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LOCALIZATION VS. DELOCALIZATION OF RANDOM DISCRETE MEASURES*

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To the memory of B. V. Gnedenko

Abstract. Sequences of discrete measures $\mu^{(n)}$ with random atoms $\{\mu_i^{(n)}, i=1,2,\ldots\}$ such that $\sum_i \mu_i^{(n)} = 1$ are considered. The notions of (complete) asymptotic localization vs. delocalization of such measures in the weak (mean or probability) and strong (with probability 1) sense are proposed and analyzed, proceeding from the standpoint of the largest atoms' behavior as $n\to\infty$. In this framework, the class of measures with the atoms of the form $\mu_i^{(n)} = X_i/S_n$ $(i=1,\ldots,n)$ is studied, where $X_1, X_2, \ldots$ is a sequence of positive, independent, identically distributed random variables (with a common distribution function F) and $S_n = X_1+\cdots+X_n$. If $\mathbf{E}[X_1]<\infty$, then the law of large numbers implies that $\mu^{(n)}$ is strongly delocalized. The case where $\mathbf{E}[X_1]=\infty$ is studied under the standard assumption that F has a regularly varying upper tail (with exponent $0\leqq\alpha\leqq 1$). It is shown that for $\alpha<1$, weak localization occurs. In the critical point $\alpha=1$, the weak delocalization is established. For $\alpha=0$, localization is strong unless the tail decay is "hardly slow."

Key words. random measures, localization, delocalization, extreme terms, order statistics, law of large numbers, regular variation

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1. Introduction. The present paper is devoted to the asymptotic study of the sequences of random discrete measures

$$\mu^{(n)} = \sum_i \mu_i^{(n)} \delta_{x_i^{(n)}}, \quad n=1,2,\ldots, \tag{1}$$

where δ_x stands for a Dirac (unit) measure concentrated at point x, and $\mu_i^{(n)}$ $(i=1,2,\ldots)$ are positive random variables which determine the masses of the measure atoms located at (random) points $x_i^{(n)}$, respectively. (In our approach, we will actually not be concerned with the structure of the set $\{x_i^{(n)}, i=1,2,\ldots\}$, that is, the support of $\mu^{(n)}$, and thus we identify $\mu^{(n)}$ with the aggregate $\{\mu_i^{(n)}, i=1,2,\ldots\}$. Such a standpoint is familiar in probability theory where, for example, the distribution of a discrete random variable may be considered apart from the probability space on which the latter is defined. (See also the remark below, in the paragraph preceding equation (3).) It is assumed that all the $\mu^{(n)}$ are normalized to unity:

$$\operatorname{Var}\mu^{(n)} = \sum_i \mu_i^{(n)} = 1, \quad n=1,2,\ldots\,. \tag{2}$$

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The question addressed in this work is whether asymptotic *localization* (in the physical literature, the term "nonself-averaging" is also used [6]) of the measures $\mu^{(n)}$ takes place, that is, their discreteness is preserved (in some sense) as $n \to \infty$. Topics of that kind arise in various contexts in the theory of disordered systems where such measures may appear as probability distributions generated in random structures (see, e.g., [6], [4], and [1]).

Let us point out that in general the limiting (random) measure $\mu = \lim_{n\to\infty} \mu^{(n)}$ may not exist. Moreover, in many applications the support of $\mu^{(n)}$ is not specified, the atoms $\{\mu_i^{(n)},\, i = 1, 2, \ldots\}$ being considered as the statistical weights of the system configurations, or "states" (cf. [4] and [6]). Therefore, in order to furnish the most broad framework for the localization effects, one is led to working in the "phase space" via setting the problem purely in terms of the measure atoms $\{\mu_i^{(n)}\}$.

A possible way of doing this is to study certain "integral" functionals of $\mu^{(n)}$. For instance, a known device is the variation of the sequence $\mu^{(n)}$ of order $1+0$. More precisely, for $\lambda > 0$ consider the variation of $\mu^{(n)}$ of order $1+\lambda$:

$$m_\lambda^{(n)} := \sum_i \left(\mu_i^{(n)}\right)^{1+\lambda}. \tag{3}$$

(Note that, according to (2), we have $m_\lambda^{(n)}|_{\lambda=0} = \operatorname{Var} \mu^{(n)} \equiv 1$.) It is easy to understand that the consecutive passing to the limit as $n \to \infty$ and then $\lambda \to +0$ can help distinguish localization from delocalization. Namely, in the delocalized case $m_\lambda^{(n)}$ should vanish in this double limit, whereas strictly positive limit indicates the presence of a residual discrete component; moreover, the approach of $m_\lambda^{(n)}$ to 1 suggests that localization is "complete." This observation lies in the background of the definition of localization and delocalization *in variation* given below.

Another convenient approach adopted in this work proceeds from the study of behavior of the *senior* (i.e., greatest in magnitude) atoms of $\mu^{(n)}$ (see [4] and [13]). Indeed, it is natural to think that whenever complete localization occurs, the bulk mass of $\mu^{(n)}$ (say, greater than $1-\varepsilon$ with $\varepsilon > 0$ arbitrarily small) stays accumulated by a few senior atoms, the number of which, as opposed to the case of delocalization, does not depend on n (although it may be dependent on ε). By specifying the type of convergence of the corresponding random quantities, we will arrive at definitions of localization and delocalization in mean, in probability, and with probability 1.

The main content of the paper is devoted to studying in this framework the class of measures with atoms of the form

$$\mu_i^{(n)} = \frac{X_i}{S_n}, \quad i = 1, \ldots, n, \tag{4}$$

where $X_1, X_2, \ldots$ is a sequence of positive, independent, identically distributed random variables and $S_n = X_1 + \cdots + X_n$. (The normalization condition (2) is automatically satisfied here.) Note that the random variables (4) are exchangeable (i.e., symmetrically dependent [7, section VII.4]). On account of (2), it follows that the expectation of each atom $\mu_i^{(n)}$ (with fixed i) is the same: $\mathbf{E}[\mu_i^{(n)}] = n^{-1}$, $i = 1, \ldots, n$ (cf. [4] and [6]). However, this fact as it stands does not imply delocalization of $\mu^{(n)}$ but indicates that localization may take place through the accumulation of the bulk mass in a few senior atoms $\mu_{1,n} \geqq \mu_{2,n} \geqq \cdots$ which occur at *random* "places" of the sample $\{\mu_1^{(n)}, \ldots, \mu_n^{(n)}\}$.

The detailed analysis of this simple model may be instructive since (4) mimics various, more realistic models which often display universal behavior (cf. [6], [4], and [13]). On the other hand, the model is of interest in its own right, in particular in connection with the role played by the greatest summands in sums of independent identically distributed random variables (see, e.g., [5], [9], [2], [12], [14], and [11]).

Some results on the behavior of the measures with atoms of the form in (4) have been announced in [3]. In the present work, the complete classification of localization vs. delocalization for this model is given with full proofs.

Let us briefly outline our results. First we discuss the general definitions of complete asymptotic localization and delocalization for the measures (1) subject to the condition (2), in the weak sense (in variation, in mean, and in probability, which prove to be equivalent to each other) and in the strong sense (with probability 1). We then study the main model (4). In the simplest case where the expectation of the random variable X_1 is finite, it is easy to prove *strong delocalization* (Theorem 1). The case of infinite expectation is studied under the standard assumption that the upper tail of the distribution function $F(x) = \mathbf{P}\{X_1 \leqq x\}$ is regularly varying at infinity (with exponent $0 \leqq \alpha \leqq 1$). Theorem 3 then shows that for $0 < \alpha < 1$, *weak localization* occurs (*strong* localization is ruled out by Theorem 2 (ii)). Theorem 4 appears as a certain asymptotic version of Theorem 3. As for the critical point $\alpha = 1$, Theorem 5 asserts that there is still *delocalization*; however, unlike the case $\alpha > 1$ (covered by Theorem 1), for $\alpha = 1$ this is only *weak* as indicated by Theorem 2 (i) (of course, provided the expectation is infinite). Last, in the case $\alpha = 0$, where weak localization always takes place (Theorem 6), it turns out that, generally speaking, localization is *strong* unless the decay of the distribution tail is "hardly slow" (Theorem 7).

