Limit Laws for Norms of IID Samples with Weibull Tails

Leonid Bogachev[1,2]

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We are concerned with the limit distribution of l_t-norms $R_N(t) = \|\mathbf{X}_N\|_t$ (of order t) of samples $\mathbf{X}_N = (X_1, \dots, X_N)$ of i.i.d. positive random variables, as $N \to \infty$, $t \to \infty$. The problem was first considered by Schlather[(10)], but the case where $\{X_i\}$ belong to the domain of attraction of Gumbel's double exponential law (in the sense of extreme value theory) has largely remained open (even for an exponential distribution). In this paper, it is assumed that the log-tail distribution function $h(x) = -\log \mathsf{P}\{X_1 > x\}$ is regularly varying at infinity with index $0 < \varrho < \infty$. We proceed from studying the limit distribution of the sums $S_N(t) = \sum_{i=1}^N X_i^t$, which is of interest in its own right. A proper growth scale of N relative to t appears to be of the form $N \sim e^{\alpha t/\varrho}$ $(0 < \alpha < \infty)$. We show that there are two critical points, $\alpha_1 = 1$ and $\alpha_2 = 2$, below which the law of large numbers and the central limit theorem, respectively, break down. For $\alpha < 2$, under a slightly stronger condition of normalized regular variation of h, we prove that the limit laws for $S_N(t)$ are stable, with characteristic exponent $\alpha \in (0, 2)$ and skewness parameter $\beta \equiv 1$. A complete picture of the limit laws for the norms $R_N(t) = S_N(t)^{1/t}$ is then derived. In particular, our results corroborate a conjecture in Ref. 10 regarding the "endpoints" $\alpha \to \infty$, $\alpha \to 0$.

KEY WORDS: Sums of independent random variables; weak limit theorems; central limit theorem; infinitely divisible laws; stable laws; l-norms.

1. INTRODUCTION

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. positive random variables, and consider the l_t-norms (of order $t > 0$) of the random sample $\mathbf{X}_N = (X_1, \dots, X_N)$:

$$R_N(t) := \|\mathbf{X}_N\|_t = \left(\sum_{i=1}^N X_i^t \right)^{1/t}. \tag{1.1}$$

Our goal is to study the limiting distribution of $R_N(t)$ as both N and t tend to infinity.

Clearly, the asymptotic behavior of $R_N(t)$ depends heavily on the relationship between the parameters N and t. If, for instance, one lets N tend to infinity with t fixed or growing slowly enough, then, under appropriate moment conditions, the usual law of large numbers (LLN) and the central limit theorem (CLT) should be valid. In contrast, if the growth rate of N is small enough as compared to t, then the asymptotics of $R_N(t)$ is dominated by the maximum of the sample $X_1, \dots, X_N$. We shall see that when both N and t tend to infinity, a rich intermediate picture emerges made up of various limit regimes.

[1] Department of Statistics, University of Leeds, Leeds LS2 9JT, United Kingdom. E-mail: bogachev@maths.leeds.ac.uk

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One can also anticipate that results will depend on the structure of the upper distribution tail of X. In this paper we will focus on a particular case where X is unbounded above and has the upper tail of Weibull's form described heuristically by

$$\mathsf{P}\{X > x\} \approx \exp(-cx^{\varrho}) \qquad (x \to +\infty), \tag{1.2}$$

where $0 < \varrho < \infty$. More precisely, we assume that the log-tail distribution function $h(x) = -\log \mathsf{P}\{X > x\}$ is regularly varying with index $\varrho \in (0, \infty)$ as $x \to +\infty$. For example, an exponential distribution is contained in this class with $\varrho = 1$.

The above problem is closely related to the limiting behavior of the partial sums

$$S_N(t) = \sum_{i=1}^{N} X_i^t, \tag{1.3}$$

which is of interest in its own right. As we shall see, it is convenient to obtain the limit of the sums $S_N(t)$ first, using well-elaborated classical techniques, and then derive results for the norms $R_N(t) = S_N(t)^{1/t}$ using an elementary "transfer" lemma (see Lemma 9.1 below).

Limits of norms of the form (1.1) were first considered in a recent paper by Schlather[(10)] with the aim to combine CLT with limit theorems in extreme value theory. Qualitatively speaking, Schlather has demonstrated that under a suitable parametrization of the functional relation between the norm order t and the sample size N, there exists a "homotopy" for the limit distributions of $R_N(t)$ extending from the CLT to a limit law for extreme values. The situation where both N and t tend to infinity arises in Ref. 10 in Theorem 2.2, where the random variables X_i are assumed to be bounded above and, in the sense of extreme value theory, in the domain of attraction of the Weibull distribution $\Psi_\alpha(x) = \exp\left(-(-x)^\alpha\right)$ ($\alpha > 0$, $x < 0$). In contrast, in Theorem 2.3 of Ref. 10, where X_i are unbounded and belong to the domain of attraction of the Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha})$ ($\alpha > 0$, $x > 0$), the norm order t is supposed to be fixed.

Let us point out that for random variables X_i with Weibull tails of the form (1.2), the distribution of the maximum of the sample $X_1, \dots, X_N$ can be shown to converge (under a slightly more restrictive assumption of *normalized regular variation*, see below) to Gumbel's double exponential distribution Note that in this case Schlather has obtained a partial result and only for exponential random variables (see Ref. 10, Theorem 2.4, p. 867). However, he has conjectured (see Ref. 10, p. 867) that in a general case of attraction to Λ, a weak limit of the properly centered and normalized $R_N(t)$ does exist; moreover, the endpoints of the parametric family of the limits should be represented by the normal distribution and the Gumbel distribution (heuristically, corresponding to the cases $N \gg t$ and $t \gg N$, respectively).

The results obtained in the present paper do corroborate this conjecture. Moreover, we find explicitly the full spectrum of the limiting laws for the underlying sums $S_N(t)$ (and hence for $R_N(t)$). In particular, we show that non-Gaussian limits of $S_N(t)$ belong to a certain family of stable laws $\mathcal{F}_\alpha$ with characteristic exponent $\alpha \in (0, 2)$ and skewness parameter $\beta \equiv 1$.

Another class of examples within such a setting is provided by Ben Arous, Bogachev and Molchanov[3, 4] who considered the sums (1.3) with exponential terms of the form $X = \exp(\tilde{X})$, where $\tilde{X}$ is either unbounded above and has a Weibull tail similar to (1.2),

$$\mathsf{P}\{\tilde{X} > x\} \approx \exp\left(-cx^{\tilde{\varrho}}\right) \qquad (x \to +\infty),$$

with index $1 < \tilde{\varrho} < \infty$ (*case B*) or is bounded above (say, $\tilde{X} < 0$) and has a Fréchet tail

$$\mathsf{P}\{\tilde{X} > x\} \approx \exp\left(-c\,(-x)^{-\tilde{\varrho}}\right) \qquad (x \to 0-), \tag{1.4}$$

with $0 < \tilde{\varrho} < \infty$ (*case A*). Note that if X satisfies (1.2) then

$$\mathsf{P}\{\tilde{X} > x\} = \mathsf{P}\{\log X > x\} = \mathsf{P}\{X > e^x\} \approx \exp\left(-c\,e^{\varrho x}\right) \qquad (x \to +\infty),$$

which formally corresponds to case B of Refs. 3, 4 with the "limiting" value $\tilde{\varrho} = \infty$.

As shown by Ben Arous *et al.*,[3] under the assumption of normalized regular variation of $h(\cdot)$ the random exponentials $\exp(\tilde{X}_i)$ belong to the domain of attraction of the double exponential distribution Λ (in both cases A and B), and the implications for the l_t-norms $R_N(t)$ are again in line with Schlather's conjecture. Interestingly, the family of the limiting distributions for $S_N(t)$ (and hence for $R_N(t)$) appears to be exactly the same as in the present paper. Such universality is quite remarkable and should be studied in more detail.

