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Asymptotics of branching symmetric random walk on the lattice with a single source

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Abstract. We study the long-time asymptotics of continuous-time branching random walk on $\mathbb{Z}^d$ $(d \geq 1)$ with a single source (i.e., branching site). The random walk is assumed homogeneous, symmetric, irreducible, and having zero mean and finite variance of jumps. We find the limiting extinction probability and the asymptotics of all integer moments for the total population size and for the number of particles at a fixed site. © Académie des Sciences/Elsevier, Paris

Asymptotique des marches aléatoires symétriques avec branchement sur un réseau avec une source unique

Résumé. *Nous étudions l'asymptotique pour les grands temps d'une marche aléatoire à temps continu avec branchement sur $\mathbb{Z}^d$ $(d \geq 1)$, avec une source unique. Les sauts sont supposés homogènes, symétriques, irréductibles, de moyenne zéro et de variance finie. Nous déterminons la probabilité d'extinction ainsi que l'asymptotique de tous les moments de la population totale et du nombre de particules en chaque point du réseau.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Nous étudions l'asymptotique pour les grand temps d'une marche aléatoire (m.a.) avec branchement sur le réseau $\mathbb{Z}^d$ $(d \geq 1)$ avec une *seule* source de branchement. Les propriétés principales de ce modèle sont les suivantes : (i) l'espace des états $(\mathbb{Z}^d)$ est *illimité*, (ii) la dimension d est *arbitraire*, (iii) l'environnement de branchement est *inhomogène*, (iv) les moments d'ordres *arbitraire* sont étudiés (*voir*, par exemple, [8], [1]).

Soit $A = (a(x, x'))_{x,x' \in \mathbb{Z}^d}$ le générateur de la m.a., avec $\sum_{x' \neq x} a(x, x') = -a(x, x) < \infty$, et a tel que $a(x, x') = a(0, x' - x)$ (homogénéité) et $a(x, x') = a(x', x)$ (symétrie). Nous supposons que la m.a. est irréductible (c'est-à-dire que chaque point $x' \in \mathbb{Z}^d$ peut être atteint) et que la moyenne (resp. variance) des sauts est zéro (resp. bornée). Sous ces hypothèses les probabilités de transition, $p(t, x, y)$, se comportent comme const $\cdot\, t^{-d/2}$ pour $t \to \infty$ (*voir* [10]). Le mécanisme de branchement

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Note présentée par Paul MALLIAVIN.

est déterminé par la fonction génératrice $f(u) := \sum_{n=0}^{\infty} b_n u^n$, où $\sum_{n \neq 1} b_n = -b_1 < \infty$. Donc la particule qui part de x dans le temps $\Delta t \to 0$ saute avec probabilité $a(x, x')\Delta t + \mathrm{o}(\Delta t)$ au point $x' \in \mathbb{Z}^d$ $(x' \neq x)$ ou disparaît avec probabilité $\delta_0(x)[b_n \Delta t + \mathrm{o}(\Delta t)]$ en laissant un nombre $n \neq 1$ de descendants. Comme d'habitude, nous supposons que chaque nouvelle particule est gouvernée par la même loi, d'une façon indépendante des autres particules.

Nous désignons par $\mu_t(y)$ le nombre (aléatoire) de particules au point y et au temps t, donc $\mu_t := \sum_{y \in \mathbb{Z}^d} \mu_t(y)$ est la population totale au temps t. Soit $\mathtt{P}_x$ (resp. $\mathtt{E}_x$) la probabilité (resp. l'espérance) sous la condition $\mu_0(\cdot) = \delta_x(\cdot)$. Soit $G_\lambda(x, y) := \int_0^\infty \mathrm{e}^{-\lambda t} p(t, x, y) \, \mathrm{d}t$ la fonction de Green de la m.a.

Nous étudions la probabilité d'extinction $q(t, x) := \mathtt{P}_x\{\mu_t = 0\}$ et les moments $m_k(t, x, y) := \mathtt{E}_x \mu_t^k(y)$ et $m_k(t, x) := \mathtt{E}_x \mu_t^k$ $(k \in \mathbb{N})$. L'équation du type de Skorohod (*voir* [9]) pour les fonctions génératrices $F_s(t, x) := \mathtt{E}_x s^{\mu_t}$ et $F_s(t, x, y) := \mathtt{E}_x s^{\mu_t(y)}$ est de la forme $\partial F_s / \partial t = A F_s + \delta_0(x) f(F_s)$, avec condition initiale $F_s(0, x) \equiv s$ et $F_s(0, x, y) = 1 + \delta_y(x)(s - 1)$, respectivement. L'opérateur A affecte seulement la variable x par $A\psi(x) = \sum_{x'} a(x, x')\psi(x')$ et est défini sur $\ell^p(\mathbb{Z}^d)$, $1 \leq p \leq \infty$ (ce qui suit par le test de Schur [6]).