The paper is organized as follows. In section 2, the general definitions of various forms of weak and strong localization and delocalization are given (Definitions 1, 2) and mutual relations between them are stated (Propositions 1, 2). The proof of Proposition 1, being technical and quite routine, is found in the Appendix (section 6). In section 3, we formulate the results (Theorems 1–7) for the model (4). A few technical lemmas, mostly well known, are recorded in section 4, and the proofs of the theorems are gathered in section 5.

2. Various forms of localization and delocalization. We need to introduce some notation. We arrange the random atoms $\{\mu_i^{(n)}, i = 1, 2, \dots\}$ in *decreasing* order $\mu_{1,n} \geqq \mu_{2,n} \geqq \cdots$. For fixed $k \geqq 1$ we put

$$\sigma_{k,n} := \sum_{i=1}^{k} \mu_{i,n}. \tag{5}$$

We also recall that the quantity $m_\lambda^{(n)}$, with $\lambda > 0$, is defined in (3).

Definition 1. *We shall say that for a sequence of measures* $\mu^{(n)} = \{\mu_i^{(n)}, i = 1, 2, \dots\}$ *subject to the normalization condition* (2), *one has* (*complete*) *asymptotic* localization *or* delocalization:
in variation *if*

$$\lim_{\lambda \to +0} \liminf_{n \to \infty} \mathbf{E}[m_\lambda^{(n)}] = 1 \tag{L1}$$

or, *respectively*,

$$\lim_{\lambda \to +0} \limsup_{n \to \infty} \mathbf{E}[m_\lambda^{(n)}] = 0; \tag{D1}$$

in mean *if*

(L2) $$\lim_{k\to\infty}\liminf_{n\to\infty}\mathbf{E}[\sigma_{k,n}]=1$$

or, *respectively*,

(D2) $$\lim_{k\to\infty}\limsup_{n\to\infty}\mathbf{E}[\sigma_{k,n}]=0;$$

in probability *if*

(L3) $$\lim_{k\to\infty}\liminf_{n\to\infty}\mathbf{P}\{\sigma_{k,n}>1-\varepsilon\}=1\quad\forall\,\varepsilon>0,$$

or, *respectively*,

(D3) $$\lim_{k\to\infty}\limsup_{n\to\infty}\mathbf{P}\{\sigma_{k,n}>\varepsilon\}=0\quad\forall\,\varepsilon>0;$$

with probability 1 *if* (*in this part of the definition*, *all random variables* $\{\mu_i^{(n)}i$, $n=1,2,\dots\}$ *are of course assumed to be defined on a common probability space*)

(L4) $$\mathbf{P}\left\{\lim_{k\to\infty}\liminf_{n\to\infty}\sigma_{k,n}=1\right\}=1,$$

or, *respectively*,

(D4) $$\mathbf{P}\left\{\lim_{k\to\infty}\limsup_{n\to\infty}\sigma_{k,n}=0\right\}=1.$$

PROPOSITION 1. *Conditions* (L1), (L2), *and* (L3) *are equivalent to each other and*, *in turn*, *follow from condition* (L4). *Similarly*, (D1), (D2), *and* (D3) *are mutually equivalent and are implied by* (D4).

The proof of Proposition 1 is given in the Appendix.

Conditions (D2)–(D4) can be brought to a simpler form in terms of a *single*, maximal atom $\mu_{1,n}$:

(D2′) $$\lim_{n\to\infty}\mathbf{E}[\mu_{1,n}]=0,$$

(D3′) $$\lim_{n\to\infty}\mathbf{P}\{\mu_{1,n}>\varepsilon\}=0\quad\forall\,\varepsilon>0,$$

(D4′) $$\mathbf{P}\left\{\lim_{n\to\infty}\mu_{1,n}=0\right\}=1.$$

PROPOSITION 2. *The conditions* (D2′), (D3′), *and* (D4′) *are equivalent to* (D2), (D3), *and* (D4), *respectively.*

This easily follows from the obvious inequalities $\mu_{1,n}\leqq\sigma_{k,n}\leqq k\mu_{1,n}$.

In the sequel, we will use the following terminology.

DEFINITION 2. *In the case where the* (*equivalent*) *conditions* (L1)–(L3) *are satisfied*, *we shall speak of* weak *localization, whereas localization in the form* (L4) *will be called* strong. *Analogously*, *delocalization in the form* (D1)–(D3) (*and also* (D2′), (D3′)) *or* (D4), (D4′) *will be referred to as* weak *or* strong, *respectively.*

3. Statements of the results. We now consider the class of measures $\mu^{(n)}$ with the atoms of the form (4), where the random variables $X_1,X_2,\dots$ are assumed positive, independent, and identically distributed (with a common distribution function F), and

$S_n = X_1 + \cdots + X_n$. Note that the decreasingly ordered atoms $\mu_{1,n} \geqq \cdots \geqq \mu_{n,n}$ (cf. section 2) are represented as

$$\mu_{i,n} = \frac{X_{i,n}}{S_n}, \quad i = 1, \ldots, n, \tag{6}$$

where $X_{1,n} \geqq \cdots \geqq X_{n,n}$ stand for the order statistics of the sample $X_1, \ldots, X_n$.

We begin with the simplest case where the expectation of the random variables X_i is finite (cf. [11, Lemma 4.1]).

THEOREM 1. *Assume that* $\mathbf{E}[X_1] < \infty$. *Then in model* (4), *the largest atom* $\mu_{1,n}$ *with probability* 1 *tends to zero as* $n \to \infty$; *that is*, *condition* (D4$'$) *is satisfied and hence strong delocalization takes place.*

The case where the expectation of the random variables X_i is infinite appears more complicated. We will study this case under the standard additional assumption of regular variation of the distribution function F at infinity [7], [15].

DEFINITION 3. *The distribution function* $F(x) = \mathbf{P}\{X_1 \leqq x\}$ *will be said to have a* regular (upper) tail (*with exponent* $0 \leqq \alpha \leqq +\infty$) *if the function* $\varphi(x) := 1 - F(x)$ *is regularly varying at infinity*; *that is*, *for every fixed* $t > 0$

$$\lim_{x \to +\infty} \frac{\varphi(tx)}{\varphi(x)} = t^{-\alpha}, \tag{7}$$

where for $\alpha = +\infty$ *the symbol* $t^{-\infty}$ *is defined as*

$$t^{-\infty} = \begin{cases} 0, & t > 1, \\ 1, & t = 1, \\ \infty, & 0 < t < 1. \end{cases}$$

Note that if F has a regular tail with the exponent α, then $\mathbf{E}[X_1^p] < \infty$ for any $p < \alpha$ and $\mathbf{E}[X_1^p] = \infty$ for any $p > \alpha$ (see, e.g., [10, section 36]).

THEOREM 2. (i) *Assume that* $\mathbf{E}[X_1] = \infty$. *Then with probability* 1

$$\limsup_{n \to \infty} \mu_{1,n} = 1. \tag{8}$$

(ii) *Let the distribution function* F *have a regular tail with exponent* $\alpha > 0$. *Then with probability* 1

$$\liminf_{n \to \infty} \mu_{1,n} = 0. \tag{9}$$

This theorem amounts to saying that with probability 1 there exists a (random) subsequence of numbers n' such that $\mu^{(n')}$ is localized inasmuch as all the mass is eventually concentrated in a *single* atom (Theorem 2 (i)). On the other hand, also with probability 1, there exists a random subsequence n'' such that $\mu^{(n'')}$ is delocalized, as all the atoms $\{\mu_i^{(n'')},\ i = 1, \ldots, n''\}$ are uniformly small (Theorem 2 (ii)).

Remark 1. In view of Theorem 1, the statement (ii) of Theorem 2 is of interest only for $\alpha \leqq 1$; otherwise $\mathbf{E}[X_1] < \infty$.

THEOREM 3. *Assume that the distribution function* F *has a regular tail with exponent* $0 < \alpha < 1$. *Then condition* (L2) *is fulfilled*, *that is*, *weak localization occurs.*

Note that from Theorem 2 (ii) it follows that for $0 < \alpha \leqq 1$ strong localization is ruled out.

