

ON BOUNDED SOLUTIONS OF THE BALANCED GENERALIZED PANTOGRAPH EQUATION

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Abstract. The question about the existence and characterization of bounded solutions to linear functional-differential equations with both advanced and delayed arguments was posed in the early 1970s by T. Kato in connection with the analysis of the pantograph equation, $y'(x) = ay(qx) + by(x)$. In the present paper, we answer this question for the *balanced* generalized pantograph equation of the form $-a_2y''(x) + a_1y'(x) + y(x) = \int_0^\infty y(\alpha x)\mu(d\alpha)$, where $a_1 \geq 0$, $a_2 \geq 0$, $a_1^2 + a_2^2 > 0$, and μ is a probability measure. By setting $K := \int_0^\infty \ln \alpha \mu(d\alpha)$, we prove that if $K \leq 0$ then the equation does not have nontrivial (i.e., nonconstant) bounded solutions, while if $K > 0$ then such a solution exists. The result in the critical case, $K = 0$, settles a long-standing problem. The proof exploits the link with the theory of Markov processes, in that any solution of the balanced pantograph equation is an \mathcal{L} -harmonic function relative to the generator \mathcal{L} of a certain diffusion process with “multiplication” jumps. The paper also includes three “elementary” proofs for the simple prototype equation $y'(x) + y(x) = \frac{1}{2}y(qx) + \frac{1}{2}y(x/q)$, based on perturbation, analytical, and probabilistic techniques, respectively, which may appear useful in other situations as efficient exploratory tools.

Key words. Pantograph equation, functional-differential equations, integro-differential equations, balance condition, bounded solutions, WKB expansion, q -difference equations, ruin problem, Markov processes, jump diffusions, \mathcal{L} -harmonic functions, martingales.

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1. Introduction. The classical *pantograph equation* is the linear first-order functional-differential equation (with rescaled argument) of the form

$$y'(x) = ay(qx) + by(x), \quad (1.1)$$

where a, b are constant coefficients (real or complex) and $q > 0$ is a rescaling parameter. Historically,¹ the term “pantograph” dates back to the seminal paper of 1971 by Ockendon and Tayler [24], where such equations² emerged

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¹The name “pantograph equation” was not in wide use until it was coined by Iserles [14] for a more general class of functional-differential equations.

²To be more precise, a certain vector analog of Eq. (1.1).

in a mathematical model for the dynamics of an overhead current collection system on an electric locomotive (with the physically relevant value $q < 1$). At about the same time, a systematic analysis of solutions to the pantograph equation was started by Fox *et al.* [11], where various analytical, perturbation, and numerical techniques were discussed at length (for both $q < 1$ and $q > 1$).

It is interesting to note that an equation of the form (1.1) (with $q > 1$) was derived more than 25 years earlier by Ambartsumian [2] to describe the absorption of light by the interstellar matter. Some particular cases of Eq. (1.1) are also found in early work by Mahler [19] on a certain partition problem in number theory (where Eq. (1.1), with $a = 1$, $b = 0$, $q < 1$, appears as a limit of a similar functional-difference equation) and by Gaver [12] on a special ruin problem (with $a = 1$, $b = -1$, $q > 1$). Subsequently, the pantograph equation has appeared in numerous applications ranging from the problem of coherent states in quantum theory [26] to cell-growth modeling in biology [28] (see further references in Refs. [7, 8, 14, 20]). These and other examples suggest that, typically, the pantograph equation and similar functional-differential equations with rescaling are relevant as long as the systems in question possess some kind of self-similarity.

Since its introduction into the mathematical literature in the early 1970s, the theory of the pantograph equation (and some of its natural generalizations) has been the subject of persistent attention and research effort, yielding over years a number of significant developments. In particular, the classification of Eq. (1.1) with regard to various domains of the parameters,³ including existence and uniqueness theorems, and an extensive asymptotic analysis of the corresponding solutions have been given by Kato and McLeod [17] and Kato [16]. The investigation of such equations in the complex domain was initiated by Morris *et al.* [22] and Oberg [23] and continued by Derfel and Iserles [5] and Marshall *et al.* [20]. A systematic treatment of the generalized first-order pantograph equation (with matrix coefficients and also allowing for a term with rescaled derivative) is contained in the influential paper by Iserles [14], where in particular a fine geometric structure of almost-periodic solutions has been described. Asymptotics for equations with variable coefficients have been studied by Derfel and Vogl [6].

Higher-order generalizations of the pantograph equation (1.1) lead to the class of linear functional-differential equations with rescaling,

$$\sum_{j=1}^{\ell} \sum_{k=0}^m a_{jk} y^{(k)}(\alpha_j x + \beta_j) = 0 \quad (1.2)$$

³Depending on whether $q < 1$ or $q > 1$ and also on the cases $\Re b < 0$, $\Re b > 0$, and $\Re b = 0$.

(see Ref. [4] and references therein).⁴ Kato [16] posed a problem of asymptotic analysis of Eq. (1.2), including the question of existence and characterization of *bounded* solutions. Some partial answers to the latter question have been given by Derfel [7, 4] and Derfel and Molchanov [8].

In particular, Derfel [7] considered the “balanced” generalized first-order pantograph equation of the form

$$y'(x) + y(x) = \sum_{j=1}^{\ell} p_j y(\alpha_j x), \quad (1.3)$$

subject to the condition

$$\sum_{j=1}^{\ell} p_j = 1, \quad p_j > 0 \quad (j = 1, \dots, \ell), \quad (1.4)$$

so that the weights p_j of the rescaled y -terms on the right-hand side of Eq. (1.3) match the unit coefficient of the $y(x)$ on the left. Note that, owing to the balance condition (1.4), Eq. (1.3) always has a trivial solution $y = \text{const}$. The question of existence of *nontrivial* (i.e., nonconstant) *bounded solutions* is most interesting (and most difficult) in the case where the right-hand side of Eq. (1.3) involves both “advanced” ($\alpha_j > 1$) and “delayed” ($0 < \alpha_j < 1$) arguments. It turns out that the answer depends crucially on the quantity

$$K := \sum_{j=1}^{\ell} p_j \ln \alpha_j. \quad (1.5)$$

Derfel [7] has proved that if $K < 0$ then Eq. (1.3) has no nontrivial bounded solutions, whereas if $K > 0$ then such a solution always exists. In the “critical” case $K = 0$, this question has remained open as yet.