In conclusion of this Introduction, let us remark that, along with the *unbounded* case represented by (1.2), it may seem natural to consider its *bounded* counterpart (say, with $X \leqslant 1$), where the upper tail is described (by analogy with (1.2)) as

$$\mathsf{P}\{X > x\} \approx \exp(-c(1-x)^{-\varrho}) \qquad (x \to 1-).$$

However, it is not difficult to see that in fact this amounts to case A of Refs. 3, 4. Indeed, after the transformation $\tilde{X} = \log X$ we would have (cf. (1.4))

$$\mathsf{P}\{\tilde{X} > x\} = \mathsf{P}\{X > e^x\} \approx \exp(-c(1-e^x)^{-\varrho}) \approx \exp(-c(-x)^{-\varrho}) \qquad (x \to 0-).$$

The remainder of the paper is laid out as follows. In Section 2 we state our results about the limit laws for sums $S_N(t)$ (Theorems 2.1, 2.2, 2.4, 2.5 and 2.6) and norms $R_N(t)$ (Theorem 2.7). In Section 3 we specify the regularity assumption on the distribution tail of X and formulate Kasahara's Tauberian theorem. Theorems 2.1 and 2.2 (LLN and CLT above the critical points) are proved in Section 4. In Section 5 we discuss the condition of normalized regular variation. Section 6 deals with asymptotics of truncated moments, and in Section 7 we establish weak convergence of $S_N(t)$ to a stable law (Theorem 2.4). In Section 8 we prove Theorems 2.5 and 2.6 (LLN and CLT at the critical points), and Section 9 contains the proof of Theorem 2.7. Finally, a discussion of our results in the context of Schlather[10] is included in Section 10. For illustration we also provide here more explicit versions of our results in the simple case of the exponential distribution, which is of some historical interest in the light of Ref. 10.

2. STATEMENT OF THE RESULTS

We assume throughout that the log-tail distribution function $h(x) = -\log \mathsf{P}\{X > x\}$ is regularly varying at infinity with index $\varrho \in (0, \infty)$ (we write $h \in R_\varrho$; see Condition 3.1 below). It follows that the moment function

$$m(t) := \mathsf{E}\, X^t \tag{2.1}$$

is well defined for all $t \geqslant 0$. Note that the expected value of the sum $S_N(t)$ is given by

$$\mathsf{E}\, S_N(t) = \sum_{i=1}^{N} \mathsf{E}\, X_i^t = N m(t),$$

suggesting that the moment function $m(t)$ should be relevant to the appropriate scale for the number of terms $N = N(t)$. However, Kasahara's Tauberian theorem (see Ref. 7 and Lemma 3.3 below) shows that $m(t)$ grows like $e^{(t/\varrho)\log t}$, while a suitable rate function should be chosen as $e^{\alpha t/\varrho}$. We will see that the values $\alpha_1 = 1$ and $\alpha_2 = 2$ are critical with respect to this scale, in that the LLN and CLT break down below α_1 and α_2, respectively. Moreover, it will be shown that α plays the role of characteristic exponent in the limit laws.

The first two theorems state that $S_N(t)$ satisfies LLN and CLT in their conventional form, providing that the number of terms N in the sum $S_N(t)$ grows fast enough (roughly speaking, $N \gg e^{t/\varrho}$ and $N \gg e^{2t/\varrho}$, respectively). More precisely, set

$$\alpha := \liminf_{t\to\infty} \frac{\varrho \log N}{t}\,. \tag{2.2}$$

Theorem 2.1. Suppose that $h \in R_\varrho$ and $\alpha > 1$. Set

$$S_N^*(t) := \frac{S_N(t)}{\mathsf{E}\, S_N(t)} = \frac{1}{N m(t)} \sum_{i=1}^{N} X_i^t.$$

Then

$$S_N^*(t) \xrightarrow{\ p\ } 1 \qquad (t \to \infty).$$

Theorem 2.2. Suppose that $h \in R_\varrho$ and $\alpha > 2$. Then

$$\frac{S_N(t) - \mathsf{E}\, S_N(t)}{\sqrt{\mathsf{Var}\, S_N(t)}} \xrightarrow{\ d\ } \mathcal{N}(0,1) \qquad (t \to \infty),$$

where $\mathcal{N}(0,1)$ is the standard normal distribution.

Below the critical points, the rate of growth of $N = N(t)$ must be specified more accurately. Namely, we will require the following

Condition 2.3 (Scaling Assumption). The number N of terms in the sum $S_N(t)$ satisfies the condition

$$\lim_{t\to\infty} N(t)\, e^{-\alpha t/\varrho} = 1, \tag{2.3}$$

where $\alpha > 0$ is a parameter.

We also need to impose a slightly stronger condition on regularity of the log-tail distribution function h — that of *normalized regular variation,* $h \in NR_\varrho$ (see Condition 5.1 below). This property will be discussed in detail in Section 5; in particular, it ensures that h is ultimately strictly increasing. Let $\eta_1(t)$ be the (unique) solution of the equation

$$\varrho h(\eta_1(t)) = \alpha t. \tag{2.4}$$

Theorem 2.4. Assume that $h \in NR_\varrho$ and the scaling condition (2.3) is fulfilled. Suppose that $0 < \alpha < 2$ and set

$$B(t) := \eta_1(t)^t, \tag{2.5}$$

$$A(t) := \begin{cases} Nm(t) & (1 < \alpha < 2), \\ Nm_1(t) & (\alpha = 1), \\ \quad 0 & (0 < \alpha < 1), \end{cases} \tag{2.6}$$

where $m(t)$ is defined in (2.1) and $m_1(t)$ is a truncated moment function,

$$m_1(t) := \mathsf{E}\bigl[X^t \mathbf{1}_{\{X \leqslant \eta_1(t)\}}\bigr]. \tag{2.7}$$

Then, as $t \to \infty$,

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{\ d\ } \mathcal{F}_\alpha, \tag{2.8}$$

where $\mathcal{F}_\alpha$ is a stable law with characteristic exponent α and skewness parameter $\beta \equiv 1$. The characteristic function ϕ_α of the law $\mathcal{F}_\alpha$ is given by

$$\phi_\alpha(u) = \begin{cases} \exp\left\{-\Gamma(1-\alpha)|u|^\alpha \exp\left(-\dfrac{i\pi\alpha}{2}\operatorname{sgn} u\right)\right\} & (\alpha \neq 1), \\ \exp\left\{iu(1-\gamma) - \dfrac{\pi}{2}|u|\left(1 + i\operatorname{sgn} u \cdot \dfrac{2}{\pi}\log|u|\right)\right\} & (\alpha = 1), \end{cases} \tag{2.9}$$

where $\gamma = 0.5772\ldots$ is the Euler constant.

Remark. For $1 < \alpha < 2$, we use the analytic continuation of the gamma function in (2.9), given by the formula $\Gamma(1-\alpha) = \Gamma(2-\alpha)/(1-\alpha)$.

At the critical points, $\alpha = 1$ and $\alpha = 2$, the LLN and CLT, respectively, prove to be valid; however, the normalizing constants require some truncation.

Theorem 2.5. Under the hypotheses of Theorem 2.4, let $\alpha = 1$. Then

$$\frac{S_N(t)}{N m_1(t)} \xrightarrow{\,p\,} 1, \tag{2.10}$$

where $m_1(t)$ is given by (2.7).

Theorem 2.6. Under the hypotheses of Theorem 2.4, let $\alpha = 2$. Then

$$\frac{S_N(t) - \mathsf{E}\, S_N(t)}{\sqrt{N m_2(t)}} \xrightarrow{\,d\,} \mathcal{N}(0,1),$$

where $m_2(t)$ is a truncated moment function of the "second order",

$$m_2(t) := \mathsf{E}\bigl[X^{2t}\mathbf{1}_{\{X \leqslant \eta_1(t)\}}\bigr]. \tag{2.11}$$

Limits for the norms $R_N(t)$ follow from the corresponding results for the sums $S_N(t)$.