The next theorem comes as an asymptotic version of Theorem 3.

THEOREM 4. *Let* $1 \leqq k_n \leqq n$ *be an arbitrary* (*nonrandom*) *sequence of integers going to* ∞. *Then under the conditions of Theorem* 3, $\lim_{n\to\infty} \mathbf{E}[\sigma_{k_n,n}] = 1$.

The situation at the critical point $\alpha = 1$ is described by the following.

THEOREM 5. *Suppose that the distribution function* F *has a regularly varying tail with* $\alpha = 1$. *Then the condition* (D3$'$) *holds and hence weak delocalization takes place.*

It should be mentioned that if $\alpha = 1$ and $\mathbf{E}[X_1] < \infty$, then delocalization is in fact strong (Theorem 1). It is of interest to note, however, that whenever $\mathbf{E}[X_1] = \infty$, this is not the case by Theorem 2 (i).

Finally, consider the case $\alpha = 0$ (*slow* variation), where one can all the more anticipate localization. In the weak version, localization immediately follows from the known results by Darling [5] and Arov and Bobrov [2] which imply that the maximal term $X_{1,n}$ completely dominates the entire sum S_n. We record this as the following theorem.

THEOREM 6. *Assume that the distribution function* F *has a slowly varying tail*; *that is*, (7) *is satisfied with* $\alpha = 0$. *Then condition* (L2) *of weak localization holds. Moreover*, $\mu_{1,n} \to 1$ *in mean*:

$$\lim_{n\to\infty} \mathbf{E}\big[|\mu_{1,n} - 1|^p\big] = 0, \tag{10}$$

with arbitrary $p > 0$, *and hence in probability*:

$$\lim_{n\to\infty} \mathbf{P}\{\mu_{1,n} > 1 - \varepsilon\} = 1 \quad \forall\, \varepsilon > 0. \tag{11}$$

Remark 2. It is worth mentioning that, conversely, in the case of positive summands the slow variation proves as well to be a *necessary* condition for (11) (see [11, Theorem 1]).

As far as strong localization is concerned, the situation appears to be more delicate. We are able to give the complete answer to this question by adapting to our case the results of Pruitt [14] on necessary and sufficient conditions for a.s.-domination. In particular, it turns out that for $\alpha = 0$, strong localization may still *not* take place.

Following [14], let us put

$$v_n = \mathbf{P}\{2^n < X_1 \leqq 2^{n+1} \mid X_1 > 2^n\} = \frac{\varphi(2^n) - \varphi(2^{n+1})}{\varphi\,(2^n)}\,, \tag{12}$$

where $\varphi(x) = 1 - F(x)$. Clearly, v_n are fashioned to provide information about the distribution tail inasmuch as the slower the tail decay, the greater the v_n's. Define the integer $0 \leqq k^* \leqq +\infty$ as

$$k^* := \min\left\{k \geqq 0\colon \sum_{n=1}^{\infty} v_n^{k+1} < \infty\right\}, \tag{13}$$

with the convention that $\min\varnothing = +\infty$. It is easy to show that the series $\sum_n v_n$ is always divergent so that $k^* \geqq 1$. Note that in the case of regular variation, from (12) and (7) it readily follows that $v_n \to 1 - 2^{-\alpha}$ and hence $k^* \equiv +\infty$ for $\alpha > 0$. So, unlike the regularity exponent α which ascribes the single value $\alpha = 0$ to all slowly varying functions, the critical index k^* sets a scale for slow variation, meanwhile "sticking" all regularly varying functions (with $\alpha > 0$) at $k^* = +\infty$.

THEOREM 7. (i) *If* $k^* = 1$ ("*very slow decay*"), *then with probability* 1

$$\lim_{n\to\infty} \mu_{1,n} = 1, \tag{14}$$

which implies condition (L4) *of strong localization.*

(ii) *If* $2 \leqq k^* < +\infty$ (*"fairly slow decay"*), *then with probability* 1

$$\lim_{n\to\infty} \sigma_{k^*,n} = 1, \tag{15}$$

so that strong localization takes place as well. However, *unlike case* (i), *with probability* 1

$$\liminf_{n\to\infty} \mu_{1,n} = \frac{1}{k^*}, \tag{16}$$

$$\limsup_{n\to\infty} \mu_{1,n} = 1, \tag{17}$$

and furthermore,

$$\limsup_{n\to\infty} \mu_{k^*,n} = \frac{1}{k^*}, \tag{18}$$

$$\liminf_{n\to\infty} \sigma_{k^*-1,n} = 1 - \frac{1}{k^*}. \tag{19}$$

(iii) *If* $k^* = +\infty$ (*"hardly slow decay"*), *then with probability* 1

$$\liminf_{n\to\infty} \mu_{1,n} = 0, \tag{20}$$

which rules out strong localization.

Remark 3. Qualitatively speaking, this theorem indicates that with growth of the critical index k^*, localization becomes, so to say, more and more weak since the mass is essentially concentrated in exactly k^* atoms, as evidenced by (15) and (19). In particular, in the intermediate case $2 \leqq k^* < \infty$, localization cannot take place in the "amplified" form (14) because of (16). Moreover, (18) means that for any $\varepsilon > 0$

$$\mathbf{P}\left\{\mu_{1,n} > \frac{1-\varepsilon}{k^*}\,,\ \ldots\,,\ \mu_{k^*,n} > \frac{1-\varepsilon}{k^*} \text{ i.o.}\right\} = 1,$$

(here and below, we use the abbreviation "i.o." for "infinitely often"; more precisely, for a sequence of events A_n, the event (A_n i.o.) is defined to be equal to $\limsup A_n = \cap_n \cup_{m\geqq n} A_m$) which amounts to saying that a.s. there exists a (random) subsequence of numbers (n') such that for $\mu^{(n')}$ almost the entire mass is shared approximately in equal parts among the k^* senior atoms.

Remark 4. Note that in the case of regular variation, statement (iii) implies Theorem 2 (ii) since, as mentioned above, $k^* = +\infty$ for all $\alpha > 0$.

To conclude this section, let us show by examples that in the case $\alpha = 0$ all the possibilities (i)–(iii) are realizable. In the following example (see [14]) the index k^* can assume each of the finite values $1, 2, \ldots$.

Example 1. Let $\varphi(x) = \exp(-(\log x)^a)$, $x \geqq 1$, where $0 < a < 1$. It is straightforward to check by definition (7) that φ is a slowly varying function. From (12) one easily obtains

$$\begin{aligned} v_n &= 1 - \exp\left(-(\log 2^n)^a + (\log 2^{n+1})^a\right) \\ &= 1 - \exp\left(-(n\log 2)^a\left[\left(1+\frac{1}{n}\right)^a - 1\right]\right) \sim a(\log 2)^a n^{a-1} \quad \text{as } n\to\infty, \end{aligned}$$

which in view of (13) implies that $k^* = 1$ if $0 < a < \frac{1}{2}$ and $k^* = k \geqq 2$ whenever $(k-1)/k \leqq a < k/(k+1)$.

Finally, the next example demonstrates that the slow variation is compatible with the possibility $k^* = +\infty$; that is, for $\alpha = 0$ it may be the case that localization is weak rather than strong (see (20)).

Example 2. For $x \geqq e$, let

$$\varphi(x) = \exp\left(-\int_e^x \frac{dy}{y \log\log y}\right) = \exp\left(-\int_1^{\log x} \frac{dz}{\log z}\right).$$

Application of the so-called Karamata representation (e.g., [15, section 0.4, Corollary to Theorem 0.6]; see also Lemma 4 below) readily shows that φ is slowly varying. Furthermore,

$$v_n = 1 - \exp\left(-\int_{n\log 2}^{(n+1)\log 2} \frac{dz}{\log z}\right) \sim \int_{n\log 2}^{(n+1)\log 2} \frac{dz}{\log z} \sim \frac{\log 2}{\log n},$$

so that $\sum_n v_n^k = \infty$ for all $k > 0$ and hence $k^* = +\infty$.

4. Auxiliary lemmas. We record here a few (mostly well known) auxiliary assertions which will be instrumental in the sequel.