In the present paper, we consider a more general integro-differential equation⁵ of the pantograph type, namely,

$$-a_2 y''(x) + a_1 y'(x) + y(x) = \int_0^{\infty} y(\alpha x) \mu(d\alpha), \quad (1.6)$$

where $a_1 \geq 0$, $a_2 \geq 0$, $a_1^2 + a_2^2 > 0$ (so that a_1, a_2 do not vanish simultaneously), and μ is a probability measure on $(0, \infty)$,

$$\mu(0, \infty) = \int_0^{\infty} \mu(d\alpha) = 1. \quad (1.7)$$

⁴Note that the theory of such equations is closely related to the theory of q -difference equations developed by Birkhoff [3] and Adams [1] (see also Section 3 below).

⁵In fact, the results of the paper [7] mentioned above include first-order equations of the form (1.6), i.e., with $a_2 = 0$. Let us also remark that more general first-order integro-differential equations (but with delayed arguments only, i.e., $\alpha \in (0, 1)$) were considered by Iserles and Liu [15].

The parameter α in Eq. (1.6) can be viewed as a random variable, with values in $(0, \infty)$ and the probability distribution given by the measure μ , i.e., $\mathbb{P}\{\alpha \in A\} = \mu(A)$, $A \subset (0, \infty)$. Note that Eq. (1.6) is balanced in the same sense as Eq. (1.3), since the mean contribution of the distributed rescaled term $y(\alpha x)$ is matched by that of $y(x)$. Moreover, Eq. (1.6) reduces to Eq. (1.3) when $a_1 = 1$, $a_2 = 0$, and the measure μ is discrete, with atoms $p_j = \mu(\alpha_j)$, $j = 1, \dots, \ell$ (i.e., α is a discrete random variable, with the distribution $\mathbb{P}\{\alpha = \alpha_j\} = p_j$, $j = 1, \dots, \ell$).

As already mentioned, due to the balance condition (1.7) any constant satisfies Eq. (1.6), and by linearity of the equation one can assume, without loss of generality, that $y(0) = 0$. Moreover, if $x > 0$ ($x < 0$) then the right-hand side of Eq. (1.6) is determined solely by the values of the function $y(u)$ with $u > 0$ (respectively, $u < 0$). Therefore, the two-sided equation (1.6) is decoupled at $x = 0$ into two one-sided boundary value problems,

$$\begin{aligned} -a_2 y''(x) + a_1 y'(x) + y(x) &= \int_0^\infty y(\alpha x) \mu(d\alpha), & x \geq 0, \\ y(0) &= 0. \end{aligned} \quad (1.8)$$

For Eq. (1.6), the analog of Eq. (1.5) is given by

$$K := \int_0^\infty \ln \alpha \mu(d\alpha) = \mathbb{E}[\ln \alpha]. \quad (1.9)$$

Our main result is the following theorem, which resolves the problem of nontrivial bounded solutions in the critical case, $K = 0$ (and also recovers and extends the result of Ref. [7] for the case $K < 0$).

THEOREM 1.1. *Assume that $0 \neq \mathbb{E}|\ln \alpha| < \infty$, so that K in Eq. (1.9) is well defined and the measure μ is not concentrated at the point $\alpha = 1$, i.e., the random variable α does not degenerate to the constant 1. Under these hypotheses, the condition $K \leq 0$ implies that any bounded solution of equation (1.6) is trivial, i.e., $y(x) \equiv \text{const}$, $x \in \mathbb{R}$.*

The apparent probabilistic structure of Eq. (1.6) is crucial for our proof of this result. The main idea is to construct a certain diffusion process X_t , with negative drift and “multiplication” jumps (i.e., of the form $x \mapsto \alpha x$), such that Eq. (1.6) can be rewritten as $\mathcal{L}y = 0$, where \mathcal{L} is the infinitesimal generator of the Markov process X_t . That is to say, the class of bounded solutions of Eq. (1.6) coincides with the set of bounded \mathcal{L} -harmonic functions. This link brings in the powerful tool kit of Markov processes; particularly instrumental is the well-known fact (see, e.g., Ref. [9]) that for any \mathcal{L} -harmonic function $f(x)$, the random process $f(X_t)$ is a martingale, and hence, for any $t \geq 0$,

$$f(x) = \mathbb{E}[f(X_t)|X_0 = x], \quad x \in \mathbb{R}. \quad (1.10)$$

On the other hand, due to the multiplication structure of independent consecutive jumps of the process X_t , its position after n jumps is expressed

in terms of a background random walk $S_k = \xi_1 + \dots + \xi_k$ ($0 \leq k \leq n$), where ξ_i 's are independent random variables with the same distribution as $\ln \alpha$. The hypothesis $K \leq 0$ of Theorem 1.1 implies that, almost surely (a.s.), the random walk S_n travels arbitrarily far to the left. Using an optional stopping theorem (whereby the boundedness of $f(x)$ is important), we can apply the martingale identity (1.10) at the suitably chosen stopping (first-passage) times, which eventually leads to the conclusion that $f'(x) \equiv 0$ and hence $f(x) \equiv \text{const}$.

This approach also allows us to give an example of a nontrivial bounded solution to equation (1.6) in the case $K > 0$ (thus extending the result by Derfel [7] to the second-order pantograph equation).

THEOREM 1.2. *Suppose that $K > 0$, and set*

$$f_\infty(x) := \mathbf{P}\left\{\liminf_{t \rightarrow \infty} X_t = +\infty \mid X_0 = x\right\}, \quad x \in \mathbb{R}, \quad (1.11)$$

where X_t is the random process constructed in the proof of Theorem 1.1. Then the function $f_\infty(x)$ is \mathcal{L} -harmonic and such that $f_\infty(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $f_\infty(x) \rightarrow 1$ as $x \rightarrow +\infty$.

When $a_2 = 0$, Eq. (1.6) becomes

$$a_1 y'(x) + y(x) = \int_0^\infty y(\alpha x) \mu(d\alpha), \quad x \in \mathbb{R}. \quad (1.12)$$

In this case, the diffusion component of the random process X_t is switched off, and it follows, due to the negative drift and multiplication jumps (see details in Section 4), that if $X_0 = x \leq 0$, then $X_t \leq 0$ for all $t \geq 0$. That is to say, the negative semi-axis $(-\infty, 0]$ is an absorbing set for the process X_t , and hence the function $f_\infty(x)$, defined by Eq. (1.11) as the probability to escape to $+\infty$ starting from x , vanishes for all $x \leq 0$. This leads to the following interesting specification of the example in Theorem 1.2.