Theorem 2.7. Under the conditions of Theorems 2.2 ($\alpha > 2$) and 2.4 ($\alpha \leqslant 2$):
(a) For $\alpha \geqslant 2$,

$$\frac{t\sqrt{N}m(t)}{\sqrt{\tilde{m}_\alpha(t)}}\left(\frac{R_N(t)}{(Nm(t))^{1/t}} - 1\right) \xrightarrow{\,d\,} \mathcal{N}(0,1) \qquad (t \to \infty),$$

where $\tilde{m}_\alpha(t) = m(2t)$ for $\alpha > 2$ and $\tilde{m}_2(t) = m_2(t)$, and the functions $m(\cdot)$ and $m_2(\cdot)$ are given by (2.1) and (2.11), respectively.

(b) For $1 \leqslant \alpha < 2$,

$$\frac{tN\tilde{m}_\alpha(t)}{\eta_1(t)^t}\left(\frac{R_N(t)}{(N\tilde{m}_\alpha(t))^{1/t}} - 1\right) \xrightarrow{\,d\,} \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $\tilde{m}_\alpha(t) = m(t)$ for $1 < \alpha < 2$ and $\tilde{m}_1(t) = m_1(t)$, with $m_1(\cdot)$ given by (2.7).

(c) For $0 < \alpha < 1$,

$$t\left(\frac{R_N(t)}{\eta_1(t)} - 1\right) \xrightarrow{\,d\,} \log \mathcal{F}_\alpha \qquad (t \to \infty),$$

where $\log \mathcal{F}_\alpha$ stands for the distribution of $\log Z_\alpha$ with Z_α having the law $\mathcal{F}_\alpha$.

3. REGULAR VARIATION AND KASAHARA'S TAUBERIAN THEOREM

Consider the log-tail distribution function

$$h(x) := -\log \mathsf{P}\{X > x\}, \qquad x \in (0, \infty). \tag{3.1}$$

Condition 3.1 (Regularity Assumption). The function h is *regularly varying at infinity with index* $\varrho \in (0,\infty)$ (we write $h \in R_\varrho$). That is, for every constant $\kappa > 0$

$$\lim_{x\to\infty} \frac{h(\kappa x)}{h(x)} = \kappa^\varrho. \tag{3.2}$$

The following result, known as the *Uniform Convergence Theorem* (UCT), is a useful extension of the definition of regular variation (see Ref. 2, Theorem 1.5.2, p. 22).

Lemma 3.2 (UCT). If $h \in R_\varrho$ with $\varrho > 0$ then (3.2) holds uniformly in κ on each interval $(0, \kappa_1]$.

The link between the asymptotic behavior of $h(x)$ and the moment function $m(t)$ is characterized by a Tauberian theorem by Kasahara.(7) Recall that a quasi-inverse of h can be defined by $h^{\leftarrow}(y) := \inf\{x : h(x) > y\}$ (see Ref. 2, Sect. 1.5.7, p. 28). One can show that $h \in R_\varrho$ if and only if $h^{\leftarrow} \in R_{1/\varrho}$ (see Ref. 2, Theorem 1.5.12, p. 28).

Lemma 3.3 (Kasahara's Tauberian theorem). Let $m(t)$ be given by (2.1) and $h(x)$ by (3.1). Then $h \in R_\varrho$ with $0 < \varrho < \infty$ if and only if

$$\frac{\log m(t)}{t} - \log h^{\leftarrow}(t) \to -\frac{1+\log\varrho}{\varrho} \qquad (t\to\infty). \tag{3.3}$$

A useful implication of this result is

Lemma 3.4. Suppose that $h \in R_\varrho$. Then for any constant $r > 0$

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{m(rt)}{m(t)^r} = \frac{r}{\varrho} \log r.$$

Proof. Applying formula (3.3) and recalling that $h^{\leftarrow} \in R_{1/\varrho}$ we obtain

$$\log \frac{m(rt)}{m(t)^r} = rt \log \frac{h^{\leftarrow}(rt)}{h^{\leftarrow}(t)} + o(t) = rt \log r^{1/\varrho} + o(t),$$

and the lemma follows. □

4. PROOF OF THEOREMS 2.1 AND 2.2

In this section, the parameter α is defined by (2.2).

Proof of Theorem 2.1. It suffices to check that for some $r > 1$

$$\lim_{t\to\infty} \mathsf{E}\,|S_N^*(t) - 1|^r = 0. \tag{4.1}$$

By an inequality of von Bahr and Esseen,[1] for any $r \in [1,2]$

$$\mathsf{E}\,|S_N^* - 1|^r \leqslant \frac{2}{N^{r-1}}\,\mathsf{E}\biggl(\frac{X^t}{m(t)} + 1\biggr)^r. \tag{4.2}$$

Furthermore, by the elementary inequality $(x+1)^r \leqslant 2^{r-1}(x^r+1)$ ($x \geqslant 0$, $r \geqslant 1$), which easily follows from Jensen's inequality applied to x^r, the right-hand side of (4.2) is bounded from above by

$$\frac{2^r}{N^{r-1}}\,\mathsf{E}\biggl(\frac{X^{rt}}{m(t)^r} + 1\biggr) = \frac{O(1)}{N^{r-1}} \cdot \frac{m(rt)}{m(t)^r}\,. \tag{4.3}$$

Using Lemma 3.4 and the hypothesis of the theorem, we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \frac{m(rt)}{N^{r-1}\,m(t)^r} &= -(r-1)\liminf_{t\to\infty} \frac{\log N}{t} + \lim_{t\to\infty} \frac{1}{t} \log \frac{m(rt)}{m(t)^r} \\
&= -\frac{(r-1)\alpha}{\varrho} + \frac{r}{\varrho}\log r \\
&\leqslant -\frac{(r-1)\alpha}{\varrho} + \frac{r}{\varrho}\,(r-1) \\
&= -\frac{(r-1)(\alpha - r)}{\varrho} < 0,
\end{aligned} \tag{4.4}$$

if $1 < r < \alpha$. This implies that the right-hand side of (4.3) tends to zero as $t \to \infty$, and (4.1) follows. □

Proof of Theorem 2.2. First of all, note that by Lemma 3.4,

$$\mathsf{Var}\,X^t = m(2t) - m(t)^2 \sim m(2t) \qquad (t \to \infty), \tag{4.5}$$

so one can replace the normalization $\sqrt{N\,\mathsf{Var}\,X^t}$ by $\sqrt{Nm(2t)}$.

By the Lyapunov theorem (see, e.g., Ref. 9, Theorem 4.9, p. 126), it suffices to check that for an appropriate $r > 1$

$$\frac{\mathsf{E}|X^t - m(t)|^{2r}}{N^{r-1}\,m(2t)^r} \to 0 \qquad (t \to \infty). \tag{4.6}$$

Arguing as in the proof of Theorem 2.1, one can show that the left-hand side of (4.6) is dominated by

$$\frac{2^{2r-1}}{N^{r-1}\,m(2t)^r}\bigl(m(2rt) + m(t)^{2r}\bigr) \sim \frac{2^{2r-1}}{N^{r-1}} \cdot \frac{m(2rt)}{m(2t)^r}. \tag{4.7}$$

Similarly to (4.4) we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \frac{m(2rt)}{N^{r-1}\,m(2t)^r} &= -\frac{(r-1)\alpha}{\varrho} + \frac{2r}{\varrho}\log r \\
&\leqslant -\frac{(r-1)\alpha}{\varrho} + \frac{2r(r-1)}{\varrho} = -\frac{(r-1)(\alpha - 2r)}{\varrho} < 0,
\end{aligned}$$

if $1 < r < \alpha/2$. Hence, the right-hand side of (4.7) vanishes in the limit $t \to \infty$, which proves (4.6). □

5. NORMALIZED REGULAR VARIATION

As mentioned in Section 2, to characterize the limiting behavior of $S_N(t)$ in the zone $\alpha \leqslant 2$, we need slightly more regularity. From now on we impose the following

Condition 5.1 (Normalized Regularity Assumption). The log-tail distribution function h is *normalized regularly varying* at infinity, $h \in NR_\varrho$ (with $0 < \varrho < \infty$). The latter means that for every $\varepsilon > 0$ the function $h(x)/x^{\varrho-\varepsilon}$ is ultimately increasing and the function $h(x)/x^{\varrho+\varepsilon}$ is ultimately decreasing (see Ref. 2, p. 24).