Let us define the *generalized inverse* of the function $\varphi(x) = 1 - F(x)$ as (cf. [15, section 0.2])

$$\varphi^{\leftarrow}(u) := \inf\{x \geqq 0\colon \varphi(x) \leqq u\}, \quad 0 \leqq u \leqq 1, \tag{21}$$

with the convention that $\inf\varnothing = +\infty$. It is easily checked that $\varphi^{\leftarrow}$ is nonincreasing and right-continuous. From the definition, it also follows that

$$\varphi^{\leftarrow}(u) \leqq x \quad \text{if and only if } u \geqq \varphi(x). \tag{22}$$

The following simple fact (see, e.g., [15, section 0.2]) allows one to make the so-called quantile transformation and to pass from the random variables X_i to *uniform* random variables U_i and, moreover, from the original order statistics $X_{1,n} \geqq \cdots \geqq X_{n,n}$, involved in (6), to the corresponding *uniform* order statistics, $U_{1,n} \leqq \cdots \leqq U_{n,n}$ (for convenience of further notation, we work with $\{U_1, \dots, U_n\}$ arranged in *increasing* order).

LEMMA 1. *Let U be a random variable with uniform distribution in* $[0,1]$. *Then the random variable $X = \varphi^{\leftarrow}(U)$ has the distribution function F.*

The proof is straightforward:

$$\mathbf{P}\{\varphi^{\leftarrow}(U) \leqq x\} = \mathbf{P}\{U \geqq \varphi(x)\} = 1 - \varphi(x) = F(x),$$

where we have used property (22).

Thanks to this lemma, we may assume without loss of generality that the given sequence $(X_n,\ n \geqq 1)$ is represented as $(\varphi^{\leftarrow}(U_n),\ n \geqq 1)$, where $U_1, U_2, \dots$ is an auxiliary sequence of independent uniform random variables. In particular, it follows that the joint distribution of $\varphi^{\leftarrow}(U_{1,n}), \dots, \varphi^{\leftarrow}(U_{n,n})$ coincides with that of the original order statistics $X_{1,n}, \dots, X_{n,n}$.

In turn, it is often handy to realize the uniform order statistics as the consecutive ratios of the sums of exponential random variables (see, e.g., [7, section III.3]).

LEMMA 2. *The joint distribution of the uniform order statistics $U_{1,n}, \dots, U_{n,n}$ coincides with that of the random variables*

$$\frac{T_1}{T_{n+1}}, \dots, \frac{T_n}{T_{n+1}},$$

where $T_i = Y_1 + \cdots + Y_i$ $(i = 1, \dots, n+1)$ *and* $Y_1, \dots, Y_{n+1}$ *are independent random variables with common exponential distribution* (*with parameter* 1):

$$\mathbf{P}\{Y > x\} = e^{-x}, \quad x > 0.$$

The following lemma (see, e.g., [15, section 0.4, Proposition 0.8 (v)]) guarantees that passing from X_i to U_i preserves the property of regular variation.

LEMMA 3. *Suppose that the function* $\varphi(x) = 1 - F(x)$ *is regularly varying at infinity with the exponent* $0 \leqq \alpha \leqq \infty$; *that is*, *condition* (7) *holds. Then the inverse function* $\varphi^{\leftarrow}$ *defined by* (21) *is regularly varying at zero with the exponent* $1/\alpha$, *that is*, *it satisfies the condition*

$$\lim_{u \to +0} \frac{\varphi^{\leftarrow}(tu)}{\varphi^{\leftarrow}(u)} = t^{-1/\alpha} \quad \forall\, t > 0.$$

The following basic formula, known as the Karamata representation, describes the general form of regularly varying functions (see, e.g., [7, section VIII.9] or [15, section 0.4, Corollary to Theorem 0.6]). We shall cite this result in the form convenient for our purposes.

LEMMA 4. *The function* $\varphi^{\leftarrow}$ *is regularly varying at zero* (*with exponent* $1/\alpha$) *if and only if it can be represented in the form*

$$\varphi^{\leftarrow}(u) = u^{-1/\alpha} c(u) \exp\left(\int_u^1 v^{-1} \gamma(v) dv \right), \tag{23}$$

where

$$\lim_{u \to +0} c(u) = c \quad (0 < c < \infty), \qquad \lim_{u \to +0} \gamma(u) = 0.$$

Actually, we will need the following consequence from the Karamata representation, which is a modification of a lemma in [7, section VIII.8] to the case of the ratio of two values of the function (cf. [15, section 0.4, Proposition 0.8 (ii)]).

LEMMA 5. *Under the conditions of Lemma* 4, *for any* $\lambda > 0$ *there exists such* $\delta > 0$ *that for any* $0 < u_1 < u_2 < \delta$ *the following inequalities hold*:

$$\frac{1}{1+\lambda} \left(\frac{u_1}{u_2} \right)^{1/\alpha + \lambda} \leqq \frac{\varphi^{\leftarrow}(u_2)}{\varphi^{\leftarrow}(u_1)} \leqq (1+\lambda) \left(\frac{u_1}{u_2} \right)^{1/\alpha - \lambda}. \tag{24}$$

Proof. Given $\lambda > 0$, by Lemma 4 we can find such $\delta > 0$ that, whenever $0 < u < \delta$,

$$\frac{2}{2+\lambda} = 1 - \frac{\lambda}{2+\lambda} \leqq \frac{c(u)}{c} \leqq 1 + \frac{\lambda}{2+\lambda} = \frac{2(1+\lambda)}{2+\lambda}$$

and $-\lambda \leqq \gamma(u) \leqq \lambda$. Substituting these inequalities into (23), we obtain

$$\frac{1}{1+\lambda} \left(\frac{u_1}{u_2} \right)^{1/\alpha} \exp\left(-\lambda \int_{u_1}^{u_2} \frac{dv}{v} \right) \leqq \frac{\varphi^{\leftarrow}(u_2)}{\varphi^{\leftarrow}(u_1)} \leqq (1+\lambda) \left(\frac{u_1}{u_2} \right)^{1/\alpha} \exp\left(\lambda \int_{u_1}^{u_2} \frac{dv}{v} \right),$$

which, on evaluation of the integral, yields (24).

We will also require the following general fact about the asymptotic behavior of the maximal order statistic (see [8, section 4.3, Corollary 4.3.1]).

LEMMA 6. *Let* $(X_n,\ n \geqq 1)$ *be independent identically distributed random variables* (*not necessarily positive*), *let* $X_{1,n} = \max_{1 \leqq i \leqq n} X_i$, *and let* $(r_n,\ n \geqq 1)$ *be a nondecreasing sequence of numbers. Then*

$$\mathbf{P}\{X_{1,n} \geqq r_n \text{ i.o.}\} = 0 \quad \textit{or } 1$$

according to whether

$$\sum_{n=1}^{\infty} \mathbf{P}\{X_n \geqq r_n\} < \infty \quad or \; = \infty.$$

Remark 5. We will need only the first part of this lemma.

5. Proofs of the theorems.

Proof of Theorem 1. According to (6), the greatest atom $\mu_{1,n}$ is represented as

$$\mu_{1,n} = \frac{X_{1,n}}{S_n} = \frac{X_{1,n} n^{-1}}{S_n n^{-1}}. \tag{25}$$

By the strong law of large numbers, the denominator on the right-hand side of (25) a.s. tends to the expectation $0 < \mathbf{E}[X_1] < \infty$ as $n \to \infty$. Let us show that $X_{1,n} n^{-1} \to 0$ with probability 1. For this, it suffices to verify that for each $\varepsilon > 0$

$$\mathbf{P}\{X_{1,n} > n\varepsilon \text{ i.o.}\} = 0. \tag{26}$$

By Lemma 6, the condition (26) holds if and only if

$$\sum_{n=1}^{\infty} \mathbf{P}\{X_1 > n\varepsilon\} < \infty. \tag{27}$$

In our case, the convergence of the series (27) follows from the finiteness of the expectation $\mathbf{E}[X_1]$. Indeed, integrating by parts we have

$$\begin{aligned} \mathbf{E}[X_1] &= \int_0^{\infty} x dF(x) = -\int_0^{\infty} x d\varphi(x) = \int_0^{\infty} \varphi(x) dx \\ &= \sum_{n=1}^{\infty} \int_{(n-1)\varepsilon}^{n\varepsilon} \varphi(x) dx \geqq \sum_{n=1}^{\infty} \varepsilon \varphi(n\varepsilon) = \varepsilon \sum_{n=1}^{\infty} \mathbf{P}\{X_1 > n\varepsilon\}. \end{aligned} \tag{28}$$

Note that in the outcome of the integration by parts in (28), the out-of-integral term vanishes thanks to $\mathbf{E}[X_1] < \infty$:

$$b\varphi(b) \leqq \int_b^{\infty} x d\varphi(x) = o(1) \quad \text{as } b \to +\infty.$$

Thus, $\mu_{1,n} \to 0$ a.s. and we are done.