COROLLARY 1.1. *If $a_2 = 0$ in Eq. (1.6) then $f_\infty(x) \equiv 0$ for all $x \leq 0$. Moreover, Eq. (1.6) implies that all derivatives of the function $f_\infty(x)$ vanish at zero, $f_\infty^{(k)}(0) = 0$ ($k = 1, 2, \dots$).*

Before elaborating on the ideas outlined above, we would like to make a short digression in order to consider the simple prototype example of Eq. (1.3), namely,

$$y'(x) + y(x) = \frac{1}{2} y(qx) + \frac{1}{2} y(q^{-1}x) \quad (q \neq 1), \quad (1.13)$$

and to give several different “sketch” proofs of Theorem 1.1 in this case. Note that, according to Eq. (1.5), we have

$$K = \frac{1}{2} \ln q + \frac{1}{2} \ln \frac{1}{q} = 0,$$

so Eq. (1.13) falls in the (most interesting) critical case. In fact, this example was the starting point of our work and a kind of mathematical test-tube

to try various approaches and ideas. Although not strictly necessary for the exposition, after some deliberation we have cautiously decided to include our early proofs (based on perturbation, analytical, and probabilistic arguments,⁶ respectively), partly because this will hopefully equip the reader with some insight into validity of the result, and also because these methods may appear useful as exploratory tools in other situations.

The rest of the paper is laid out as follows. In Sections 2, 3, and 4, we discuss the three approaches to equation (1.13) as just mentioned. In Section 5 we start a more systematic treatment by describing the construction of a suitable diffusion process with multiplication jumps. In Section 6, we discuss the corresponding \mathcal{L} -harmonic functions and obtain an *a priori* bound for the derivative of a solution. Finally, in Section 7 we prove our main Theorems 1.1 and 1.2.

2. Perturbative proof. Following the ideas used by Ockendon and Tayler [24] and Fox *et al.* [11] in the case of the original pantograph equation (1.1), we start by observing that if $q = 1$ then Eq. (1.13) is reduced to the equation $y' = 0$, which has constant solutions only. Therefore, when the parameter q is close to 1, it is reasonable to seek solutions of Eq. (1.13) in a form that involves “superposition” of (exponentially) small oscillations (*fast variation*) on top of an almost constant (polynomial) function (*slow variation*). This leads to a WKB-type asymptotic expansion of the solution in terms of perturbation parameter $\varepsilon \approx 0$ (see Ref. [25]), which in the first-order approximation yields two first-order differential equations: a nonlinear equation (called the *eikonal equation*) for the fast variation and a linear equation (called the *transport equation*) for the slow variation. For simplicity of presentation, we will restrict ourselves to the first-order approximation, but in principle one can go on to the analysis of higher-order terms, which are described by linear equations and therefore can be determined without much trouble.

To implement this approach, set $q = 1 \pm \varepsilon$ ($\varepsilon > 0$), $x = \varepsilon^{-1}u$, and $y(x) = y(\varepsilon^{-1}u) =: f(u)$. Then

$$y'(x) = \frac{df}{du} \cdot \frac{du}{dx} = \varepsilon f'(u),$$

and Eq. (1.13) takes the form

$$\varepsilon f'(u) + f(u) = \frac{1}{2} f((1 \pm \varepsilon)u) + \frac{1}{2} f((1 \pm \varepsilon)^{-1}u). \quad (2.1)$$

As explained in the Introduction, without loss of generality we may assume that $f(0) = 0$.

⁶It is amusing that these three methods represent nicely the traditional organization of British mathematics into Applied Mathematics, Pure Mathematics, and Statistics, which is reflected in the names of mathematical departments in most universities in the UK.

Now, suppose that, for small ε , the function $f(u)$ admits a WKB-type expansion,

$$f(u) \sim (A_0(u) + \varepsilon A_1(u) + \dots) \exp(\varepsilon^{-1}V(u)), \quad (2.2)$$

which in principle should be valid uniformly for all u , including the limiting values $u \rightarrow 0$ and $u \rightarrow \infty$. Differentiation of Eq. (2.2) yields

$$\varepsilon f'(u) \sim (A_0(u)V'(u) + \varepsilon(A_0'(u) + A_1(u)V'(u)) + \dots) \exp(\varepsilon^{-1}V(u)). \quad (2.3)$$

From Eq. (2.2) we also obtain

$$\begin{aligned} f((1 \pm \varepsilon)u) &\sim (A_0(u) + \varepsilon(A_1(u) \pm uA_0'(u)) + \dots) \\ &\times \exp\left(\frac{V(u)}{\varepsilon} \pm uV'(u) + \frac{\varepsilon u^2 V''(u)}{2} + \dots\right), \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} f((1 \pm \varepsilon)^{-1}u) &\sim (A_0(u) + \varepsilon(A_1(u) \mp uA_0'(u)) + \dots) \\ &\times \exp\left(\frac{V(u)}{\varepsilon} \mp uV'(u) + \varepsilon\left(uV'(u) + \frac{u^2 V''(u)}{2}\right) + \dots\right). \end{aligned} \quad (2.5)$$

Substituting the expansions (2.3), (2.4) and (2.5) into Eq. (2.1), canceling out the common factor $\exp(V(u)/\varepsilon)$, and collecting the terms that remain after setting $\varepsilon = 0$, we get

$$A_0(u)V'(u) + A_0(u) = \frac{1}{2} A_0(u) \exp(\pm uV'(u)) + \frac{1}{2} A_0(u) \exp(\mp uV'(u)).$$

Assuming that $A_0(u) \neq 0$, this gives the equation

$$1 + V'(u) = \cosh(uV'(u)), \quad (2.6)$$

or equivalently

$$u = \frac{uV'(u)}{\cosh(uV'(u)) - 1}. \quad (2.7)$$

Similarly, equating the terms of order of ε and noting that $A_1(u)$ cancels out owing to Eq. (2.6), we obtain

$$\begin{aligned} A_0'(u) &= A_0'(u) u \sinh(uV'(u)) + \frac{1}{2} A_0(u) u^2 V''(u) \cosh(uV'(u)) \\ &\quad + \frac{1}{2} A_0(u) uV'(u) \exp(\mp uV'(u)) \end{aligned} \quad (2.8)$$