Faisant tendre s vers $+0$ dans l'équation de $F_s(t, x)$ on obtient que q satisfait à l'équation $\partial q / \partial t = Aq + \delta_0(x) f(q)$, avec condition initiale $q(0, x) \equiv 0$ (Proposition 1).

Les équations pour les moments m_k peuvent être dérivées d'une façon standard par une différentiation appropriée de F_s au point $s = 1$ (*voir* [8], [2]). Soit $\beta_r := f^{(r)}(1)$ (nous supposons $\beta_r < \infty$, $r \in \mathbb{N}$) et $\beta := \beta_1$. La proposition 2 affirme que $m_k(t, x, y)$ et $m_k(t, x)$ satisfont à la chaîne d'équations linéaires :

$$\frac{\partial m_k}{\partial t} = H m_k + \delta_0(x) \sum_{r=2}^{k} \frac{\beta_r}{r!} \sum_{(i_1, \ldots, i_r)} \frac{k!}{i_1! \cdots i_r!} m_{i_1} \cdots m_{i_r}, \tag{1}$$

avec conditions initiales $m_k(0, \cdot, y) = \delta_y(\cdot)$ (resp. $m_k(0, \cdot) \equiv 1$). On a posé $H := A + \beta\delta_0$ et la deuxième somme dans (1) porte sur les r-tuples d'entiers $i_1, \ldots, i_r > 0$, $i_1 + \cdots + i_r = k$. Pour $k = 1$ on a $\partial m_1 / \partial t = H m_1$.

Il est facile de voir que l'opérateur A dans $\ell^2(\mathbb{Z}^d)$ a un spectre continu et $\sigma(A) = [\min_\theta \phi(\theta), 0]$, avec $\phi(\theta) := \sum_x a(x, 0)\mathrm{e}^{i(x, \theta)}$, $\theta \in [-\pi, \pi]^d$, et le spectre essentiel de H coïncide avec $\sigma(A)$. Posons $\beta_{\mathrm{c}} := 1/G_0(0, 0)$. On a alors $\beta_{\mathrm{c}} > 0$ pour $d \geq 3$, $\beta_{\mathrm{c}} = 0$ pour $d = 1, 2$. Donc pour $\beta \geq \beta_{\mathrm{c}}$ l'équation $\beta G_\lambda(0, 0) = 1$ a une unique racine $\lambda_0 = \lambda_0(\beta) \geq 0$ (« paramètre malthusien »). On peut vérifier que λ_0 est une valeur propre de H (dans le cas $\beta = \beta_{\mathrm{c}}$ où $\lambda_0 = 0$, ce qui vaut pour $d \geq 5$).

Nos résultats les plus importants sont les suivants :

THÉORÈME 1. – *Soit* $q(x) := \lim_{t \to \infty} q(t, x)$ *la probabilité d'extinction limite. Si* $d = 1, 2$, *alors* $q(x)$ *ne dépend pas de* x *et coïncide avec la plus petite solution non négative de l'équation* $f(q) = 0$. *Si* $d \geq 3$, $q(0)$ *est la plus petite solution non négative de l'équation* $f(q) = q/G_0(0, 0)$, *et* $q(x) = q(0) G_0(x, 0)/G_0(0, 0)$.