Proof of Theorem 2. Part (i) is an immediate corollary to a general result by Kesten [12] stating that, with probability 1,

$$\limsup_{n\to\infty} \frac{|X_n|}{|S_{n-1}|} = \infty$$

whenever $(X_n)_{n\geqq 1}$ (not necessarily positive) are independent and identically distributed with $\mathbf{E}[|X_1|] = \infty$. Indeed, for positive summands this statement amounts to

$$\limsup_{n\to\infty} \frac{X_n}{S_n} = 1,$$

whence (8) obviously follows, with the role of $X_{1,n}$ played by the X_n.

For the proof of part (ii), we shall require a lemma on the a.s.-behavior of the uniform order statistics $U_{1,n} < \cdots < U_{n,n}$ (with probability 1, all the inequalities are strict). Let us partition the interval $(0,1]$ by the points $t_i = i/n$ into the "boxes" $\Delta_{i,n} = (t_{i-1}, t_i]$, $i = 1, \dots, n$, and set

$$\nu_{i,n} := \sum_{j=1}^{n} I\{U_j \in \Delta_{i,n}\}, \quad i = 1, \dots, n, \tag{29}$$

(here and below $I(A)$ stands for the indicator of event A). That is to say, $\nu_{i,n}$ is the number of random points $U_1, \dots, U_n$ falling into the box $\Delta_{i,n}$. In the following lemma we shall use the notation $\log_{(2)} := \log\log$, $\log_{(3)} := \log\log\log$, etc., where logarithms are taken to base 2.

LEMMA 7. *Define the events*

$$A_n := \{\nu_{1,n} = 0\} = \left\{U_{1,n} > n^{-1}\right\}, \tag{30}$$

$$B_n := \{\nu_{2,n} \geqq r_n\}, \tag{31}$$

where

$$r_n := \left[\frac{\log_{(3)} n}{\log_{(4)} n}\right], \tag{32}$$

with $[\,\cdot\,]$ *denoting the "integer part." Then*

$$\mathbf{P}(A_n \cap B_n \text{ i.o.}) = 1.$$

Proof of Lemma 7. Wishing to apply the Borel–Cantelli lemma, we need to pass from the events $A_n \cap B_n$ to suitably constructed independent events. Consider the subsequence $n_k = 2^{2^k}$; obviously, it suffices to prove the statement of the lemma for such n. Observe that

$$\mathbf{P}\{U_{1,n_{k-1}} \leqq n_k^{-1} \text{ i.o.}\} = 0. \tag{33}$$

Indeed, note that $n_k = n_{k-1}^2$ and that

$$\mathbf{P}\{U_{1,n} \leqq n^{-2} \text{ i.o.}\} = 0.$$

The latter follows from Lemma 6 as applied to random variables $1/U_i$, because

$$\sum_n \mathbf{P}\left\{\frac{1}{U_1 \geqq n^2}\right\} = \sum_n \mathbf{P}\left\{U_1 \leqq \frac{1}{n^2}\right\} = \sum_n n^{-2} < \infty.$$

On the other hand, in view of (29)

$$\nu_{2,n_k} \geqq \widetilde{\nu}_{2,n_k} := \sum_{j=n_{k-1}+1}^{n_k} I\left\{U_j \in \Delta_{2,n_k}\right\}. \tag{34}$$

By combining (33) and (34), it is seen that our lemma will be proved once we show that

$$\mathbf{P}(\widetilde{A}_k \cap \widetilde{B}_k \text{ i.o.}) = 1$$

with $\widetilde{A}_k$, $\widetilde{B}_k$ defined as

$$\widetilde{A}_k := \left\{\min_{n_{k-1}+1 \leqq j \leqq n_k} U_j > n_k^{-1}\right\},$$

$$\widetilde{B}_k := \{\widetilde{\nu}_{2,n_k} \geqq r_{n_k}\}$$

(cf. (30), (31)). However, the events $\widetilde{A}_k \cap \widetilde{B}_k$ belong to the σ-algebras generated by the random variables $\{U_j,\ n_{k-1}+1 \leqq j \leqq n_k\}$, respectively, and hence they are independent, as desired. By the Borel–Cantelli lemma, it remains to verify that the series $\sum_k \mathbf{P}(\widetilde{A}_k \cap \widetilde{B}_k)$ is divergent.

Let us notice that, given the condition $\widetilde{A}_k$, the conditional distribution of each random variable U_j, $n_{k-1}+1 \leqq j \leqq n_k$, is uniform in the interval $(n_k^{-1}, 1]$. Indeed, for $n_k^{-1} < x \leqq 1$ we have

$$\mathbf{P}\{U_j \leqq x \mid \widetilde{A}_k\} = \mathbf{P}\{U_j \leqq x \mid U_j > n_k^{-1}\} = \frac{\mathbf{P}\{n_k^{-1} < U_j \leqq x\}}{\mathbf{P}\{U_j > n_k^{-1}\}} = \frac{x - n_k^{-1}}{1 - n_k^{-1}}\,.$$

Similar computation shows that these random variables are conditionally independent.

Therefore,

$$\mathbf{P}(\widetilde{A}_k \cap \widetilde{B}_k) = \mathbf{P}(\widetilde{A}_k)\cdot \mathbf{P}(\widetilde{B}_k \mid \widetilde{A}_k) = \left(1 - \frac{1}{n_k}\right)^{\widetilde{n}_k} \sum_{i=r_{n_k}}^{\widetilde{n}_k} \binom{\widetilde{n}_k}{i} p_k^i (1-p_k)^{\widetilde{n}_k - i}, \tag{35}$$

where

$$\widetilde{n}_k := n_k - n_{k-1} = n_k - \sqrt{n_k}, \tag{36}$$

$$p_k := \mathbf{P}\{n_k^{-1} < U_1 \leqq 2n_k^{-1} \mid U_1 > n_k^{-1}\} = \frac{n_k^{-1}}{1 - n_k^{-1}} = \frac{1}{n_k - 1}\,, \tag{37}$$

and $\binom{n}{i} = n!/(i!(n-i)!)$ are the binomial coefficients.

To make the formulas less cumbersome, let us for a moment omit the subindices and write n, $\widetilde{n}$, p, and r in place of n_k, $\widetilde{n}_k$, p_k, and r_{n_k}, respectively. Hence, (36) and (37) are rewritten as $\widetilde{n} = n - \sqrt{n}$ and $p = 1/(n-1)$. We also note from (32) that

$$r = \left[\frac{\log_{(3)} n}{\log_{(4)} n}\right] = o(\sqrt{n}).$$

Then, using Stirling's formula, the right-hand side of (35) can be estimated from below by

$$\begin{aligned}\left(1 - \frac{1}{n}\right)^n \binom{\widetilde{n}}{i} p^r (1-p)^n &\sim \text{const.}\cdot \widetilde{n}^{\widetilde{n}+1/2} r^{-r-1/2} (\widetilde{n} - r)^{-\widetilde{n}+r-1/2} p^r \\ &\geqq \text{const.}\cdot r^{-r-1/2} (\widetilde{n}p)^r \sim \text{const.}\cdot r^{-r-1/2} \\ &= \text{const.}\cdot 2^{-(r+1/2)\log r} \geqq \text{const.}\cdot 2^{-\log_{(3)} n - 1/2 \log_{(4)} n}.\end{aligned} \tag{38}$$

Recalling that $n = n_k = 2^{2^k}$, the right-hand side of (38) amounts to

$$\text{const.}\cdot 2^{-\log k - (1/2)\log_{(2)} k} = \frac{\text{const.}}{k\sqrt{\log k}},$$

which is the term of a divergent series, as required.