We can now check that the formal expansion (2.2) is compatible with the zero initial condition, $f(0) = 0$. Equation (2.7) implies that if $u \rightarrow 0$ then $uV'(u) \rightarrow \infty$, and moreover

$$uV'(u) \sim -\ln u + \ln \ln \frac{1}{u} + \dots, \quad (2.9)$$

whence

$$V(u) \sim -\frac{\ln^2 u}{2} + \ln u \cdot \ln \ln \frac{1}{u} + \dots \quad (2.10)$$

Furthermore, differentiation of Eq. (2.9) gives

$$u^2 V''(u) \sim \ln u - \ln \ln \frac{1}{u} + \dots \quad (2.11)$$

Inserting formulas (2.9) and (2.11) into Eq. (2.8), we obtain for $A_0(u)$ the asymptotic differential equation

$$\frac{A_0'(u)}{A_0(u)} \sim \mp \frac{1}{2u} \left(\ln u - \ln \ln \frac{1}{u} + \dots \right),$$

which solves to

$$\ln A_0(u) \sim \mp \frac{\ln u}{4} \left(\ln u - 2 \ln \ln \frac{1}{u} + \dots \right). \quad (2.12)$$

Finally, substituting the expansions (2.10) and (2.12) into Eq. (2.2) we obtain that $f(u) \rightarrow 0$ as $u \rightarrow 0$, as required.

Let us now explore the behavior of the solution as $u \rightarrow \infty$. In this limit, Eq. (2.7) gives $uV'(u) \rightarrow 0$, and moreover

$$u \sim \frac{2}{uV'(u)} - \frac{uV'(u)}{6} + \dots,$$

whence

$$uV'(u) \sim \frac{2}{u} - \frac{2}{3u^3} + \dots, \quad (2.13)$$

$$u^2 V''(u) \sim -\frac{4}{u} + \frac{8}{3u^3} + \dots \quad (2.14)$$

From (2.13), we also find

$$V(u) \sim C - \frac{2}{u} + \frac{2}{9u^3} + \dots, \quad (2.15)$$

where $C = \text{const.}$ Inserting the expansions (2.13) and (2.14) into Eq. (2.8), we obtain

$$\frac{A_0'(u)}{A_0(u)} \sim \frac{1}{u} \pm \frac{2}{u^2} + \dots,$$

and hence

$$A_0(u) \sim C_0 u \exp \left(\mp \frac{2}{u} + \dots \right). \quad (2.16)$$

The case $C_0 \neq 0$ is unsuitable, since Eq. (2.16) would imply that $A_0(u) \rightarrow \infty$ as $u \rightarrow \infty$ and, in view of formulas (2.15) and (2.2), the solution $f(u)$ appears to be unbounded, which contradicts our assumption. Therefore, $C_0 = 0$ and hence $A_0(u) = 0$, thus reducing the expansion (2.2) to

$$f(u) \sim \varepsilon(A_1(u) + \varepsilon A_2(u) + \dots) \exp(\varepsilon^{-1}V(u)).$$

Arguing as above, we successively obtain $A_1(u) = 0$, $A_2(u) = 0$, etc. This indicates that $f(u) = 0$, which was our aim.

3. Analytical proof. In this section, we demonstrate how the theory of q -difference equations (see Refs. [1, 3]) can be used to show that Eq. (1.13) has no nontrivial bounded solutions. In what follows, we assume that $q \neq 1$.

As explained in the Introduction (see Eq. (1.8)), Eq. (1.13) splits into two (similar) one-sided equations, so it suffices to consider the boundary-value problem

$$\begin{aligned} y'(x) + y(x) &= \frac{1}{2} y(qx) + \frac{1}{2} y(q^{-1}x), & x \geq 0, \\ y(0) &= 0. \end{aligned} \quad (3.1)$$

Assume that $y(x)$ is a bounded solution of Eq. (3.1), $|y(x)| \leq B$ ($x \geq 0$), and let $\hat{y}(s)$ be the Laplace transform of $y(x)$,

$$\hat{y}(s) := \int_0^\infty e^{-sx} y(x) dx,$$

then $\hat{y}(s)$ is analytic in the right half-plane, $\Re s > 0$, and

$$|\hat{y}(s)| \leq \frac{B}{\Re s}, \quad \Re s > 0. \quad (3.2)$$

On account of the boundary condition $y(0) = 0$, Eq. (3.1) transforms into

$$(1+s)\hat{y}(s) = \frac{1}{2q}\hat{y}(q^{-1}s) + \frac{q}{2}\hat{y}(qs), \quad (3.3)$$

or, after the substitution $\varphi(s) := s\hat{y}(s)$,

$$(1+s)\varphi(s) = \frac{1}{2}\varphi(q^{-1}s) + \frac{1}{2}\varphi(qs). \quad (3.4)$$

Note that the estimate (3.2) implies

$$|\varphi(s)| \leq \frac{B|s|}{\Re s}, \quad \Re s > 0, \quad (3.5)$$

and in particular $\varphi(s)$ is bounded in the vicinity of the origin.

Let us rewrite Eq. (3.4) in the form

$$\varphi(q^2s) - 2(1 + qs)\varphi(qs) + \varphi(s) = 0. \quad (3.6)$$

Equation (3.6) is a linear q -difference equation of order 2. According to the general theory of such equations (see Ref. [1]), the characteristic equation for Eq. (3.6) (in the vicinity of $s = 0$) reads

$$q^{2\rho} - 2q^\rho + 1 = 0,$$

and $\rho_{1,2} = 0$ is its multiple root. The corresponding fundamental set of solutions to Eq. (3.6) is given by

$$\begin{aligned} \varphi_1(s) &= P_1(s), \\ \varphi_2(s) &= P_2(s) + \frac{\ln s}{\ln q} P_3(s), \end{aligned}$$

where $P_1(s)$, $P_2(s)$, and $P_3(s)$ are generic power series convergent in some neighborhood of zero, and $\ln s$ denotes the principal branch of the logarithm. Note that the solution $\varphi_2(s)$ is unsuitable because it is unbounded near $s = 0$ (see Eq. (3.5)). On the other hand, the function $\varphi_1(s)$ is analytic in the vicinity of zero and, moreover, it can be analytically continued, step by step, into the whole complex plane \mathbb{C} by means of Eq. (3.4). For instance, if $q > 1$ then the analytic continuation from a disk $|s| \leq a$ to the bigger disk $|s| \leq qa$ is furnished by the formula $\varphi(qs) = 2(1 + s)\varphi(s) - \varphi(q^{-1}s)$ (see Eq. (3.4)), and so on.