THÉORÈME 2. – *Pour* $t \to \infty$ *on a* $m_k(t, x, y) \sim C_k^{d,\beta}(x, y)\, u_k(t)$, $m_k(t, x) \sim C_k^{d,\beta}(x)\, v_k(t)$, *où toutes les constantes* $C_k^{d,\beta}(x, y)$, $C_k^{d,\beta}(x)$, *dépendent de la dimension d et du paramétre* β, *sont définies récursivement en k, et les fonctions* u_k, v_k *ont la forme :*

(a) $\beta > \beta_{\mathrm{c}}$: $u_k(t) = v_k(t) = \mathrm{e}^{k\lambda_0 t}$;
(b) $\beta = \beta_{\mathrm{c}}$: $d \geq 5$: $u_k(t) = t^{k-1}$, $v_k(t) = t^{2k-1}$;
$d = 4$: $u_k(t) = t^{k-1}(\ln t)^{1-2k}$, $v_k(t) = t^{2k-1}(\ln t)^{1-2k}$;
$d = 3$: $u_k(t) = t^{-1/2}(\ln t)^{k-1}$, $v_k(t) = t^{k-1/2}$;
$d = 1, 2$: $u_k(t) = t^{-d/2}$, $v_k(t) \equiv 1$;
(c) $\beta < \beta_{\mathrm{c}}$: $u_k(t) = t^{-d/2}$, $v_k(t) \equiv 1$.

L'affirmation (a) du théorème 2 est une conséquence du théorème spectral. Les affirmations (b) et (c) sont une conséquence de l'asymptotique des transformées de Laplace $\widehat{m}_k(\lambda, x, y)$ et $\widehat{m}_k(\lambda, x)$ pour $\lambda \to +0$, en utilisant le théorème taubérien de Karamata pour les densités (*voir*, par exemple, [4]).

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1. Introduction

In this Note, we study the long-time asymptotics of continuous-time branching random walk (r.w.) on the integer lattice $\mathbb{Z}^d$ ($d \geq 1$) with a *single* source, that is the site (say, the origin $x = 0$) where the branching takes place. More specifically, we are concerned with the limiting extinction probability (Theorem 1) and with the asymptotics of all the moments (of integer orders) for the total population size as well as for the number of particles at a fixed site (Theorem 2).

Since the seminal paper by Sevastyanov [7], there have been many works treating branching processes with motion of particles (*see*, e.g., [8], Ch. 10, and [1], Part C, for systematic exposition, and [11], Sec. 11, for a survey and further bibliography). The main points that, in conjunction, distinguish the present work are the following (*see* [8], [1]): (i) the phase space where the particles perform their motion, $\mathbb{Z}^d$, is *unbounded*, (ii) its dimension d is *arbitrary*, (iii) the branching medium is spatially *inhomogeneous*, and (iv) the moments of *all* orders are studied.

2. The model

Let $A = (a(x, x'))_{x, x' \in \mathbb{Z}^d}$ be the matrix of the r.w. transition intensities, so that $a(x, x') \geq 0$ for $x \neq x'$, $a(x, x) < 0$, and $\sum_{x' \neq x} a(x, x') = -a(x, x) < \infty$. We suppose that the r.w. is spatially homogeneous ($a(x, x') = a(0, x' - x)$), symmetric ($a(x, x') = a(x', x)$), and irreducible (i.e., every point $x' \in \mathbb{Z}^d$ is reachable. The r.w. is also assumed to have zero mean and finite variance of jumps (which are not necessarily bounded). A familiar representative of this class is the simple r.w. with $a(x, x') = a/(2d)$ if $|x' - x| = 1$ and $a(x, x') = 0$ otherwise.

It is well known that under these conditions, the r.w. transition probability has the asymptotics :

$$p(t, x, y) \sim \gamma_d \cdot t^{-d/2} \qquad \text{as } t \to \infty, \tag{2}$$

where γ_d is a constant depending on the space dimension and the determinant of the r.w. covariance matrix (*see* [10]). Denote by $G_\lambda(x, y) := \int_0^\infty \mathrm{e}^{-\lambda t} p(t, x, y)\, \mathrm{d}t$ the Green function of the r.w. It follows that $G_\lambda(0, 0)|_{\lambda = 0}$ is finite if and only if $d \geq 3$. Therefore, our r.w. is transient if $d \geq 3$ and recurrent otherwise.

The branching mechanism at the source, being independent of the r.w., is governed by the infinitesimal generating function $f(u) := \sum_{n=0}^\infty b_n u^n$, where $b_n \geq 0$ for $n \neq 1$, $b_1 < 0$, and $\sum_{n \neq 1} b_n = -b_1 < \infty$.