In particular, writing $h(x) = x^{\varrho-\varepsilon}\,(h(x)/x^{\varrho-\varepsilon})$ with $0 < \varepsilon < \varrho$ we see that $h(x)$ is ultimately (strictly) increasing and therefore is invertible for x large enough.

The next lemma gives an important characterization of the class NR_ϱ (cf. Ref. 2, p. 15).

Lemma 5.2. Let h be a positive (measurable) function. Then $h \in NR_\varrho$ if and only if h is absolutely continuous (and hence a.e. differentiable) and

$$\frac{xh'(x)}{h(x)} \to \varrho \qquad (x \to \infty). \tag{5.1}$$

Integration of (5.1) shows that the function $h \in NR_\varrho$ can be represented in the form

$$h(x) = h(0) + \int_0^x \frac{h(u)}{u}\,\big(\varrho + \varepsilon(u)\big)\,du, \tag{5.2}$$

where $\varepsilon(x) \to 0$ as $x \to \infty$.

The next lemma can be viewed as a refinement of the UCT (Lemma 3.2) for the case of normalized regular variation.

Lemma 5.3. If $h \in NR_\varrho$ with $\varrho > 0$ then, uniformly in κ on each interval $[\kappa_0, \kappa_1] \subset (0, \infty)$,

$$h(\kappa x) - h(x) = h(x)(\kappa^\varrho - 1)(1 + o(1)) \qquad (x \to \infty).$$

Proof. Using the representation (5.2), after the substitution $u = xy$ we have

$$\frac{h(\kappa x) - h(x)}{h(x)} = \int_1^\kappa \frac{h(xy)}{h(x)y}\,\big(\varrho + \varepsilon(xy)\big)\,dy. \tag{5.3}$$

By Lemma 3.2, the integrand in (5.3) tends to $\varrho y^{\varrho-1}$, as $x \to \infty$, uniformly in $y \in [\kappa_0, \kappa_1]$. Therefore, the integral (5.3) converges, uniformly in $\kappa \in [\kappa_0, \kappa_1]$, to $\int_1^\kappa \varrho y^{\varrho-1}\,dy = \kappa^\varrho - 1$. $\square$

For $\tau > 0$, set $\eta_\tau(t) = \tau^{1/t}\eta_1(t)$, where $\eta_1(t)$ is defined by

$$\eta_1(t) = h^{-1}(\alpha t/\varrho) \tag{5.4}$$

and hence satisfies equation (2.4). The next lemma will play a crucial role.

Lemma 5.4. Uniformly in τ on each interval $[\tau_0, \tau_1] \subset (0, \infty)$,

$$\lim_{t\to\infty} [h(\eta_\tau(t)) - h(\eta_1(t))] = \alpha \log \tau.$$

Proof. Note that

$$\kappa_\tau(t) := \frac{\eta_\tau(t)}{\eta_1(t)} = \tau^{1/t} \to 1 \qquad (t \to \infty),$$

uniformly in $\tau \in [\tau_0, \tau_1]$. Therefore, for all large enough t the function $\kappa_\tau(t)$ is uniformly bounded, $0 < \kappa_0 \leqslant \kappa_\tau(t) \leqslant \kappa_1 < \infty$. Applying Lemma 5.3, in the limit $t \to \infty$ we obtain, uniformly in τ,

$$h(\eta_\tau) - h(\eta_1) \sim h(\eta_1)(\kappa_\tau^\varrho - 1) = \frac{\alpha t}{\varrho}\left(\tau^{\varrho/t} - 1\right) \sim \frac{\alpha t}{\varrho} \cdot \frac{\varrho \log \tau}{t} = \alpha \log \tau,$$

where we also used the identity (2.4). □

6. ASYMPTOTICS OF TRUNCATED MOMENTS

Throughout this section we assume that the hypotheses of Theorem 2.4 are satisfied.

Lemma 6.1. For any $\tau > 0$ and $p > 0$,

$$\lim_{t\to\infty} \frac{e^{\alpha t/\varrho}}{\eta_1(t)^{pt}} \mathsf{E}\big[X^{pt}\big(\mathbf{1}_{\{X\leqslant\eta_\tau(t)\}} - \mathbf{1}_{\{X\leqslant\eta_1(t)\}}\big)\big] = \begin{cases} \dfrac{\alpha}{p-\alpha}\,(\tau^{p-\alpha} - 1), & p \neq \alpha, \\ \alpha \log \tau, & p = \alpha. \end{cases} \tag{6.1}$$

Proof. The case $\tau = 1$ is obvious. Suppose that $\tau > 1$. Integration by parts yields

$$\begin{aligned} \mathsf{E}\big[X^{pt}\mathbf{1}_{\{\eta_1 < X \leqslant \eta_\tau\}}\big] &= \int_{\eta_1}^{\eta_\tau} x^{pt}\, d\big(1 - e^{-h(x)}\big) = -\int_{\eta_1}^{\eta_\tau} x^{pt}\, de^{-h(x)} \\ &= \eta_1^{pt} e^{-h(\eta_1)} - \eta_1^{pt}\tau^p e^{-h(\eta_\tau)} + \int_{\eta_1}^{\eta_\tau} e^{-h(x)}\, dx^{pt}. \end{aligned}$$

Using the substitution $x = \eta_y \equiv \eta_1 y^{1/t}$ and identity (2.4) we obtain

$$\eta_1^{-pt} e^{\alpha t/\varrho}\, \mathsf{E}\big[X^{pt}\mathbf{1}_{\{\eta_1 < X \leqslant \eta_\tau\}}\big] = 1 - \tau^p e^{h(\eta_1) - h(\eta_\tau)} + \int_1^\tau e^{h(\eta_1) - h(\eta_y)}\, dy^p. \tag{6.2}$$

By Lemma 5.4, $h(\eta_1) - h(\eta_y) \to -\alpha \log y$ as $t \to \infty$, uniformly in $y \in [1, \tau]$. Hence, the right-hand side of (6.2) tends to

$$1 - \tau^p e^{-\alpha \log \tau} + \int_1^\tau e^{-\alpha \log y}\, dy^p = 1 - \tau^{p-\alpha} + p \int_1^\tau y^{p-\alpha-1}\, dy,$$

and (6.1) follows. The case $0 < \tau < 1$ is considered analogously. □

Lemma 6.2. (i) For any $\tau > 0$ and each $p > \alpha$,

$$\lim_{t\to\infty} \frac{e^{\alpha t/\varrho}}{\eta_1(t)^{pt}} \,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X\leqslant \eta_\tau(t)\}}\big] = \frac{\alpha}{p-\alpha}\,\tau^{p-\alpha}. \tag{6.3}$$

(ii) For any $\tau > 0$ and each $p < \alpha$,

$$\lim_{t\to\infty} \frac{e^{\alpha t/\varrho}}{\eta_1(t)^{pt}} \,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X> \eta_\tau(t)\}}\big] = \frac{\alpha}{\alpha-p}\,\tau^{p-\alpha}. \tag{6.4}$$

Proof. In view of Lemma 6.1 it suffices to consider the case $\tau = 1$.
(i) Set $\theta := e^{-(\alpha+\delta)/p\varrho}$ with $\delta > 0$, then

$$\eta_1^{-pt} e^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X\leqslant\theta\eta_1\}}\big] \leqslant e^{\alpha t/\varrho}\theta^{pt} = e^{-\delta t/\varrho} = o(1).$$