It is now easy to complete the proof of part (ii). Let us take a (random) subsequence n'' of the numbers n provided by Lemma 7, for which, with probability 1, $U_{1,n} > 1/n$ and $\nu_{2,n} \geqq r_n \to \infty$. Using Lemma 1, we can write

$$\mu_{1,n} = \frac{X_{1,n}}{S_n} = \left(\sum_{j=1}^{n} \frac{X_{j,n}}{X_{1,n}}\right)^{-1} = \left(\sum_{j=1}^{n} \frac{\varphi^{\leftarrow}(U_{j,n})}{\varphi^{\leftarrow}(U_{1,n})}\right)^{-1} \leqq \left(\sum_{j=1}^{\nu_{2,n}} \frac{\varphi^{\leftarrow}(U_{j,n})}{\varphi^{\leftarrow}(U_{1,n})}\right)^{-1}. \tag{39}$$

Note that

$$U_{j,n} \leqq \frac{2}{n} \leqq 2U_{1,n}, \tag{40}$$

because both $U_{1,n}$ and $U_{j,n}$ are contained in Δ_2. As the function $\varphi^{\leftarrow}$ is nonincreasing, we can insert the inequality (40) into (39) to obtain

$$\mu_{1,n} \leqq \frac{1}{\nu_{2,n}} \cdot \frac{\varphi^{\leftarrow}(U_{1,n})}{\varphi^{\leftarrow}(2U_{1,n})}.$$

Finally, since $U_{1,n} \to 0$ a.s., Lemma 3 is applied to yield

$$\mu_{1,n} \leqq \frac{1}{\nu_{2,n}} \cdot \frac{\varphi^{\leftarrow}(U_{1,n})}{\varphi^{\leftarrow}(2U_{1,n})} \sim \frac{2^{1/\alpha}}{\nu_{2,n}} \longrightarrow 0,$$

and the theorem is proved.

Proof of Theorem 3. We have to verify that the expectation of the "tail"

$$0 \leqq 1 - \sigma_{k,n} = \sum_{i>k} \mu_{i,n} = \frac{1}{S_n} \sum_{i>k} X_{i,n}$$

vanishes as n and, afterward, k go to infinity. Consider the truncated random variables

$$X_i^{(b)} = X_i \cdot I\{X_i \leqq b\}, \tag{41}$$

where $b > 0$ will be specified later, and split the set of indices $\{i:\ k < i \leqq n\}$ into two parts:

$$\mathcal{I}_1 := \{i > k\colon\ X_{i,n} \leqq b\}, \tag{42}$$

$$\mathcal{I}_2 := \{i > k\colon\ X_{i,n} > b\}. \tag{43}$$

Note that

$$\mathbf{P}\{X_{1,n} \leqq b \text{ i.o.}\} = 0,$$

because $X_{1,n} = \varphi^{\leftarrow}(U_{1,n}) \longrightarrow +\infty$ a.s. Therefore, without loss of generality one may assume that $1 \in \mathcal{I}_2$.

Correspondingly, we can write

$$\begin{aligned} \sum_{i>k} \mu_{i,n} &= \sum_{i \in \mathcal{I}_1} \mu_{i,n} + \sum_{i \in \mathcal{I}_2} \mu_{i,n} \leqq \frac{1}{S_n} \sum_{i=1}^{n} X_i^{(b)} + \sum_{i \in \mathcal{I}_2} \frac{X_{i,n}}{X_{1,n}} \\ &= \frac{S_n^{(b)} n^{-1}}{S_n n^{-1}} + \sum_{i \in \mathcal{I}_2} \frac{\varphi^{\leftarrow}(U_{i,n})}{\varphi^{\leftarrow}(U_{1,n})}, \end{aligned} \tag{44}$$

where

$$S_n^{(b)} = \sum_{i=1}^{n} X_i^{(b)}.$$

We remark that for large n, the first term on the right-hand side of (44) is small with probability 1 (and also in mean, as it is bounded). Indeed, since the expectation of $X_i^{(b)}$ is finite, by the strong law of large numbers with probability 1

$$\frac{S_n^{(b)}}{n} \longrightarrow \mathbf{E}\left[X_1^{(b)}\right] < \infty,$$

whereas S_n/n, thanks to positivity of the summands, a.s. goes to infinity:

$$\liminf_{n\to\infty} \frac{S_n}{n} \geqq \lim_{n\to\infty} \frac{S_n^{(b)}}{n} = \mathbf{E}[X_1^{(b)}] \uparrow +\infty \quad \text{as} \quad b \uparrow +\infty.$$

Turning to the second term in (44) where $U_{i,n}$ are supposed to be small, we wish to make use of a regular variation of the function $\varphi^{\leftarrow}$ (see Lemma 3) and, roughly speaking, to replace the ratio $\varphi^{\leftarrow}(U_{i,n})/\varphi^{\leftarrow}(U_{1,n})$ by $(U_{i,n}/U_{1,n})^{-1/\alpha}$. More precisely, pick a $\lambda > 0$ such that $\lambda < 1/\alpha - 1$ and define the number ρ as

$$\rho := \frac{1}{\alpha} - \lambda > 1. \tag{45}$$

Then, by Lemma 5, there exists a $\delta > 0$ such that for $0 < u_1 < u_2 < \delta$

$$\frac{\varphi^{\leftarrow}(u_2)}{\varphi^{\leftarrow}(u_1)} \leqq (1+\lambda) \left(\frac{u_1}{u_2}\right)^{1/\alpha-\lambda} = (1+\lambda) \left(\frac{u_1}{u_2}\right)^{\rho}. \tag{46}$$

Finally, let us put

$$b := \varphi^{\leftarrow}(\delta), \tag{47}$$

thus choosing the truncation parameter used in (41). Then for each $i \in \mathcal{I}_2$ we have that $0 < U_{i,n} < \delta$. Indeed, assume to the contrary that $U_{i,n} \geqq \delta$; then by monotonicity of the function $\varphi^{\leftarrow}$ we would get

$$X_{i,n} = \varphi^{\leftarrow}(U_{i,n}) \leqq \varphi^{\leftarrow}(\delta) = b,$$

which contradicts definition (43) of the set $\mathcal{I}_2$. Thus, we may apply inequality (46) to the pair $U_{1,n}$, $U_{i,n}$ to obtain

$$\mathbf{E}\left[\sum_{i\in\mathcal{I}_2} \frac{\varphi^{\leftarrow}(U_{i,n})}{\varphi^{\leftarrow}(U_{1,n})}\right] \leqq \mathbf{E}\left[\sum_{i\in\mathcal{I}_2} (1+\lambda) \left(\frac{U_{1,n}}{U_{i,n}}\right)^{\rho}\right] \leqq (1+\lambda) \sum_{i=k+1}^{n} \mathbf{E}\left[\left(\frac{U_{1,n}}{U_{i,n}}\right)^{\rho}\right]. \tag{48}$$

By Lemma 2, we can rewrite the right-hand side of (48) (omitting the constant factor) as

$$\begin{aligned} \sum_{i=k+1}^{n} \mathbf{E}\left[\left(\frac{T_1}{T_i}\right)^{\rho}\right] &\leqq \sum_{i=k+1}^{n} \mathbf{E}\left[T_1^{\rho} \cdot (T_i - T_1)^{-\rho}\right] \\ &\leqq \sum_{i=k+1}^{n} \mathbf{E}\left[T_1^{\rho}\right] \cdot \mathbf{E}\left[(T_i - T_1)^{-\rho}\right], \end{aligned} \tag{49}$$

because the random variables $T_1 = Y_1$ and $T_i - T_1 = Y_2 + \cdots + Y_i$ are independent of each other. Using that the random variables T_i have the gamma distribution (see, e.g., [7, section III.3]) with the density

$$f_i(x) = \frac{1}{\Gamma(i)} x^{i-1} e^{-x}, \quad x > 0,$$

the expectations on the right-hand side of (49) can be easily computed:

$$\mathbf{E}\left[T_1^{\rho}\right] = \int_0^{\infty} x^{\rho} f_1(x) dx = \int_0^{\infty} x^{\rho} e^{-x} dx = \Gamma(\rho+1),$$

$$\mathbf{E}\left[(T_i - T_1)^{-\rho}\right] = \int_0^{\infty} x^{-\rho} f_{i-1}(x)\, dx = \frac{1}{\Gamma(i-1)} \int_0^{\infty} x^{i-\rho-2} e^{-x} dx = \frac{\Gamma(i-\rho-1)}{\Gamma(i-1)}$$