Thus, the particle's evolution at the microscopic time scale proceeds as follows: found at point x, the particle during time $\Delta t \to 0$ with probability $a(x, x')\Delta t + \mathrm{o}(\Delta t)$ can jump to a point $x' \in \mathbb{Z}^d$ ($x' \neq x$), with probability $\delta_0(x)[b_n \Delta t + \mathrm{o}(\Delta t)]$ it dies leaving $n \neq 1$ descendants, or otherwise, with probability $1 - \sum_{x' \neq x} a(x, x')\Delta t - \delta_0(x) \sum_{n \neq 1} b_n \Delta t + \mathrm{o}(\Delta t)$, the particle stays inact at x during the whole time Δt. As usual, it is assumed that each of the new particles evolves according to the same law, independently of the others.

Note that such a process can also be viewed as the age-dependent (Bellman–Harris) branching process (*see* [2]) with the particle's lifetime determined by the successive returns of the r.w. to the origin until eventual branching. However, the standard theory does not seem to apply readily (e.g., for $d \geq 3$ the lifetime distribution is defective, due to the transience of the r.w.).

3. The main equations and results

Denote the (random) number of particles at site y at time t by $\mu_t(y)$, then $\mu_t := \sum_{y\in\mathbb{Z}^d}\mu_t(y)$ is the population size at time t. Let P_x and E_x stand respectively for the probability and the expectation under the condition $\mu_0(\cdot) = \delta_x(\cdot)$.

We will be concerned with the extinction probability $q(t,x) := \mathrm{P}_x\{\mu_t = 0\}$ and the moments $m_k(t,x,y) := \mathrm{E}_x\mu_t^k(y)$ and $m_k(t,x) := \mathrm{E}_x\mu_t^k$ ($k \in \mathbb{N}$). The Skorohod type equation (*see* [9]) for the generating functions $F_s(t,x) := \mathrm{E}_x s^{\mu_t}$ and $F_s(t,x,y) := \mathrm{E}_x s^{\mu_t(y)}$ is of the form:

$$\frac{\partial F_s}{\partial t} = AF_s + \delta_0(x)f(F_s), \tag{3}$$

with the initial condition $F_s(0,x) \equiv s$ and $F_s(0,x,y) = 1 + \delta_y(x)(s-1)$, respectively, where the linear operator A acts with respect to variable x as $A\psi(x) = \sum_{x'} a(x,x')\psi(x')$. (By the known Schur test [6], it can be verified that A is defined on $\ell^p(\mathbb{Z}^d)$ for any $1 \le p \le \infty$.)

Letting $s \to +0$ in equation (3) for $F_s(t,x)$ yields:

Proposition 1. – *The extinction probability $q(t,x)$ is the solution of the Cauchy problem:*

$$\frac{\partial q}{\partial t} = Aq + \delta_0(x)f(q), \qquad q(0,x) \equiv 0. \tag{4}$$

The equations for the moments m_k can be derived in the usual way via an appropriate differentiation of F_s at $s = 1$ (*see* [2]). Suppose that $\beta_r := f^{(r)}(1) < \infty$ for all $r \in \mathbb{N}$, that is, the particle's offspring has finite moments of all orders, and put $\beta := \beta_1$.

Proposition 2. – *The moments $m_k(t,x,y)$ and $m_k(t,x)$ satisfy the chain of evolution equations:*

$$\frac{\partial m_k}{\partial t} = Hm_k + \delta_0(x)g_k(m_1,\ldots,m_{k-1}), \tag{5}$$

with the initial conditions $m_k(0,\cdot,y) = \delta_y(\cdot)$, $m_k(0,\cdot) \equiv 1$, respectively. Here $H := A + \beta\delta_0(x)$ and

$$g_k(m_1,\ldots,m_{k-1}) := \sum_{r=2}^{k}\frac{\beta_r}{r!}\sum_{(i_1,\ldots,i_r)}\frac{k!}{i_1!\cdots i_r!}m_{i_1}\cdots m_{i_r},$$

where the second sum is over the integer r-tuples with $i_1,\ldots,i_r > 0$, $i_1 + \cdots + i_r = k$. For $k = 1$, we have $g_1 \equiv 0$, so that $\partial m_1/\partial t = Hm_1$.

Let us put $\beta_c := 1/G_0(0,0)$. Obviously, $\beta_c = 0$ for $d = 1$, 2, and $\beta_c > 0$ for $d \ge 3$. Hence, for $\beta > \beta_c$ the equation $\beta G_\lambda(0,0) = 1$ with respect to the variable λ has a unique root $\lambda_0 = \lambda_0(\beta) > 0$ (i.e., the Malthusian parameter).