That is to say, $\varphi_1(s)$ can be extended to an entire function $\varphi(s)$, which by construction satisfies Eq. (3.6) for all $s \in \mathbb{C}$. But then, according to one result by Mason [21], the entire function $\varphi(s)$ must be of *zero order*, and consequently (see Ref. [27, §8.7.3]) it is *unbounded* on any ray (in particular, for $s \in (0, \infty)$), unless it is a constant. However, the unboundedness for real $s > 0$ contradicts the estimate (3.5). Hence, $\varphi(s) = \text{const}$, so that $\hat{y}(s) = \text{const} \cdot s^{-1}$, and by the uniqueness theorem for the Laplace transform this implies that $y(x) \equiv \text{const}$, i.e. $y(x) \equiv y(0) = 0$, as claimed.

4. Probabilistic proof. In this section, we give a probabilistic interpretation of Eq. (1.13) via a certain ruin problem, and prove that the corresponding solution is constant using elementary probabilistic considerations. Although our argument does not cover the whole class of bounded solutions, it contains some ideas that we will use in the second half of the paper to give a complete proof of our general result.

Let us consider the following “double-or-half” gambling model (in continuous time). Suppose that a player spends his initial capital, x , at rate v per unit time, so that after time t he is left with capital $x - vt$. However, at a random time τ (with exponential distribution), he gambles by putting the remaining capital at stake, whereby he can either *double* his money or lose *half* of it, both with probability $1/2$. After that, the process continues

in a similar fashion, independently of the past history. If the capital reaches zero and then moves down to become negative, this is interpreted as borrowing, so the process proceeds in the same way without termination. In that case, gambling will either double or halve the debt, and in particular the capital will remain negative forever.

More generally, if X_t denotes the player's capital at time $t \geq 0$, starting with the initial amount $X_0 = x$, then the random process X_t moves with constant negative drift $(-v)$, interrupted at random time instants σ_i by random multiplication jumps from its location $x_i = X_{\sigma_i-0}$ (i.e., immediately before the jump) to either qx_i or $q^{-1}x_i$ ($q \neq 1$), both with probability $1/2$. We assume that the jumps occur at the arrival times $\sigma_1, \sigma_2, \dots$ of an auxiliary Poisson process with parameter $\lambda > 0$, so that the waiting times until the next jump, $\tau_i = \sigma_i - \sigma_{i-1}$ ($\sigma_0 := 0$), are independent identically distributed (i.i.d.) random variables, each with the exponential distribution

$$\mathbb{P}\{\tau > t\} = e^{-\lambda t}, \quad t > 0.$$

According to this description, (X_t) is a Markov process, in that the probability law of its future development is completely determined by its current state, but not by the past history ("lack of memory") (see, e.g., Refs. [9, 10]).

We are concerned with the *ruin problem* for this model.⁷ Namely, consider the probability $f_0(x)$ of becoming bankrupt starting with the initial capital x ,

$$f_0(x) := \mathbb{P}\left\{\liminf_{t \rightarrow \infty} X_t \leq 0 \mid X_0 = x\right\} \equiv \mathbb{P}_x\{T_0 < \infty\}, \quad x \in \mathbb{R},$$

where $T_0 := \min\{t \geq 0 : X_t \leq 0\}$ is the random time to bankruptcy and \mathbb{P}_x denotes the probability measure conditioned on the initial state $X_0 = x$.

From the definition of the process X_t , it is clear that if $x \leq 0$ then $T_0 = 0$ and so $f_0(x) = 1$. For $x > 0$, we note that if the first jump does not occur prior to time x/v , then the process will simply drift down to 0, in which case $T_0 = x/v < \infty$. Otherwise (i.e., if a jump does happen before time x/v), the ruin problem may be reformulated by treating the landing point after the jump as a new starting point (thanks to the Markov property). More precisely, by conditioning on the first jump instant σ_1 ($= \tau_1$) and using the (strong) Markov property, we obtain

⁷A similar ruin problem for the process with deterministic multiplication jumps of the form $x_i \mapsto qx_i$ ($q > 1$), was first considered by Gaver [12], leading to the equation $y'(x) + y(x) = y(qx)$ (i.e., with advanced argument, cf. Eq. (1.3)). The systematic theory of general processes with multiplication jumps was developed by Lev [18].

$$\begin{aligned}
f_0(x) &= \mathbf{P}_x\{\sigma_1 > x/v\} + \int_0^{x/v} \lambda e^{-\lambda s} \mathbf{P}_x(T_0 < \infty \mid \sigma_1 = s) ds \\
&= e^{-\lambda x/v} + \lambda \int_0^{x/v} e^{-\lambda s} \frac{1}{2} \left(f_0(q(x-vs)) + f_0(q^{-1}(x-vs)) \right) ds \quad (4.1) \\
&= e^{-\lambda x/v} \left(1 + \frac{\lambda}{2v} \int_0^x e^{\lambda u/v} (f_0(qu) + f_0(q^{-1}u)) du \right),
\end{aligned}$$

where in the last line we have made the substitution $u = x - vs$. The representation (4.1) implies that the function $f_0(x)$ is continuous and, moreover, (infinitely) differentiable, and by differentiation of Eq. (4.1) with respect to x , it follows that the function $y = f_0(x)$ satisfies the generalized pantograph equation (cf. Eq. (1.13))

$$\frac{v}{\lambda} y'(x) + y(x) = \frac{1}{2} y(qx) + \frac{1}{2} y(q^{-1}x). \quad (4.2)$$

It is easy to see that, in fact, this equation is satisfied on the whole axis, $x \in \mathbb{R}$. As we have mentioned, $f_0(x) = 1$ for all $x \leq 0$, and it is now our aim to show that the same is true for all $x > 0$, which would mean that the solution $y = f_0(x)$ to equation (1.13) is a constant, $f_0(x) \equiv 1$, $x \in \mathbb{R}$.