Further, similarly as in the proof of Lemma 6.1 (see (6.2)), integrating by parts and using the substitution $x = \eta_y$ and identity (2.4) we obtain

$$\eta_1^{-pt} e^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{\theta\eta_1<X\leqslant\eta_1\}}\big] = O(1)\,e^{-\delta t/\varrho} - 1 + \int_0^1 f_t(y)\,dy^p, \tag{6.5}$$

where $f_t(y) := e^{h(\eta_1)-h(\eta_y)}\,\mathbf{1}_{\{\theta^t<y\}}$. Noting that $\theta^t \to 0$ and using Lemma 5.4, it is easy to see that $\lim_{t\to\infty} f_t(y) = y^{-\alpha}$ for each $y > 0$. We also note that $\eta_y/\eta_1 = y^{1/t} \in [\theta, 1]$ for $y \in [\theta^t, 1]$, and so Lemma 5.3 implies that for any $\varepsilon > 0$, all t large enough and all $y \in [\theta^t, 1]$

$$h(\eta_1) - h(\eta_y) \leqslant h(\eta_1)(1 - y^{\varrho/t})(1+\varepsilon) \leqslant \frac{\alpha t}{\varrho}\cdot\frac{-\varrho\log y}{t}\,(1+\varepsilon) = -\alpha(1+\varepsilon)\log y.$$

It follows that $f_t(y)$ is bounded by the function $y^{-\alpha(1+\varepsilon)}$, which is integrable on $[0, 1]$ with respect to dy^p if $p > \alpha$ and ε is sufficiently small. Hence, by Lebesgue's dominated convergence theorem the limit of (6.5) equals

$$-1 + \int_0^1 y^{-\alpha}\,dy^p = -1 + \frac{p}{p-\alpha} = \frac{\alpha}{p-\alpha}\,,$$

in accord with (6.3).

(ii) We start by showing that for any $\theta > 1$

$$\lim_{t\to\infty} e^{\alpha t/\varrho}\eta_1^{-pt}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\theta\eta_1\}}\big] = 0. \tag{6.6}$$

Indeed, using that $p - \alpha < 0$ we have

$$\mathsf{E}\big[X^p\mathbf{1}_{\{X>\theta\eta_1\}}\big] \leqslant (\theta\eta_1)^{(p-\alpha)t}\,\mathsf{E}\,X^{\alpha t} = \theta^{(p-\alpha)t}\eta_1^{(p-\alpha)t}m(\alpha t).$$

By Lemma 3.3 we have

$$\frac{1}{\alpha t}\log\big(e^{\alpha t/\varrho}\eta_1^{-\alpha t}m(\alpha t)\big) = \log\frac{h^{-1}(\alpha t)}{\eta_1(t)} - \frac{\log\varrho}{\varrho} + o(1). \tag{6.7}$$

Substituting (5.4) and using that $h^{-1}\in R_{1/\varrho}$, it is easy to see that the right-hand side of (6.7) tends to 0 as $t\to\infty$. Hence,

$$e^{\alpha t/\varrho}\eta_1^{-pt}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\theta\eta_1\}}\big]=e^{-Ct(1+o(1))}=o(1),$$

where $C:=(\alpha-p)\log\theta>0$, and (6.6) follows.

Similarly to (6.5), integration by parts gives

$$\eta_1^{-pt}e^{\alpha t/\varrho}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{\eta_1<X\leqslant\theta\eta_1\}}\big]=-\theta^{pt}e^{h(\eta_1)-h(\theta\eta_1)}+1+\int_1^\infty\bar f_t(y)\,dy^p,\qquad(6.8)$$

where $\bar f_t(y):=e^{h(\eta_1)-h(\eta_y)}\,\mathbf{1}_{\{1<y\leqslant\theta^t\}}$. Using Lemma 5.3 and the identity (2.4) we note that

$$\lim_{t\to\infty}\frac{h(\eta_1)-h(\theta\eta_1)}{t}=-\frac{\alpha}{\rho}\,(\theta^\varrho-1)\leqslant-\alpha\log\theta,$$

so the first term in (6.8) is estimated by $e^{-(\alpha-p)t\log\theta\,(1+o(1))}=o(1)$. Next, noting that $\theta^t\to\infty$ and using Lemma 5.4, we obtain $\lim_{t\to\infty}\bar f_t(y)=y^{-\alpha}$ for each $y>1$. Moreover, similarly to the proof of part (i) one can show that the integral in (6.8) converges to

$$\int_1^\infty y^{-\alpha}\,dy^p=\frac{p}{\alpha-p}\,.$$

The limit (6.4) now follows, and the proof is complete. □

In the case $p=\alpha$ not covered by Lemma 6.2, we obtain one crude estimate that will nevertheless be sufficient for our purposes below.

Lemma 6.3. As $t\to\infty$,

$$b_\alpha(t):=\frac{e^{\alpha t/\varrho}}{\eta_1(t)^{\alpha t}}\,\mathsf{E}\big[X^{\alpha t}\mathbf{1}_{\{X\leqslant\eta_1(t)\}}\big]\to+\infty.$$

Proof. For any $\delta\in(0,1)$, using Lemma 6.1 we have

$$b_\alpha(t)\geqslant\eta_1^{-\alpha t}e^{\alpha t/\varrho}\,\mathsf{E}\big[X^{\alpha t}\mathbf{1}_{\{\eta_\delta<X\leqslant\eta_1\}}\big]\to-\alpha\log\delta,$$

hence

$$\liminf_{t\to\infty}b_\alpha(t)\geqslant-\alpha\log\delta.$$

Setting $\delta\downarrow0$, we obtain $\lim_{t\to\infty}b_\alpha(t)=+\infty$, as claimed. □

For convenience of reference, let us record a few more estimates for truncated moments. Denote

$$m_\alpha(t):=\mathsf{E}\big[X^{\alpha t}\mathbf{1}_{\{X\leqslant\eta_1(t)\}}\big]\qquad(6.9)$$

and set

$$\tilde Y\equiv\tilde Y(t):=\frac{X^t}{(Nm_\alpha(t))^{1/\alpha}}.\qquad(6.10)$$

From (6.10) it is seen that the inequality $\tilde{Y}(t) > \tau$ is equivalent to $X > \eta_{\alpha,\tau}(t)$, where

$$\tilde{\eta}_{\alpha,\tau} \equiv \tilde{\eta}_{\alpha,\tau}(t) := \tau^{1/t}(N m_\alpha(t))^{1/\alpha t}. \tag{6.11}$$

Using the scaling condition (2.3) and Lemma 6.3, one can check that

$$\log \tilde{\eta}_{\alpha,t} = \log \eta_1 + \frac{\log b_\alpha}{\alpha t}\big(1 + o(1)\big).$$

Hence, for all sufficiently large t

$$\tilde{\eta}_{\alpha,\tau}(t) > \eta_1(t). \tag{6.12}$$

Lemma 6.4. For any p such that $0 \leqslant p < \alpha$ and each $\tau > 0$

$$\lim_{t\to\infty} N\,\mathsf{E}\big[\tilde{Y}(t)^p \mathbf{1}_{\{\tilde{Y}(t)>\tau\}}\big] = 0.$$