Our main results are the following two theorems.

Theorem 1. – *Let $q(x) := \lim_{t\to\infty} q(t,x)$ be the limiting extinction probability. If $d = 1, 2$, then $q(x)$ does not depend on x and coincides with the least non-negative root of the equation $f(q) = 0$. If $d \ge 3$, then $q(0)$ is the least non-negative root of the equation $f(q) = q/G_0(0,0)$, and $q(x) = q(0)G_0(x,0)/G_0(0,0)$.*

Theorem 2. – *As $t \to \infty$, the moments m_k ($k \in \mathbb{N}$) have the asymptotics:*

$$m_k(t,x,y) \sim C_k^{d,\beta}(x,y)\,u_k(t), \qquad m_k(t,x) \sim C_k^{d,\beta}(x)\,v_k(t), \tag{6}$$

where the constants $C_k^{d,\beta}(x,y)$, $C_k^{d,\beta}(x)$, depending on the dimension d and the parameter β, are defined recursively in k, and the functions u_k, v_k are of the form:

(a) $\beta > \beta_c$: $u_k(t) = v_k(t) = e^{k\lambda_0 t}$;
(b) $\beta = \beta_c$: $d \geq 5$: $u_k(t) = t^{k-1}$, $v_k(t) = t^{2k-1}$;
$d = 4$: $u_k(t) = t^{k-1}(\ln t)^{1-2k}$, $v_k(t) = t^{2k-1}(\ln t)^{1-2k}$;
$d = 3$: $u_k(t) = t^{-1/2}(\ln t)^{k-1}$, $v_k(t) = t^{k-1/2}$;
$d = 1, 2$: $u_k(t) = t^{-d/2}$, $v_k(t) \equiv 1$;
(c) $\beta < \beta_c$: $u_k(t) = t^{-d/2}$, $v_k(t) \equiv 1$.

The spectral analysis immediately reveals the critical point $\beta = \beta_c$. Namely, it is easy to see that the operator A in $\ell^2(\mathbb{Z}^d)$ has continuous spectrum and $\sigma(A) = [\min_\theta \phi(\theta), 0]$, where $\phi(\theta) := \sum_x a(x,0)e^{i(x,\theta)}$, $\theta \in [-\pi,\pi]^d$, and also that the essential spectrum of H coincides with $\sigma(A)$. Furthermore, it can be checked that the Malthusian parameter λ_0, which exists for $\beta \geq \beta_c$, is the largest non-negative eigenvalue of H; to be more precise, in the case $\beta = \beta_c$, where $\lambda_0 = 0$, the latter is valid provided $d \geq 5$. These facts enlighten the statements of Theorem 2, in particular in the supercritical regime, and also in the critical case where there is bifurcation with respect to the dimension.

4. Integral equations

It is convenient to rewrite the Cauchy problems of Propositions 1, 2 in the form of integral equations. This can be done by representing the solution $u(t,x;u_0)$ of a "perturbed" equation of the form $\partial u/\partial t = Tu + g(t,x,u)$ with initial condition $u|_{t=0} = u_0$ as:

$$u(t,x\,;\,u_0) = \underline{u}(t,x;u_0) + \int_0^t u(t-s,x\,;\,g(s,x,u(s,x\,;\,u_0)))\,\mathrm{d}s, \tag{7}$$

where $\underline{u}(t,x\,;\,u_0)$ is the solution of the unperturbed problem (with $g \equiv 0$) (*see* [3], Ch. 2).

In what follows, we use the notation $\psi_1(t) * \psi_2(t) := \int_0^t \psi_1(t-s)\psi_2(s)\,\mathrm{d}s$.

LEMMA 1. – *The extinction probability* $q(t,x)$ *satisfies the equation* $q(t,x) = p(t,x,0) * f(q(t,0))$.