To this end, note that the position of the process after n jumps is given by

$$X_{\sigma_n} = (X_{\sigma_{n-1}} - v\tau_n) q^{\xi_n}, \quad n = 1, 2, \dots,$$

where ξ_n 's are i.i.d. random variables taking the values ± 1 with probabilities $1/2$. By iterations (using that $X_0 = x$), we obtain⁸

$$\begin{aligned}
X_{\sigma_n} &= \left(((x - v\tau_1) q^{\xi_1} - v\tau_2) q^{\xi_2} - \dots - v\tau_n \right) q^{\xi_n} \\
&= (x - v\tau_1) q^{\xi_1 + \xi_2 + \dots + \xi_n} - v\tau_2 q^{\xi_2 + \dots + \xi_n} - \dots - v\tau_n q^{\xi_n} \quad (4.3) \\
&= q^{S_n} \left(x - v \sum_{i=1}^n \tau_i q^{-S_{i-1}} \right),
\end{aligned}$$

where $S_n := \xi_1 + \xi_2 + \dots + \xi_n$, $S_0 := 0$. Note that S_n can be interpreted as a (simple) random walk, which in our case is symmetric (i.e., $\mathbf{P}\{\xi_i = 1\} = \mathbf{P}\{\xi_i = -1\} = 1/2$) and therefore *recurrent* (see, e.g., Ref. [10]). In particular, the events $A_n := \{S_{n-1} = 0\}$ ($n = 1, 2, \dots$) occur infinitely often, with probability 1. Furthermore, setting $B_n := \{\tau_n > 1\}$, we note that the events $A_n \cap B_n$ ($n = 1, 2, \dots$) are conditionally independent, given the realization of the random walk $\{S_k, k \geq 1\}$. Since the random variables τ_n (and therefore the events B_n) are independent of $\{S_k\}$, we have, with probability 1,

⁸Similar random sums as in Eq. (4.3) arise in products of certain random matrices in relation to random walks on the group of affine transformations of the line (see Ref. [13]).

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(A_n \cap B_n | \{S_k\}) &= \sum_{n=1}^{\infty} \mathbb{P}(B_n) \mathbf{1}_{A_n} \\ &= e^{-\lambda} \#\{n : A_n \text{ occurs}\} = \infty, \end{aligned}$$

where $\mathbf{1}_{\{\dots\}}$ denotes the indicator of an event. Hence, Borel-Cancelli's lemma (see, e.g., Ref. [10]) implies

$$\mathbb{P}(A_n \cap B_n \text{ occur infinitely often} | \{S_k\}) = 1 \quad \text{a.s.}, \quad (4.4)$$

and by taking the expectation in Eq. (4.4) (with respect to the distribution of the sequence $\{S_k\}$), the same is true in the unconditional form,

$$\mathbb{P}(A_n \cap B_n \text{ occur infinitely often}) = 1.$$

As a consequence, the terms in the random series

$$\sum_{n=1}^{\infty} \tau_n q^{-S_{n-1}}$$

will infinitely often exceed the value 1, all other terms being nonnegative. Therefore, the series diverges to $+\infty$ a.s., and from (4.3) it follows that

$$\liminf_{n \rightarrow \infty} X_{\sigma_n} \leq 0 \quad \text{a.s.}$$

In turn, this implies that $T_0 < \infty$ a.s., and so $f_0(x) = 1$ for all $x > 0$, as claimed.

5. Jump diffusions. We now pursue a more general (and more systematic) approach. Equations of the form (1.2) are linked in a natural way with certain continuous-time Markov processes (more specifically, diffusions with multiplication jumps). To describe this class of processes, let us consider a Brownian motion $B_t^{\kappa, v}$, starting at the origin, with diffusion coefficient $\kappa \geq 0$ and nonpositive (constant) drift $-v \leq 0$,

$$dB_t^{\kappa, v} = \kappa dB_t - v dt, \quad B_0^{\kappa, v} = 0,$$

or equivalently

$$B_t^{\kappa, v} = \kappa B_t - vt, \quad t \geq 0,$$

where $B_t = B_t^{1, 0}$ is a standard Brownian motion (with continuous sample paths). We assume that $\kappa^2 + v^2 > 0$, so that $B_t^{\kappa, v}$ does not degenerate to a (zero) constant.

The random process $B_t^{\kappa, v}$ determines the underlying diffusion dynamics for a process with jumps, $(X_t, t \geq 0)$, which is defined as follows. Suppose that the jump instants are given by the arrival times $\sigma_1, \sigma_2, \dots$ of

an auxiliary Poisson process with parameter $\lambda > 0$, so that $\tau_i = \sigma_i - \sigma_{i-1}$ ($i = 1, 2, \dots$) are i.i.d. random variables with exponential distribution,

$$P\{\tau > t\} = e^{-\lambda t}, \quad t > 0$$

(we set formally $\sigma_0 := 0$). Furthermore, suppose that the successive jumps are determined by the rescaling coefficients α_i of the form $\alpha_i = e^{\xi_i}$, where ξ_i 's are i.i.d. random variables. Then, the (right-continuous) sample paths of the process X_t are defined inductively by

$$X_t = \begin{cases} x + B_t^{\kappa, v}, & 0 = \sigma_0 \leq t < \sigma_1, \\ e^{\xi_i} X_{\sigma_i-0} + B_t^{\kappa, v} - B_{\sigma_i}^{\kappa, v}, & \sigma_i \leq t < \sigma_{i+1}, \quad i = 1, 2, \dots \end{cases} \quad (5.1)$$

That is to say, the process $(X_t, t \geq 0)$ starts at point x and moves as $X_t = x + B_t^{\kappa, v}$ until a random time σ_1 , when it jumps to a random point

$$X_{\sigma_1} = e^{\xi_1} X_{\sigma_1-0} = e^{\xi_1} (x + B_{\sigma_1}^{\kappa, v}) = e^{\xi_1} x + e^{\xi_1} \zeta_1,$$

where $\zeta_1 := B_{\sigma_1}^{\kappa, v} - B_{\sigma_0}^{\kappa, v} = B_{\sigma_1}^{\kappa, v} - B_{\sigma_0}^{\kappa, v}$. Thereafter, the process proceeds in a diffusive way as $X_t = X_{\sigma_1} + B_t^{\kappa, v} - B_{\sigma_1}^{\kappa, v}$ until a random time σ_2 , when it makes the next jump to

$$\begin{aligned} X_{\sigma_2} &= e^{\xi_2} X_{\sigma_2-0} = e^{\xi_2} (X_{\sigma_1} + B_{\sigma_2}^{\kappa, v} - B_{\sigma_1}^{\kappa, v}) \\ &= e^{\xi_2 + \xi_1} x + e^{\xi_2 + \xi_1} \zeta_1 + e^{\xi_2} \zeta_2, \end{aligned}$$

where $\zeta_2 := B_{\sigma_2}^{\kappa, v} - B_{\sigma_1}^{\kappa, v}$, and so on. Iterating, we obtain that the n -th jump, occurring at time σ_n , lands at the point

$$\begin{aligned} X_{\sigma_n} &= e^{\xi_n} X_{\sigma_n-0} = e^{\xi_n} (X_{\sigma_{n-1}} + B_{\sigma_n}^{\kappa, v} - B_{\sigma_{n-1}}^{\kappa, v}) \\ &= e^{S_n} (x + \zeta_1 + \zeta_2 e^{-S_1} + \dots + \zeta_n e^{-S_{n-1}}), \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} \zeta_n &:= B_{\sigma_n}^{\kappa, v} - B_{\sigma_{n-1}}^{\kappa, v} \\ &= \kappa(B_{\sigma_n} - B_{\sigma_{n-1}}) - v(\sigma_n - \sigma_{n-1}) \quad (n = 1, 2, \dots) \end{aligned} \quad (5.3)$$

is a sequence of i.i.d. random variables.