Proof. Substituting (6.10) and using inequality (6.12) we have

$$N\,\mathsf{E}\big[\tilde{Y}^p \mathbf{1}_{\{\tilde{Y}>\tau\}}\big] \leqslant \frac{N}{(N m_\alpha)^{p/\alpha}}\,\mathsf{E}\big[X^{pt}\mathbf{1}_{\{X>\eta_1\}}\big] \sim \frac{\alpha}{\alpha - p}\cdot b_\alpha^{-p/\alpha} \to 0 \qquad (t\to\infty), \tag{6.13}$$

where we also used (2.3), (6.13) and Lemmas 6.2(ii) and 6.3. □

Denote

$$y_\alpha \equiv y_\alpha(t) := \frac{\eta_1(t)^t}{(N m_\alpha(t))^{1/\alpha}}\,, \tag{6.14}$$

so that $\tilde{Y} > y_\alpha$ if and only if $X > \eta_1$. From (2.3) and Lemma 6.3 it follows that

$$y_\alpha(t) \sim b_\alpha(t)^{-1/\alpha} \to 0 \qquad (t\to\infty).$$

Lemma 6.5. Suppose that $p > 0$. Then for any $\tau > 0$

$$N\,\mathsf{E}\big[\tilde{Y}(t)^p \mathbf{1}_{\{y_\alpha(t)<\tilde{Y}(t)\leqslant\tau\}}\big] \to 0 \qquad (t\to\infty). \tag{6.15}$$

Proof. Picking a number q such that $0 < q < \min\{\alpha, p\}$, the left-hand side of (6.15) is estimated from above by

$$N\tau^{p-q}\,\mathsf{E}\big[\tilde{Y}^q \mathbf{1}_{\{y_\alpha<\tilde{Y}\}}\big] = \frac{N\tau^{p-q}}{(N m_\alpha)^{q/\alpha}}\,\mathsf{E}\big[X^{qt}\mathbf{1}_{\{X>\eta_1\}}\big] \to 0,$$

as was shown above (see (6.13)). □

7. PROOF OF THEOREM 2.4

We first establish convergence to an infinitely divisible law. Denote

$$Y_i \equiv Y_i(t) := \frac{X_i^t}{B(t)}, \qquad i = 1, 2, \dots \tag{7.1}$$

According to classical theorems (see, e.g., Ref. 8, Theorem 8, p. 81–82; cf. Theorem 7, p. 80–81), in order for the sum

$$S_N^*(t) := \sum_{i=1}^{N} Y_i(t) - \frac{A(t)}{B(t)}$$

to converge in distribution to an infinitely divisible law with characteristic function

$$\phi(u) = \exp\left\{ iau - \frac{\sigma^2 u^2}{2} + \int_{|x|>0} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) dL(x) \right\}, \tag{7.2}$$

it is sufficient that the following three conditions be fulfilled:

1) In all points of its continuity, the function $L(\cdot)$ satisfies

$$L(x) = \begin{cases} \lim\limits_{t\to\infty} N\,\mathsf{P}\{Y \leqslant x\}, & x < 0, \\ -\lim\limits_{t\to\infty} N\,\mathsf{P}\{Y > x\}, & x > 0. \end{cases}$$

2) The constant σ^2 is given by

$$\sigma^2 = \lim_{\tau\to 0+} \lim_{t\to\infty} N\,\mathsf{Var}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big]$$

3) For each $\tau > 0$, the constant a satisfies the identity

$$\lim_{t\to\infty} \left(N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] - \frac{A(t)}{B(t)} \right) = a + \int_0^\tau \frac{x^3}{1+x^2}\,dL(x) - \int_\tau^\infty \frac{x}{1+x^2}\,dL(x). \tag{7.3}$$

We will show that the distribution of $S_N^*(t)$ does converge to a law determined by the characteristic function ϕ_α given by (7.2) with

$$L(x) = \begin{cases} 0, & x < 0, \\ -x^{-\alpha}, & x > 0. \end{cases} \qquad \sigma^2 = 0, \qquad a = \begin{cases} \dfrac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & (\alpha \neq 1), \\ 0 & (\alpha = 1). \end{cases} \tag{7.4}$$

1) Since $Y \geqslant 0$, it is clear that $L(x) \equiv 0$ for $x < 0$. For $x > 0$, using (7.1), (2.3) and Lemma 5.4 we obtain

$$\begin{aligned} N\,\mathsf{P}\{Y > x\} = N\,\mathsf{P}\big\{X > x^{1/t}\eta_1\big\} &\sim e^{\alpha t/\varrho}\,\mathsf{P}\{X > \eta_x\} \\ &= e^{h(\eta_1)-h(\eta_x)} \to e^{-\alpha \log x} = x^{-\alpha}. \end{aligned}$$

2) Since $0 \leqslant \mathsf{Var}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] \leqslant \mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y\leqslant\tau\}}\big]$, it suffices to prove that

$$\lim_{\tau\to 0+}\lim_{t\to\infty} N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] = 0.$$

Recalling (7.1) and (2.3) and using Lemma 6.2(i) with $p = 2$, we obtain, as $t \to \infty$,

$$N\,\mathsf{E}\big[Y^2\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] \sim e^{\alpha t/\varrho}\eta_1^{-2t}\,\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\leqslant\eta_\tau\}}\big] \sim \frac{\alpha}{2-\alpha}\,\tau^{2-\alpha}. \tag{7.5}$$

As $\tau \to 0+$, the right-hand side of (7.5) tends to zero, since $2 - \alpha > 0$.

3) Let us show that

$$\lim_{t\to\infty}\left(N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] - \frac{A(t)}{B(t)}\right) = \begin{cases} \dfrac{\alpha}{1-\alpha}\,\tau^{1-\alpha} & (\alpha \neq 1), \\ \log\tau & (\alpha = 1). \end{cases} \tag{7.6}$$

One can then check that the function (7.6) satisfies the identity (7.3) with a given by (7.4) (see details in Ref. 4, Proposition 6.4).

If $0 < \alpha < 1$ then, using (2.3) and Lemma 6.2(i) with $p = 1$, we obtain

$$N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] = \frac{N}{\eta_1^t}\,\mathsf{E}\big[X^t\mathbf{1}_{\{X\leqslant\eta_\tau\}}\big] \sim \frac{\alpha}{1-\alpha}\,\tau^{1-\alpha}. \tag{7.7}$$

If $1 < \alpha < 2$ then, similarly to (7.7), application of Lemma 6.2(ii) with $p = 1$ yields

$$N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] - \frac{A(t)}{B(t)} = -\frac{N}{\eta_1^t}\,\mathsf{E}\big[X^t\,\mathbf{1}_{\{X>\eta_\tau\}}\big] \sim \frac{\alpha}{1-\alpha}\,\tau^{1-\alpha}.$$

Finally, for $\alpha = 1$ we obtain using condition (2.3) and Lemma 6.1 (with $\alpha = 1$):

$$N\,\mathsf{E}\big[Y\,\mathbf{1}_{\{Y\leqslant\tau\}}\big] - \frac{A(t)}{B(t)} = N\eta_1^{-t}\Big(\mathsf{E}\big[X^t\,\mathbf{1}_{\{X\leqslant\eta_\tau\}}\big] - \mathsf{E}\big[X^t\,\mathbf{1}_{\{X\leqslant\eta_1\}}\big]\Big) \to \log\tau.$$

Formulas (7.4) readily imply that ϕ_α corresponds to a stable law with characteristic exponent α (see Ref. 6, Theorem 2.2.1, p. 39–40). In fact, as shown in Ref. 4 (Theorem 6.2), the function ϕ_α can be reduced to the canonical form (2.9), which completes the proof of Theorem 2.4.

