \frac{\sum_{k=1}^{N} \exp\{\pm t h^{\leftarrow}(T_{k,N})^{\pm}\}}{\exp\{\pm t h^{\leftarrow}(T_{1,N})^{\pm}\}}. \tag{5.25}$$

Dividing both the numerator and denominator by the function $B(t)$ defined in (3.3) and applying Lemma 5.6 (with $k = 1$) and Theorem 5.8, we deduce that the right-hand side of (5.25) with probability one converges to

$$V_\alpha \mathrm{e}^{-Z_1/\alpha} = 1 + \sum_{k=2}^{\infty} k^{-1/\alpha} \mathrm{e}^{-(Z_1 - Z_k)/\alpha}. \tag{5.26}$$

From (5.6) it follows that for $k \ge 2$

$$Z_1 - Z_k = \sum_{i=1}^{k-1} \left(\frac{\zeta_i}{i} - \log \frac{i+1}{i} \right) = \sum_{i=1}^{k-1} \frac{\zeta_i}{i} - \log k. \tag{5.27}$$

Hence, (5.26) is reduced to the expression

$$1 + \sum_{k=2}^{\infty} \exp\left(-\frac{1}{\alpha} \sum_{i=1}^{k-1} \frac{\zeta_i}{i} \right),$$

which is the same as the right-hand part of (5.24). □

Remark 5.12. The random variable (5.24) can be represented as (cf. (5.23))

$$W_\alpha = \sum_{k=1}^{\infty} \left(\frac{\tau_1}{\tau_k} \right)^{1/\alpha}, \tag{5.28}$$

where (τ_k) are defined in Lemma 5.4.

Theorem 5.13. *The random variable W_α defined in* (5.24) *has the characteristic function given by*

$$f_\alpha(u) = \frac{\mathrm{e}^{iu}}{1 - \alpha \int_0^1 (\mathrm{e}^{iux} - 1)\, \mathrm{d}x / x^{\alpha+1}}. \tag{5.29}$$

Remark 5.14. Remembering that W_α has emerged in Theorem 5.11 in relation to the limit of $S_N(t)/M_{1,N}(t)$, it is worthwhile to compare Theorem 5.13 with the analogous result by Darling [8, Theorem 5.1]; see also Arov and Bobrov [1, Corollary 4], stating that if (Y_i) is a sequence of i.i.d. random variables with distribution belonging to the domain of attraction of a stable law with exponent $0 < \alpha < 1$, then for $S_N := Y_1 + \cdots + Y_N$ and $M_{1,N} := \max\{Y_1, \ldots, Y_N\}$ the ratio $S_N/M_{1,N}$ has the limit distribution with characteristic function (5.29).

Proof of Theorem 5.13. Using the observation in Remark 5.14, we will prove the theorem's statement indirectly, via establishing a representation analogous to (5.24) for the limit of the ratio $S_N/M_{1,N}$. Clearly, it suffices to do this with a suitable choice of random variables Y_i. Let us set $Y_i = \mathrm{e}^{\xi_i/\alpha}$, where (ξ_i) is an i.i.d. sequence of exponentially distributed random variables with mean 1. Note that $Y_i > 1$ and

$$\mathrm{P}\{Y_i > x\} = \mathrm{P}\{\xi_i > \alpha \log x\} = \mathrm{e}^{-\alpha \log x} = x^{-\alpha} \quad (x > 1).$$

Hence (see [15, Theorem 2.6.1]), the distribution of Y_i is in the domain of attraction of a stable law $\mathcal{F}_\alpha$ (with $\beta = 1$) and

$$\frac{S_N}{N^{1/\alpha}} = \frac{\mathrm{e}^{\xi_1/\alpha} + \cdots + \mathrm{e}^{\xi_N/\alpha}}{N^{1/\alpha}} \xrightarrow{\mathrm{d}} \mathcal{F}_\alpha \quad (N \to \infty).$$

Passing to the order statistics $\xi_{1,N} \ge \cdots \ge \xi_{N,N}$, in a way similar to the above we represent the sum S_N as

$$S_N = \sum_{k=1}^{N} \mathrm{e}^{\xi_{k,N}/\alpha} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N} \mathrm{e}^{T_{k,N}/\alpha},$$

where $T_{k,N}$ are given by (5.1). Analogously to (5.13) and (5.14), we obtain

$$\frac{S_N}{N^{1/\alpha}} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N} \mathrm{e}^{(T_{k,N}-\log N)/\alpha} \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} k^{-1/\alpha}\mathrm{e}^{Z_k/\alpha} \equiv V_\alpha \quad (N\to\infty), \tag{5.30}$$

where a term-by-term passing to the limit can be justified as before, using (5.16).