6. \mathcal{L} -harmonic functions. Let us study the properties of the process X_t in greater detail.

DEFINITION 6.1. *The infinitesimal operator (generator) \mathcal{L} of a Markov random process $(X_t, t \geq 0)$ is defined by*

$$(\mathcal{L}f)(x) := \lim_{h \rightarrow 0+} \frac{\mathbb{E}_x[f(X_h)] - f(x)}{h}, \quad (6.1)$$

with the domain $\mathcal{D}(\mathcal{L})$ consisting of functions f for which the limit in Eq. (6.1) exists.

In a standard way, by considering possible scenarios for the process (X_t) up to an infinitesimal time h (see Ref. [9]), one obtains the following.

PROPOSITION 6.1. *For the random diffusion with jumps (X_t) defined above, its generator \mathcal{L} acts on bounded C^2 -smooth functions as*

$$(\mathcal{L}f)(x) = \frac{\kappa^2}{2} f''(x) - v f'(x) + \lambda(\mathbb{E}[f(e^\xi x)] - f(x)), \quad (6.2)$$

where ξ is a random variable with the same distribution as any one of the i.i.d. random variables ξ_1, ξ_2, \dots .

DEFINITION 6.2. *A function f is called \mathcal{L} -harmonic if $\mathcal{L}f = 0$.*

Note that the expectation in Eq. (6.2) can be written as a Stieltjes integral,

$$\mathbb{E}[f(e^\xi x)] = \int_0^\infty f(zx) dF(z),$$

where $F(z)$ is the cumulative distribution function of the random variable $Z = e^\xi$, i.e., $F(z) := \mathbb{P}\{e^\xi \leq z\} = \mathbb{P}\{\xi \leq \ln z\}$ ($0 < z < \infty$). Hence, the equation $\mathcal{L}f = 0$, with \mathcal{L} given by Eq. (6.2), is equivalent to

$$-\frac{\kappa^2}{2} f''(x) + v f'(x) + \lambda f(x) = \lambda \int_0^\infty f(zx) dF(z),$$

which is a balanced generalized pantograph equation of the form (1.6).

Our aim is to study the class of bounded \mathcal{L} -harmonic functions. Let us denote by $\|\cdot\|$ the standard sup-norm on \mathbb{R} :

$$\|f\| := \sup_{x \in \mathbb{R}} |f(x)|.$$

As a first step, we estimate the derivative of an \mathcal{L} -harmonic function.

PROPOSITION 6.2. *Suppose that $v > 0$. If $\|f\| < \infty$ and $\mathcal{L}f = 0$ then $\|f'\| < \infty$.*

Proof. If $\kappa = 0$ then, according to Eq. (6.2), the equation $\mathcal{L}f = 0$ takes the form

$$v f'(x) = \lambda(\mathbb{E}[f(e^\xi x)] - f(x)),$$

which gives

$$\|f'\| \leq \frac{2\lambda}{|v|} \|f\| < \infty.$$

For $\kappa > 0$, the condition $\mathcal{L}f = 0$ is equivalent to

$$f''(x) - \gamma f'(x) = -g(x), \quad (6.3)$$

where

$$\gamma := \frac{2v}{\kappa^2} > 0, \quad g(x) := \frac{2\lambda}{\kappa^2} (\mathbb{E}[f(e^\xi x)] - f(x)).$$

Solving equation (6.3), we obtain

$$\begin{aligned} f'(x) &= \int_x^\infty g(u) e^{-\gamma(u-x)} du + C e^{\gamma x} \\ &= \int_0^\infty g(u+x) e^{-\gamma u} du + C e^{\gamma x}, \end{aligned} \tag{6.4}$$

where $C = \text{const}$. Since $\|g\| \leq 4\lambda\kappa^{-2}\|f\| < \infty$, Eq. (6.4) implies that if $C \neq 0$ then

$$f'(x) = O(1) + C e^{\gamma x} \rightarrow \infty \quad (x \rightarrow +\infty),$$

so that $\lim_{x \rightarrow +\infty} f(x) = \infty$, which contradicts the assumption $\|f\| < \infty$. Therefore, $C = 0$ and

$$\|f'\| \leq \|g\| \int_0^\infty e^{-\gamma u} du = \frac{\|g\|}{\gamma} < \infty.$$

The proof is completed. \square

7. Proof of the main results. Let $(\mathcal{F}_t, t \geq 0)$ be the natural filtration generated by the process (X_t) , i.e., $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$ is the minimal σ -algebra containing all “level” events $\{X_s \leq c\}$ ($c \in \mathbb{R}, s \leq t$). Intuitively, \mathcal{F}_t is interpreted as the collection of all the information that can be obtained by observation of the random process (X_s) up to time t . As is well known (see, e.g., Ref. [9, Ch. 4]), if a function f is \mathcal{L} -harmonic then the random process $f(X_t)$ is a *martingale* relative to (\mathcal{F}_t) , i.e., for any $0 \leq s \leq t$, with probability 1,

$$\mathbb{E}_x[f(X_t) | \mathcal{F}_s] = f(X_s). \tag{7.1}$$

In words, this means that if we are trying to predict the mean value of the martingale $f(X_t)$ at some future time t using its past history up to time $s \leq t$, then the best estimate is given by $f(X_s)$, i.e., the value of the process at the latest available time instant s .