(we assume that $i > \rho + 1$). Substituting these expressions into (49) and returning to the estimate (48), we obtain

$$\mathbf{E}\left[\sum_{i\in\mathcal{I}_2} \mu_{i,n}\right] \leqq \text{const.} \cdot \sum_{i=k+1}^{\infty} \mathbf{E}\left[\frac{\Gamma(i-\rho-1)}{\Gamma(i-1)}\right]. \tag{50}$$

By means of Stirling's formula for the gamma function

$$\Gamma(s+1) \sim \sqrt{2\pi} s^{s+\frac{1}{2}} e^{-s} \quad \text{as } s \to \infty,$$

it is easy to check that

$$\frac{\Gamma(i-\rho-1)}{\Gamma(i-1)} \sim i^{-\rho} \quad \text{as } i \to \infty,$$

and therefore the series in (50) is convergent (recall that by condition (45), $\rho > 1$). Hence, for k large enough the right-hand side of (50) becomes arbitrarily small, which completes the proof.

Proof of Theorem 4. This follows immediately from the proof of Theorem 3 since the right-hand side of the estimate (50) is independent of n.

Proof of Theorem 5. We follow the strategy of the proof of Theorem 3. Given $\varepsilon > 0$, consider the probability

$$\mathbf{P}\{\mu_{1,n} > \varepsilon\} = \mathbf{P}\left\{\frac{X_{1,n}}{S_n} > \varepsilon\right\} = \mathbf{P}\left\{\sum_{i=1}^{n} \frac{X_{i,n}}{X_{1,n}} < \frac{1}{\varepsilon}\right\}. \tag{51}$$

Let us fix an arbitrary number $\lambda > 0$. Applying Lemma 5 (with $\alpha = 1$), we find a $\delta > 0$ such that for any $0 < u_1 < u_2 < \delta$

$$\frac{\varphi^{\leftarrow}(u_2)}{\varphi^{\leftarrow}(u_1)} \geqq \frac{1}{1+\lambda}\left(\frac{u_1}{u_2}\right)^{1+\lambda}. \tag{52}$$

Define the index sets

$$\mathcal{I}_1 := \{i\colon U_{i,n} \geqq \delta\}, \qquad \mathcal{I}_2 := \{i\colon U_{i,n} < \delta\}.$$

Since $U_{1,n} \to 0$ with probability 1, we may, and will, assume that $U_{1,n} < \delta$, that is, $1 \in \mathcal{I}_2$.

Retaining in the sum on the right-hand side of (51) the summands with $i \in \mathcal{I}_2$ only and applying the inequality (52) to the pair $U_{1,n}$, $U_{i,n}$, we get

$$\limsup_{n\to\infty} \mathbf{P}\{\mu_{1,n} > \varepsilon\} \leqq \limsup_{n\to\infty} \mathbf{P}\left\{\sum_{i\in\mathcal{I}_2}\left(\frac{U_{1,n}}{U_{i,n}}\right)^{1+\lambda} < \frac{1+\lambda}{\varepsilon}\right\}. \tag{53}$$

We now show that this result can be extended to the case of the complete sum, that is, over all $i = 1, \dots, n$. Set

$$W_i := \frac{1}{U_i} \cdot I\{U_i \geqq \delta\}, \quad i = 1, \dots, n.$$

Then, as $n \to \infty$, we have that

$$\sum_{i\in\mathcal{I}_1}\left(\frac{U_{1,n}}{U_{i,n}}\right)^{1+\lambda} = nU_{1,n}^{1+\lambda} \cdot \frac{1}{n}\sum_{i=1}^{n} W_i^{1+\lambda} \longrightarrow 0$$

in probability. Indeed, by the law of large numbers

$$\frac{1}{n}\sum_{i=1}^{n} W_i^{1+\lambda} \longrightarrow \mathbf{E}[W_1^{1+\lambda}] < \infty,$$

whereas $nU_{1,n}^{1+\lambda} \longrightarrow 0$, as can be easily verified by representing $U_{1,n}$ as T_1/T_{n+1} (Lemma 2) and applying the law of large numbers:

$$n\left(\frac{T_1}{T_{n+1}}\right)^{1+\lambda} = \frac{T_1^{1+\lambda}}{n^{\lambda}}\cdot\left(\frac{T_{n+1}}{n}\right)^{-1-\lambda} \longrightarrow 0.$$

Hence, increasing the constant in the right-hand side inequality in (53), we can rewrite (53) as

$$\begin{aligned}\limsup_{n\to\infty} \mathbf{P}\{\mu_{1,n} > \varepsilon\} &\leqq \limsup_{n\to\infty} \mathbf{P}\left\{\sum_{i=1}^{n}\left(\frac{U_{1,n}}{U_{i,n}}\right)^{1+\lambda} < \frac{1+2\lambda}{\varepsilon}\right\}\\ &= \limsup_{n\to\infty} \mathbf{P}\left\{\sum_{i=1}^{n}\left(\frac{T_1}{T_i}\right)^{1+\lambda} < \frac{1+2\lambda}{\varepsilon}\right\},\end{aligned} \tag{54}$$

where we once more used Lemma 2. Let us now notice that the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{T_n}\right)^{1+\lambda}$$

is a.s.-convergent because, by the strong law of large numbers,

$$\frac{T_n}{n} = \frac{Y_1+\cdots+Y_n}{n} \longrightarrow \mathbf{E}[Y_1] = \int_0^{\infty} xe^{-x}\,dx = 1. \tag{55}$$

Hence, estimate (54) can be rewritten as

$$\limsup_{n\to\infty} \mathbf{P}\{\mu_{1,n} > \varepsilon\} \leqq \mathbf{P}\left\{Y_1^{1+\lambda}\sum_{n=1}^{\infty}\left(\frac{1}{T_n}\right)^{1+\lambda} \leqq \frac{1+2\lambda}{\varepsilon}\right\}. \tag{56}$$

Moreover, (55) implies that, with probability 1, there exists such a (random) number n_0 that $T_n \leqq 2n$ for all $n \geqq n_0$. It then follows that the probability on the right-hand side of (56) is no greater than

$$\mathbf{P}\left\{Y_1^{1+\lambda}\sum_{n=n_0}^{\infty}\left(\frac{1}{T_n}\right)^{1+\lambda} \leqq \frac{1+2\lambda}{\varepsilon}\right\} \leqq \mathbf{P}\left\{Y_1^{1+\lambda}\sum_{n=n_0}^{\infty}\left(\frac{1}{2n}\right)^{1+\lambda} \leqq \frac{1+2\lambda}{\varepsilon}\right\}. \tag{57}$$

It is easy to see that

$$\sum_{n=n_0}^{\infty}\frac{1}{n^{1+\lambda}} \geqq \int_{n_0}^{\infty}\frac{1}{x^{1+\lambda}}\,dx = \frac{1}{\lambda n_0^{\lambda}}. \tag{58}$$

Therefore, from (57) and (58) we finally get

$$\limsup_{n\to\infty} \mathbf{P}\{\mu_{1,n} > \varepsilon\} \leqq \mathbf{P}\bigl\{Y_1^{1+\lambda} \leqq (1+2\lambda)\,\varepsilon^{-1}\lambda n_0^{\lambda}2^{1+\lambda}\bigr\}. \tag{59}$$

Recall that $\lambda > 0$ is arbitrary and note that n_0, by its choice, does not depend on λ. Passing to the limit in (59) as $\lambda \to +0$, we obtain

$$\limsup_{n\to\infty} \mathbf{P}\{\mu_{1,n} > \varepsilon\} \leqq \mathbf{P}\{Y_1 \leqq 0\} = 0,$$

as required.