Similarly, one shows that

$$\frac{M_{1,N}}{N^{1/\alpha}} \stackrel{\mathrm{d}}{=} \mathrm{e}^{(T_{1,N}-\log N)/\alpha} \xrightarrow{\text{a.s.}} \mathrm{e}^{Z_1/\alpha} \quad (N\to\infty). \tag{5.31}$$

Hence, dividing (5.30) by (5.31) we obtain (cf. (5.24))

$$\frac{S_N}{M_{1,N}} \xrightarrow{\mathrm{d}} \mathrm{e}^{-Z_1/\alpha} V_\alpha \equiv W_\alpha.$$

Comparing this with the result by Darling [8] mentioned above, we conclude that W_α has distribution with the characteristic function (5.29). □

Remark 5.15. It would be interesting to derive formula (5.29), or otherwise characterize the distribution of W_α directly from representation (5.24) (or (5.28)).

The following result, being an immediate consequence of Theorem 5.13, is of interest due to the striking simplicity of the answer.

Corollary 5.16. *For $0<\alpha<1$, the expected value of W_α is given by*

$$\mathrm{E}[W_\alpha] = \frac{1}{1-\alpha}. \tag{5.32}$$

Proof. Differentiate formula (5.29) at $u=0$. □

Remark 5.17. In the Appendix, we will give three alternative proofs of identity (5.32) based on representations (5.24) or (5.28).

Remark 5.18. Taking the expectation of W_α using (5.24) (see (A.1) below) and comparing with (5.32), we arrive at the following curious identity:

$$\sum_{k=1}^{\infty}\prod_{i=1}^{k}\left(1+\frac{1}{i\alpha}\right)^{-1} = \frac{\alpha}{1-\alpha} \quad (0<\alpha<1).$$

The next assertion highlights an increasingly overwhelming role played by the maximal term $M_{1,N}(t)$ in the sum $S_N(t)$ as α tends to zero.

Proposition 5.19. *With probability one, $W_\alpha \to 1$ as $\alpha\to 0+$.*

Proof. As we know, the series (5.24) is a.s.-convergent for all $0<\alpha<1$. Since $\zeta_i\ge 0$, its terms are nondecreasing functions of α, so that for $0<\alpha\le\alpha_0<1$ each one is dominated by the respective value with $\alpha=\alpha_0$. Moreover, with probability one each term is $o(1)$ as $\alpha\to 0+$, so the Lebesgue dominated convergence theorem implies that the series in (5.24) almost surely vanishes and hence $W_\alpha\to 1$. □

Using the series representation provided by Theorem 5.10, one can easily derive a limit theorem for the stable distribution $\mathcal{F}_\alpha$ as its parameter α tends to zero.

Proposition 5.20. *Let a random variable ζ_α have the stable distribution $\mathcal{F}_\alpha$ determined by* (3.5)*, with parameters $0<\alpha<1$ and $\beta=1$. Then, as $\alpha\to 0+$, the distribution of $\alpha\log\zeta_\alpha$ weakly converges to the double exponential distribution,*

$$\lim_{\alpha\to 0+} \mathrm{P}(\alpha\log\zeta_\alpha \le x) = \exp(-\mathrm{e}^{-x}), \quad x\in\mathbb{R}. \tag{5.33}$$

PROOF. By Theorem 5.10, the random variable V_α has the distribution $\mathcal{F}_\alpha$, so it suffices to verify (5.33) for V_α. From (5.24) and using Proposition 5.19 we have $\alpha \log V_\alpha = Z_1 + \log W_\alpha \xrightarrow{\text{a.s.}} Z_1$ as $\alpha \to 0+$. To find the distribution of Z_1, note that according to the definition of τ_k in Lemma 5.4, we have $Z_1 = -\log \tau_1$, where τ_1 has the exponential distribution with mean 1. Hence, $\mathrm{P}\{Z_1 \le x\} = \mathrm{P}\{\tau_1 \ge \mathrm{e}^{-x}\} = \exp(-\mathrm{e}^{-x})$. □

Remark 5.21. The limit theorem (5.33) can be used to explore the limiting case $\alpha \to 0+$ of Shlather's conjecture concerning random norms (see [3, §9; 6; 22]). A general result of this kind was proved purely analytically by Zolotarev [24, Theorem 5; 25, Theorem 2.9.1].

We conclude this section by stating a series representation theorem for the limit in the case $\lambda_1 \le \lambda < \lambda_2$, analogous to Theorem 5.8.

Theorem 5.22. (a) *For* $\lambda_1 < \lambda < \lambda_2$,

$$\frac{1}{B(t)} \sum_{k=1}^{N} \left(\mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} - \mathrm{E}\left[\mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}}\right]\right) \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} \frac{\mathrm{e}^{Z_k/\alpha} - \mathrm{E}\left[\mathrm{e}^{Z_k/\alpha}\right]}{k^{1/\alpha}}.$$

(b) *For* $\lambda = \lambda_1$,

$$\frac{1}{B(t)} \sum_{k=1}^{N} \left(\mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} - \mathrm{E}\left[\mathrm{e}^{\pm t h^{\leftarrow}(T_{k,N})^{\pm}} \mathbf{1}_{\{T_{k,N} \le h(\eta_1^{\pm})\}}\right]\right)$$
$$\xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} \frac{\mathrm{e}^{Z_k/\alpha} - \mathrm{E}\left[\mathrm{e}^{Z_k/\alpha} \mathbf{1}_{\{Z_k \le \log k\}}\right]}{k^{1/\alpha}}.$$

Here the limiting series converge with probability one.