LEMMA 2. – *The function* $m_1(t,x,y)$ *satisfies either of the two renewal equations:*

$$m_1(t,x,y) = p(t,x,y) + \beta p(t,x,0) * m_1(t,0,y), \tag{8}$$
$$m_1(t,x,y) = p(t,x,y) + \beta m_1(t,x,0) * p(t,0,y). \tag{9}$$

Similar equation for $m_1(t,x)$ *reads:*

$$m_1(t,x) = 1 + \beta p(t,x,0) * m_1(t,0). \tag{10}$$

The higher moments $m_k(t,x,y)$ *and* $m_k(t,x)$ *satisfy respectively:*

$$m_k(t,x,y) = m_1(t,x,y) + m_1(t,x,0) * g_k(m_1(t,0,y),\ldots,m_{k-1}(t,0,y)), \tag{11}$$
$$m_k(t,x) = m_1(t,x) + m_1(t,x,0) * g_k(m_1(t,0),\ldots,m_{k-1}(t,0)). \tag{12}$$

Proof of Lemmas. – Application of (7) to (4) with $T = A$ proves Lemma 1. Equation (8) is obtained similarly from (5) with $k = 1$, taking into account that $p(t,x,y)$ is the fundamental solution of $\partial p/\partial t = Ap$. Derivation of (10) is similar. For (9), take $T = A^*$ and apply (7) to the equation $\partial m_1/\partial t = A^* m_1 + \beta\delta_0(y)m_1$, where A^* acts with respect to y. Equations (11) and (12) are derived analogously by taking $T = H$.

5. Sketch of the proofs of the theorems

Theorem 1 can be proved by a direct analysis of the equation in Lemma 1 using (4). Let us now turn to Theorem 2.

(a) For $k = 1$, the statement follows from the spectral representation:

$$m_1(t,x,y) = \int_{\sigma(H)\cap(-\infty,0]} \mathrm{e}^{\lambda t}\mathrm{d}(E_\lambda\delta_y,\delta_x) + \frac{\psi_0(x)\psi_0(y)}{\|\psi_0\|^2}\,\mathrm{e}^{\lambda_0 t}, \qquad (13)$$

where $\{E_\lambda\}$ is the spectral family of the (self-adjoint) operator H, and ψ_0 is the eigenfunction corresponding to the eigenvalue λ_0. For $m_1(t,x)$ which is not in $\ell^2(\mathbb{Z}^d)$, set $m_1(t,x) = 1 + \widetilde{m}(t,x)$ with $\widetilde{m}(t,x) \in \ell^2(\mathbb{Z}^d)$ and proceed as before.

Once the asymptotics (6) is obtained in the case $k = 1$, the asymptotics for the higher moments can be derived by induction from (11), (12), taking into account (2).

(b) We again start with $k = 1$. Using (8), (10), we find the Laplace transforms $\widehat{m}_1(\lambda,x,y)$, $\widehat{m}_1(\lambda,x)$ and study their asymptotics as $\lambda \to +0$. The behaviour of $m_1(t,x,y)$ and $m_1(t,x)$ as $t \to \infty$ can then be restored via Karamata's Tauberian theorem "for densities" (*see*, e.g., [4], Sec. XIII.5). It should be noted that this theorem requires the condition of (ultimate) monotonicity of the originals, which is not difficult to verify for $m_1(t,x)$ (e.g., via the Feynman–Kac representation, *see* [5]) but in general does not hold for $m_1(t,x,y)$. However, for $m_1(t,0,0)$ the monotonicity does take place, as can be seen from the spectral representation (see (13)) using that $\sigma(H) \subset (-\infty,0]$ if $\beta \le \beta_c$. Similarly to the case (a), one can then proceed "by induction" and derive the asymptotics of $m_1(t,x,y)$ and of the higher moments from (8)–(12).

(c) The above method works as well for $m_1(t,x)$. However, for $m_1(t,x,y)$ its Laplace transform has a non-vanishing regular part at point $\lambda = 0$, and hence one has to "singularize" $\widehat{m}_1(\lambda,x,y)$ via an appropriate differentiation with respect to λ. More specifically, take $r = \left[\frac{d}{2}\right]$ and write the identity $t^{r+1}m(t) = (r+1)\int_0^t s^r m(s)\,\mathrm{d}s + \int_0^t s^{r+1}m'(s)\,\mathrm{d}s$, where $m(t) := m_1(t,0,0)$. The Laplace transform of the integrands is identified as $(-1)^r\widehat{m}^{(r)}(\lambda)$ and $(-1)^{r+1}(\lambda\widehat{m}^{(r+1)}(\lambda)+(r+1)\widehat{m}^{(r)}(\lambda))$, respectively. It remains to find the asymptotics of these functions as $\lambda \to +0$ and apply the (integral) Tauberian theorem.

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