8. PROOF OF THEOREMS 2.5 AND 2.6

Proof of Theorem 2.5. The result follows from Theorem 2.4 (for $\alpha = 1$). Indeed, according to (2.3), (2.5), (2.6) and Lemma 6.3 we have

$$A^*(t) := \frac{A(t)}{B(t)} = N\eta_1(t)^{-t}m_1(t) \sim b_1(t) \to \infty \qquad (t\to\infty).$$

Therefore, dividing (2.8) by $A^*(t) \to \infty$ we obtain $S_N(t)/A(t) = 1 + o_p(1)$ as $t \to \infty$, which is in agreement with (2.10). □

Proof of Theorem 2.6. Denote

$$Y_i \equiv Y_i(t) := \frac{X_i^t}{\sqrt{Nm_2(t)}} \tag{8.1}$$

(cf. (6.10)). According to a classical CLT for sums with independent terms (see, e.g., Ref. 8, Theorem 18, p. 95), it suffices to check that for any $\tau > 0$ the following three conditions are satisfied as $t \to \infty$:

$$N\,\mathsf{P}\{Y(t) > \tau\} \to 0, \tag{8.2}$$

$$N\Big(\mathsf{E}\big[Y(t)^2\mathbf{1}_{\{Y(t)\leqslant\tau\}}\big] - \big(\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)\leqslant\tau\}}\big]\big)^2\Big) \to 1, \tag{8.3}$$

$$N\,\mathsf{E}\big[Y(t)\mathbf{1}_{\{Y(t)>\tau\}}\big] \to 0. \tag{8.4}$$

Firstly, note that conditions (8.2) and (8.4) are guaranteed by Lemma 6.4 (with $p = 0$ and $p = 1$, respectively). To check (8.3), let us first show that

$$\frac{\big(\mathsf{E}\big[Y\mathbf{1}_{\{Y\leqslant\tau\}}\big]\big)^2}{\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\leqslant\tau\}}\big]} = \frac{\big(\mathsf{E}\big[X^t\mathbf{1}_{\{X\leqslant\eta_{2,\tau}\}}\big]\big)^2}{\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\leqslant\eta_{2,\tau}\}}\big]} \to 0 \qquad (t\to\infty), \tag{8.5}$$

where $\eta_{2,\tau}$ is defined in (6.11). Indeed, taking into account inequality (6.12) and Lemma 3.4 (with $r = 2$), the ratio in (8.5) is estimated from above by

$$\frac{(\mathsf{E}\,X^t)^2}{\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\leqslant 2t\}}\big]} = \frac{m(t)^2}{m_2(t)} = e^{-t\,(2/\varrho)\log\varrho\,(1+o(1))} = o(1).$$

Hence, condition (8.3) amounts to

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\leqslant\tau\}}\big] \to 1 \qquad (t\to\infty). \tag{8.6}$$

Noting that, according to (8.1), (6.14) and (6.9),

$$N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{Y\leqslant y_2\}}\big] = \frac{1}{m_2}\,\mathsf{E}\big[X^{2t}\mathbf{1}_{\{X\leqslant\eta_1\}}\big] \equiv 1,$$

we can rewrite (8.6) in the form $N\,\mathsf{E}\big[Y^2\mathbf{1}_{\{y_2<Y\leqslant\tau\}}\big] \to 0$. The latter is true by Lemma 6.5, and (8.3) follows. □

9. PROOF OF THEOREM 2.7

First, let us prove a general "transfer" lemma.

Lemma 9.1. Let $\{S(t),\ t\geqslant 0\}$ be a family of positive random variables, such that for some (non-negative) functions $A(t)$ and $B(t)$,

$$S^*(t) := \frac{S(t)-A(t)}{B(t)} \xrightarrow{\ d\ } \mathcal{F} \qquad (t\to\infty). \tag{9.1}$$

Set $R(t) := S(t)^{1/t}$ and $A^*(t) := A(t)/B(t)$.

(a) If $A^*(t) \to \infty$ as $t \to \infty$, then

$$tA^*(t)\left(\frac{R(t)}{A(t)^{1/t}} - 1\right) \xrightarrow{d} \mathcal{F} \qquad (t \to \infty).$$

(b) If $A(t) \equiv 0$ then

$$t\left(\frac{R(t)}{B(t)^{1/t}} - 1\right) \xrightarrow{d} \log \mathcal{F} \qquad (t \to \infty).$$

Proof. (a) Note that $S(t)$ can be represented as

$$S(t) = A(t)\left(1 + \frac{S^*(t)}{A^*(t)}\right), \tag{9.2}$$

whence

$$R(t) = A(t)^{1/t} \exp\left(\frac{1}{t}\log\left(1 + \frac{S^*(t)}{A^*(t)}\right)\right). \tag{9.3}$$

The condition $A^*(t) \to \infty$ implies that $S^*(t)/A^*(t) = o_p(1)$, hence

$$\exp\left(\frac{1}{t}\log\left(1 + \frac{S^*(t)}{A^*(t)}\right)\right) = \exp\left(\frac{S^*(t)}{tA^*(t)}\,(1 + o_p(1))\right) = 1 + \frac{S^*(t)}{tA^*(t)}\,(1 + o_p(1)).$$

Substituting this into (9.3) yields

$$tA^*(t)\left(\frac{R(t)}{A(t)^{1/t}} - 1\right) = S^*(t)(1 + o_p(1)) \xrightarrow{d} \mathcal{F} \qquad (t \to \infty).$$

(b) We have $S(t) = S^*(t)\,B(t)$, whence

$$\frac{R(t)}{B(t)^{1/t}} = \exp\left(\frac{\log S^*(t)}{t}\right) = 1 + \frac{\log S^*(t)}{t}\,(1 + o_p(1)).$$

Therefore,

$$t\left(\frac{R(t)}{B(t)^{1/t}} - 1\right) = \log S^*(t)\,(1 + o_p(1)),$$

which converges weakly to $\log \mathcal{F}$. □

In view of Lemma 9.1, Theorem 2.7 will follow from the limit theorems for the sum $S_N(t)$ according as the function $A^*(t) = A(t)/B(t)$ tends to infinity or zero as $t \to \infty$.

(a) For $\alpha \geqslant 2$, the CLT is valid (see Theorems 2.2 and 2.6), so we have weak convergence of the form (9.1) with $A(t) = Nm(t)$ and $B(t) \leqslant \sqrt{Nm(2t)}$ (see (2.11), (4.5)). Hence, for all $\alpha \geqslant 2$

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log A^*(t)}{t} &\geqslant \liminf_{t\to\infty}\left(\frac{\log N}{2t} + \frac{1}{2t}\log\frac{m(t)^2}{m(2t)}\right)\\ &= \frac{\alpha}{2\varrho} - \frac{\log 2}{\varrho} \geqslant \frac{1 - \log 2}{\varrho} > 0,\end{aligned}$$

where we used (2.2) and Lemma 3.4 with $r = 2$. Therefore, $A^*(t) \to \infty$, and application of Lemma 9.1(a) proves part (a).

(b) If $1 \leqslant \alpha < 2$ then, according to Theorem 2.4, $B(t) = \eta_1(t)^t$ and $A(t)$ is defined in (2.6). Noting that $m(t) \geqslant m_1(t)$ and using (2.3) and Lemma 6.3, we obtain

$$A^*(t) \geqslant \frac{N m_1(t)}{\eta_1(t)^t} \sim e^{(\alpha-1)t/\varrho}\, b_1(t) \to \infty \qquad (t \to \infty).$$

Hence, Lemma 9.1(a) applies and part (b) is proved.

(c) In the case $0 < \alpha < 1$, the assertion of the theorem readily follows from Lemma 9.1(b), since by Theorem 2.4 we have $A(t) \equiv 0$, so that $A^*(t) \equiv 0$.

10. DISCUSSION AND AN EXAMPLE

In order to clarify the link with the setting in Ref. 10, let us show that under our conditions, the random variables $\{X_i\}$, in the sense of extreme value theory, belong to the domain of attraction of Gumbel's double exponential distribution Λ.