This theorem can be proved along the same lines as in the case $0 < \lambda < \lambda_1$ above. However, the proof is technically more involved and will be presented elsewhere.

Appendix A. Direct proofs of Corollary 5.16

A.1. A proof using representation (5.24). From (5.24) it follows

$$\mathrm{E}\left[W_\alpha\right] = 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \mathrm{E}\left[\mathrm{e}^{-\zeta_i/(i\alpha)}\right] = 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{i}{\alpha^{-1} + i}. \tag{A.1}$$

Note that the right-hand side of (A.1) coincides with the hypergeometric function [12, #9.100]

$$F(a, b; c; z) := 1 + \sum_{k=1}^{\infty} z^k \prod_{i=0}^{k-1} \frac{(a+i)(b+i)}{(c+i)(1+i)}$$

taken at the values $a = 1$, $b = 1$, $c = 1 + \alpha^{-1}$, $z = 1$. Furthermore, it is known [12, #9.122(1)] that for $z = 1$ one has

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c > a + b).$$

For the above specific values of the parameters this yields

$$F(1, 1; 1 + \alpha^{-1}; 1) = \frac{\Gamma(1 + \alpha^{-1})\Gamma(\alpha^{-1} - 1)}{\Gamma(\alpha^{-1})^2}. \tag{A.2}$$

Using that $\Gamma(1+x) = x\Gamma(x)$, we obtain

$$\Gamma(1+\alpha^{-1})\Gamma(\alpha^{-1}-1) = \alpha^{-1}\Gamma(\alpha^{-1})\cdot\frac{\Gamma(\alpha^{-1})}{\alpha^{-1}-1} = \frac{\Gamma(\alpha^{-1})^2}{1-\alpha},$$

so substituting this into (A.2) we get (5.32).

A.2. A proof using representation (5.28). The proof above may not seem quite satisfactory, as it relies heavily on the 'external' analytic aid from tables of formulas. Here we give a simpler, self-contained proof based on another representation of W_α given by equation (5.28). Namely, using Lemma 5.4 and relation (5.8) we deduce from (5.28) that $W_\alpha \stackrel{\mathrm{d}}{=} 1 + \sum_{k=1}^{\infty} U_{1,k}^{1/\alpha}$, where $U_{1,k}$ is the minimum of independent random variables $U_1,\dots,U_k$ with uniform distribution on $[0,1]$. Therefore,

$$\mathrm{E}\bigl[W_\alpha\bigr] = 1 + \sum_{k=1}^{\infty}\mathrm{E}\bigl[U_{1,k}^{1/\alpha}\bigr]. \tag{A.3}$$

Note that $\mathrm{P}\{U_{1,k} > x\} = (1-x)^k$ $(0 \le x \le 1)$, and hence

$$\mathrm{E}\bigl[U_{1,k}^{1/\alpha}\bigr] = \int_0^1 x^{1/\alpha} k(1-x)^{k-1}\,\mathrm{d}x.$$

Substituting this expression into (A.3), we obtain

$$\mathrm{E}\bigl[W_\alpha\bigr] = 1 + \int_0^1 x^{1/\alpha}\sum_{k=1}^{\infty} k(1-x)^{k-1}\,\mathrm{d}x = 1 + \int_0^1 x^{1/\alpha-2}\,\mathrm{d}x = \frac{1}{1-\alpha},$$

and (5.32) follows.

A.3. Yet another proof. Finally, following an elegant idea of S. A. Molchanov (private communication), we give an elementary proof of identity (5.32) proceeding directly from representation (A.1). For $0<\alpha<1$, set

$$a_1 := \frac{1}{\alpha^{-1}-1}, \qquad a_i := \frac{i}{\alpha^{-1}+i-1} \quad (i \ge 2),$$
$$A_k := \prod_{i=1}^{k} a_i \quad (k \ge 1).$$

It is easy to check that for all $k \ge 1$,

$$\prod_{i=1}^{k}\frac{i}{\alpha^{-1}+i} = \prod_{i=1}^{k} a_i\cdot(1-a_{k+1}) = A_k - A_{k+1}.$$

It follows that the series on the right-hand side of (A.1) is reduced to

$$1 + \sum_{k=1}^{\infty}(A_k - A_{k+1}) = 1 + \lim_{n\to\infty}(A_1 - A_n) = 1 + A_1,$$

since $A_n \to 0$ as $n\to\infty$. But

$$1 + A_1 = 1 + a_1 = \frac{1}{1-\alpha},$$

and formula (5.32) is proved.

Remark A.1. The different proofs of relation (5.32) given above, being of interest in their own right, may be useful for a direct proof of Theorem 5.13 (cf. Remark 5.14).

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