Proof of Theorem 6. For p being an integer, the statements (10) and (11) are contained in [5] ($p = 1$) and [2] ($p \geqq 1$). Extension to arbitrary $p > 0$ is straightforward, e.g., by Lyapunov's inequality. It remains to show that (11) implies weak localization. Indeed, from (10) with $p = 1$ we have $\lim_{n\to\infty} \mathbf{E}[\mu_{1,n}] = 1$, whence in view of the inequalities $\mu_{1,n} \leqq \sigma_{k,n} \leqq 1$, condition (L2) follows.

Proof of Theorem 7. The statements (14)–(16) and (20) are adapted from [14, Theorems 2, 3]). The statement (17) follows from Theorem 2 (i) which can be applied because for $k^* < \infty$ one has $\mathbf{E}[X_1] = \infty$. Indeed, if $\mathbf{E}[X_1] < \infty$, then by Theorem 1, $\lim_{n\to\infty} \mu_{1,n} = 0$ a.s., which contradicts (16).

To check (18), note that

$$k^* \mu_{k^*,n} \leqq \sigma_{k^*,n} \leqq (k^* - 1)\, \mu_{1,n} + \mu_{k^*,n},$$

whence

$$\sigma_{k^*,n} - (k^* - 1)\, \mu_{1,n} \leqq \mu_{k^*,n} \leqq \frac{1}{k^*} \sigma_{k^*,n}. \tag{60}$$

Combined with (15) this yields

$$\limsup_{n\to\infty} \mu_{k^*,n} \leqq \frac{1}{k^*}\,. \tag{61}$$

On the other hand, by (16), for any $\varepsilon > 0$ with probability 1 we have that

$$\mu_{1,n} < \frac{1}{k^*} + \varepsilon \quad \text{i.o.}$$

and hence from (60)

$$\mu_{k^*,n} \geqq \sigma_{k^*,n} - 1 + \frac{1}{k^*} - (k^* - 1)\, \varepsilon \quad \text{i.o.}$$

Keeping in mind (15), let $n \to \infty$ and $\varepsilon \to 0$ to obtain

$$\limsup_{n\to\infty} \mu_{k^*,n} \geqq \frac{1}{k^*}\,. \tag{62}$$

As a result, the inequalities (61) and (62) yield (18).

Next, statement (19) immediately follows from the identity $\sigma_{k^*-1,n} = \sigma_{k^*,n} - \mu_{k^*,n}$ and formulas (15) and (18).

Finally, to show that (20) inhibits condition (L4) of strong localization, it suffices to notice that $\sigma_{k,n} \leqq k\mu_{1,n}$, and so (20) implies that $\liminf_{n\to\infty} \sigma_{k,n} = 0$ with probability 1.

6. Appendix.

Proof of Proposition 1. We first show that (L1) $\Leftrightarrow$ (L2). Let us fix $k \geqq 1$ and $\lambda > 0$. Using (2), (5), and (3), we have

$$m_\lambda^{(n)} \leqq \sum_{i=1}^{k} (\mu_{i,n})^{1+\lambda} + (\mu_{k,n})^\lambda \cdot \sum_{i>k} \mu_{i,n} \leqq \sum_{i=1}^{k} \mu_{i,n} + (\mu_{k,n})^\lambda = \sigma_{k,n} + (\mu_{k,n})^\lambda. \tag{63}$$

Note that

$$1 = \sum_{i} \mu_{i,n} \geqq \sum_{i=1}^{k} \mu_{i,n} \geqq k\mu_{k,n},$$

whence

$$\mu_{k,n} \leqq k^{-1}. \tag{64}$$

Substituting (64) into (63) and taking the expectation yield

$$\mathbf{E}[m_\lambda^{(n)}] \leqq \mathbf{E}[\sigma_{k,n}] + k^{-\lambda}. \tag{65}$$

Consecutively passing to the limit in (65) as $n \to \infty$, $k \to \infty$, and $\lambda \to +0$, we obtain

$$\liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] \leqq \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}] + k^{-\lambda}, \tag{66}$$

$$\liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] \leqq \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}], \tag{67}$$

and finally,

$$\lim_{\lambda\to+0} \liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] \leqq \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}]. \tag{68}$$

On the other hand, by Hölder's inequality

$$\sigma_{k,n} = \sum_{i=1}^{k} \mu_{i,n} \leqq k^{\lambda/(1+\lambda)} \cdot \left(\sum_{i=1}^{k} (\mu_{i,n})^{1+\lambda} \right)^{1/(1+\lambda)} \leqq k^{\lambda/(1+\lambda)} \cdot \left(m_\lambda^{(n)} \right)^{1/(1+\lambda)},$$

so that

$$m_\lambda^{(n)} \geqq (\sigma_{k,n})^{1+\lambda} k^{-\lambda}. \tag{69}$$

Let us take the expectation and again pass to the limit in (69) as $n \to \infty$, $\lambda \to +0$, and $k \to \infty$ (note that the order of limits is different from that used above in (66)–(68)):

$$\begin{gathered} \liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] \geqq k^{-\lambda} \left(\liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}] \right)^{1+\lambda}, \\ \lim_{\lambda\to+0} \liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] \geqq \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}], \end{gathered} \tag{70}$$

and

$$\lim_{\lambda\to+0} \liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] \geqq \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}]. \tag{71}$$

By combining the inequalities (68) and (71), we deduce that

$$\lim_{\lambda\to+0} \liminf_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] = \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}] \tag{72}$$

and thus (L1) $\Leftrightarrow$ (L2).

Similarly, as for the derivation of (72) one can show that

$$\lim_{\lambda\to+0} \limsup_{n\to\infty} \mathbf{E}[m_\lambda^{(n)}] = \lim_{k\to\infty} \limsup_{n\to\infty} \mathbf{E}[\sigma_{k,n}],$$

so that (D1) $\Leftrightarrow$ (D2).

Further, let us verify the equivalence of (L2) and (L3). Assume that (L3) holds. By Chebyshev's inequality,

$$\mathbf{E}[\sigma_{k,n}] \geqq (1-\varepsilon) \cdot \mathbf{P}\{\sigma_{k,n} > 1-\varepsilon\}, \tag{73}$$

whence

$$\lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}] \geqq (1-\varepsilon) \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{P}\{\sigma_{k,n} > 1-\varepsilon\} = 1-\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\sigma_{k,n} \leqq 1$, (L2) follows.

Conversely, we have (L2). Again applying Chebyshev's inequality (to the random variable $1-\sigma_{k,n}$), we have

$$\mathbf{E}[1-\sigma_{k,n}] \geqq \varepsilon \mathbf{P}\{1-\sigma_{k,n} \geqq \varepsilon\} \tag{74}$$

or

$$\mathbf{E}[\sigma_{k,n}] \leqq 1-\varepsilon+\varepsilon \mathbf{P}\{1-\sigma_{k,n} \geqq \varepsilon\}.$$

Passing to the limit here as $n \to \infty,\ k \to \infty$, and using condition (L2) yield

$$1 \leqq 1-\varepsilon+\varepsilon \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{P}\{\sigma_{k,n} > 1-\varepsilon\},$$

whence (L3) follows.

The equivalence of (D2) and (D3) is established analogously, with the only modification being that in the estimates (73) and (74) one should replace ε by $1-\varepsilon$.

Finally, let us prove the implications (L4) $\Rightarrow$ (L2) and (D4) $\Rightarrow$ (D2). Using that $0 \leqq \sigma_{k,n} \leqq 1$ for all k, n, and twice applying the Fatou lemma (for $n \to \infty$ and $k \to \infty$), we get

$$1 \geqq \lim_{k\to\infty} \liminf_{n\to\infty} \mathbf{E}[\sigma_{k,n}] \geqq \lim_{k\to\infty} \mathbf{E}\left[\liminf_{n\to\infty} \sigma_{k,n}\right] \geqq \mathbf{E}\left[\lim_{k\to\infty} \liminf_{n\to\infty} \sigma_{k,n}\right]. \tag{75}$$

Whenever (L4) is fulfilled, the right-hand side of (75) is equal to 1, so that the inequalities (75) imply (L2). Similarly, using the Fatou lemma with $\limsup$ in place of $\liminf$, we obtain (D4) $\Rightarrow$ (D2).

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