Proposition 10.1. Assume that $h \in NR_\varrho$. Let $X_{1,N} := \max\{X_1, \dots, X_N\}$ and set

$$a_N := h^{-1}(\log N), \qquad b_N := \frac{a_N}{\varrho \log N}\,. \tag{10.1}$$

Then

$$\lim_{N\to\infty} \mathsf{P}\left\{\frac{X_{1,N} - a_N}{b_N} \leqslant x\right\} = \exp(-e^{-x}), \qquad x \in \mathbb{R}. \tag{10.2}$$

Proof. It is not difficult to verify available sufficient conditions for convergence of the maximum's distribution to Λ (see, e.g., Ref. 5, Theorem 2.1.3, p. 52). However, it is even simpler to prove (10.2) directly. Indeed, setting $L_N(x) := a_N + x b_N$ we have

$$\mathsf{P}\left\{\frac{X_{1,N} - a_N}{b_N} \leqslant x\right\} = \Big(\mathsf{P}\{X \leqslant a_N + x b_N\}\Big)^N = \Big(1 - e^{-h(L_N(x))}\Big)^N. \tag{10.3}$$

Note that, according to (10.1),

$$L_N(0) = a_N = h^{-1}(\log N) \to +\infty \qquad (N \to \infty), \tag{10.4}$$

since $h^{-1}(x) \in R_{1/\varrho}$, and

$$\frac{b_N}{a_N} = \frac{1}{\varrho \log N} \to 0 \qquad (N \to \infty).$$

Therefore,

$$\kappa_N(x) := \frac{L_N(x)}{L_N(0)} = \frac{a_N + x b_N}{a_N} = 1 + \frac{x b_N}{a_N} \to 1 \qquad (N \to \infty). \tag{10.5}$$

Recalling (10.3), it is then easy to see that (10.2) is reduced to

$$h(L_N(x)) - \log N \to x \qquad (N \to \infty). \tag{10.6}$$

Furthermore, (10.4) implies $h(L_N(0)) = h(a_N) = \log N$. Hence, (10.6) takes the form

$$h(L_N(x)) - h(L_N(0)) \to x \qquad (N \to \infty). \tag{10.7}$$

To show (10.7), we use (10.5) and apply Lemma 5.3 to obtain

$$\begin{aligned} h(L_N(x)) - h(L_N(0)) &\sim h(L_N(0))\big(\kappa_N(x)^\varrho - 1\big) \\ &= \log N \left(\left(1 + \frac{x b_N}{a_N}\right)^{\varrho} - 1\right) \sim \log N \cdot \frac{\varrho x b_N}{a_N} = x, \end{aligned}$$

according to the choice of b_N (see (10.1)). Thus, (10.7) is proved. □

As mentioned in the Introduction, in the case of attraction to the double exponential distribution Λ, Schlather[10] only considered a concrete example of random variables X_i with the unit exponential distribution (therefore, fitting in the class of distributions (1.2) with $\varrho = 1$). Namely, in our notation he has shown (Ref. 10, Theorem 2.4, p. 867) that under the scaling $N = e^{\alpha t}$ the limit distribution of $R_N(t)$ is Gaussian if $\alpha > 2$ and non-Gaussian if $2\log 2 < \alpha < 2$.

Note that our results (see Theorem 2.7) show that $\alpha = 2$ is indeed a critical point, in that a Gaussian law breaks down for $\alpha < 2$. (However, the value $\alpha = 2\log 2$ does not play any special role.) Furthermore, it is not difficult to check that our results corroborate a general conjecture in Ref. 10 (p. 867) asserting (in our terms) that in the case of attraction to Λ there exist functions $a(t)$, $b(t)$ such that, under an appropriate scaling $t = cp(N)$, $a(t)/b(t) = p(N)$, the distribution of $\big(R_N(t) - a(t)\big)/b(t)$ weakly converges to a distribution which, in turn, tends to Λ as $c \to +\infty$ and, properly recentered and renormalized, to $\mathcal{N}(0,1)$ as $c \to 0+$. Comparing this conjecture with our Theorem 2.7, one can see that the role of c is played by $1/\alpha$, so that $c \to +\infty$ is equivalent to $\alpha \to 0+$. As a result, normality in the limit $c \to 0+$ (i.e., $\alpha \to +\infty$) is obvious from Theorem 2.7(a).

To obtain the limit as $c \to +\infty$ (i.e., $\alpha \to 0+$), note that in Schlather's terms Theorem 2.7(c) takes the form

$$\frac{R_N(t) - B(t)^{1/t}}{B(t)^{1/t}/\alpha t} \xrightarrow{d} \alpha \log Z_\alpha \qquad (t \to \infty),$$

where Z_α has the distribution $\mathcal{F}_\alpha$.

Theorem 10.2. As $\alpha \to 0+$, the distribution of $\alpha \log Z_\alpha$ converges to the double exponential distribution:

$$\lim_{\alpha \to 0+} \mathsf{P}\{\alpha \log Z_\alpha \leqslant x\} = \exp(-e^{-x}), \qquad x \in \mathbb{R}.$$

Proof. By Zolotarev[11] (see Theorem 5, p. 447–448) we have, as $\alpha \to 0+$,

$$\mathsf{P}\{\alpha \log Z_\alpha \leqslant x\} = \mathsf{P}\{Z_\alpha \leqslant e^{1/\alpha}\} \sim \frac{1-\beta}{2} + \frac{1+\beta}{2}\, \exp\bigl(-b\, e^{-x}\bigr), \tag{10.8}$$

where, according to (2.9), $\beta = 1$ and $b = \Gamma(1-\alpha)\cos(\pi\alpha/2) \to 1$. Hence, the right-hand side of (10.8) tends to $\exp(e^{-x})$ as $\alpha \to 0$. □

Remark. An analogous result was proved by Ben Arous *et al.*[3] (see Proposition 8.29, p. 47) using a representation of the distribution $\mathcal{F}_\alpha$ via exponential order statistics.

Example 10.3. Let us specify Theorem 2.7 in the case where X has a unit exponential distribution, that is,

$$\mathsf{P}\{X > x\} = e^{-x}, \qquad x \geqslant 0.$$

Therefore, $\varrho = 1$ and $h(x) = x$. The moment function $m(t)$ defined in (2.1) is given by

$$m(t) = \int_0^\infty x^t\, e^{-x}\, dx = \Gamma(t+1),$$

and Stirling's asymptotic formula yields

$$m(t) \sim \sqrt{2\pi}\, t^{t+1/2}\, e^{-t} \qquad (t \to \infty).$$

Furthermore, from (5.4) it is seen that $\eta_1(t) = \alpha t$ and

$$m_\alpha(t) = \int_0^{\alpha t} x^{\alpha t}\, e^{-x}\, dx \sim \frac{1}{2}\,\Gamma(\alpha t + 1) \sim \sqrt{\frac{\pi}{2}}\, (\alpha t)^{\alpha t + 1/2}\, e^{-\alpha t}. \tag{10.9}$$

Hence, Theorem 2.7 implies the following result.

Proposition 10.4. Let X have the unit exponential distribution, and suppose that the scaling condition (2.3) is fulfilled. Then

(a) for $\alpha \geqslant 2$,

$$\frac{C_\alpha \pi^{1/4}\, e^{\alpha t/2}\, t^{5/4}}{2^t} \left(\frac{R_N(t)}{(N\Gamma(t+1))^{1/t}} - 1\right) \xrightarrow{d} \mathcal{N}(0,1),$$

where $C_\alpha = 1$ for $\alpha > 2$ and $C_2 = \sqrt{2}$;

(b) for $1 \leqslant \alpha < 2$,

$$\frac{\sqrt{2\pi}\, e^{(\alpha-1)t}\, t^{3/2}}{C_\alpha\, \alpha^t} \left(\frac{R_N(t)}{(N\tilde{m}_\alpha(t))^{1/t}} - 1\right) \xrightarrow{d} \mathcal{F}_\alpha,$$

where

$$\tilde{m}_\alpha(t) = \begin{cases} m(t), & 1 < \alpha < 2, \\ m_1(t), & \alpha = 1, \end{cases} \qquad C_\alpha = \begin{cases} 1, & 1 < \alpha < 2, \\ 2, & \alpha = 1, \end{cases}$$

with $m_1(\cdot)$ given by (10.9);

(c) for $0 < \alpha < 1$,

$$R_N(t) - \alpha t \xrightarrow{d} \alpha \log \mathcal{F}_\alpha.$$

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