In particular, taking expectation of both sides of Eq. (7.1) at $s = 0$ gives

$$\mathbb{E}_x[f(X_t)] = f(x), \quad t \geq 0. \tag{7.2}$$

In addition, if f is bounded then, by Doob’s optional stopping theorem (see, e.g., Ref. [29, §8]), Eq. (7.2) extends to

$$\mathbb{E}_x[f(X_T)] = f(x), \tag{7.3}$$

where T is a random *stopping time*, i.e., such that $\{T \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. In words, one should be able to decide whether the random time T has occurred by observing the process up to a given time. The martingale

property (7.3) is crucial in the proof of our main theorem below, where we will apply it to the special sequence of stopping times.

THEOREM 7.1 (cf. Theorem 1.1). *Assume that $\mathbb{E}[\xi] \leq 0$ and $\mathbb{P}\{\xi \neq 0\} > 0$. If $\mathcal{L}f = 0$ and $\|f\| < \infty$, then $f(x) \equiv \text{const}$, $x \in \mathbb{R}$.*

Proof. For $r > 0$, set $N_r := \min\{n : S_n \leq -r\}$, where $S_n = \xi_1 + \dots + \xi_n$, and define

$$T_r := \sigma_{N_r} = \sum_{n=1}^{\infty} \sigma_n \mathbf{1}_{\{N_r=n\}}, \quad (7.4)$$

where (σ_i) are the time instants of successive jumps (see Section 3). The assumption $\mathbb{E}[\xi] \leq 0$ implies (see, e.g., Ref. [10]) that $\liminf_{n \rightarrow \infty} S_n = -\infty$ a.s., so that $N_r < \infty$ a.s. and therefore the random variable T_r is well defined.

Note that T_r is a stopping time for the random process (X_t) , i.e., $\{T_r \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. Indeed, suppose that we are given a sample path of the process X_s up to time t , and in particular we know the time instants σ_i and the magnitudes of all the jumps prior to t . Then, using Eq. (5.2), we can reconstruct the corresponding values⁹ $\xi_i = \ln(X_{\sigma_i}/X_{\sigma_i-0})$. In turn, this allows us to determine if the threshold $(-r)$ has been reached by the associated random walk S_n and, therefore, whether or not the condition $T_r \leq t$ holds, as required.

Now, applying the optional stopping theorem in the form (7.3), we obtain

$$f(x) = \mathbb{E}_x[f(X_{T_r})] = \mathbb{E}[f(e^{S_{N_r}} x + \text{terms independent of } x)]. \quad (7.5)$$

First, suppose that $v > 0$. Differentiation of Eq. (7.5) with respect to x gives

$$f'(x) = \mathbb{E}[e^{S_{N_r}} f'(e^{S_{N_r}} x + \dots)], \quad (7.6)$$

whence, using Proposition 6.2, we get

$$\|f'\| \leq e^{-r} \|f'\| < \infty.$$

Letting here $r \rightarrow +\infty$, we conclude that $\|f'\| = 0$ and so $f = \text{const}$.

If $v = 0$ then, differentiating Eq. (7.6) we get

$$f''(x) = \mathbb{E}[e^{2S_{N_r}} f''(e^{S_{N_r}} x + \dots)],$$

so that

$$\|f''\| \leq e^{-2r} \|f''\| < \infty. \quad (7.7)$$

⁹There is a slight problem if $X_{\sigma_i-0} = X_{\sigma_i} = 0$, but this only happens with zero probability, so may be ignored.

Equation (6.3) (with $\gamma = 0$) implies $\|f''\| \leq \|g\| < \infty$, so taking $r \rightarrow +\infty$ in Eq. (7.7) yields $f''(x) \equiv 0$. Therefore, $f(x) = c_1x + c_0$, but since $\|f\| < \infty$, we must have $c_1 = 0$, so that $f(x) \equiv c_0 = \text{const}$. \square

PROPOSITION 7.1. Denote $A_\infty := \{\liminf_{t \rightarrow \infty} X_t = +\infty\}$, and consider the probability of the event A_∞ as a function of the initial point of the process X_t ,

$$f_\infty(x) := P_x(A_\infty) = P(A_\infty | X_0 = x), \quad x \in \mathbb{R}.$$

Then the function $f_\infty(x)$ is \mathcal{L} -harmonic, i.e.,

$$(\mathcal{L}f_\infty)(x) = 0, \quad x \in \mathbb{R},$$

where \mathcal{L} is the generator of the random process (X_t) given by Eq. (6.2).

Proof. Conditioning on X_h ($< +\infty$), by the Markov property we obtain

$$\begin{aligned} f_\infty(x) &= E_x[P_x(A_\infty | X_h)] \\ &= E_x[P_{X_h}(A_\infty)] \\ &= E_x[f_\infty(X_h)], \end{aligned}$$

whence, by the definition (6.1), it readily follows that $\mathcal{L}f_\infty = 0$. \square

THEOREM 7.2 (cf. Theorem 1.2). Suppose that $E[\xi] > 0$. Then the function $f_\infty(x)$ is a nontrivial bounded \mathcal{L} -harmonic function; in particular, $f_\infty(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $f_\infty(x) \rightarrow 1$ as $x \rightarrow +\infty$.

Proof. The function $f_\infty(x)$ is \mathcal{L} -harmonic by Proposition 7.1. In order to obtain the limits of $f_\infty(x)$ as $x \rightarrow \pm\infty$, note that

$$\begin{aligned} 1 - f_\infty(x) &= P_x\left\{\liminf_{t \rightarrow \infty} X_t < +\infty\right\} \\ &= P_x\left\{\liminf_{n \rightarrow \infty} X_{\sigma_n} < +\infty\right\} \\ &= \lim_{M \rightarrow \infty} P_x\{X_{\sigma_n} \leq M \text{ infinitely often}\}. \end{aligned} \tag{7.8}$$

According to Eq. (5.2), the condition $X_{\sigma_n} \leq M$ can be rewritten as

$$x + \zeta_1 + \zeta_2 e^{-S_1} + \cdots + \zeta_n e^{-S_{n-1}} \leq M e^{-S_n}. \tag{7.9}$$

Since $E[\xi] > 0$, the strong Law of Large Numbers implies that, with probability 1,

$$S_n \sim nE[\xi] \rightarrow +\infty \quad (n \rightarrow \infty). \tag{7.10}$$

It follows that if the inequality (7.9) holds for infinitely many n , then

$$x + \liminf_{n \rightarrow \infty} (\zeta_1 + \zeta_2 e^{-S_1} + \cdots + \zeta_n e^{-S_{n-1}}) \leq \lim_{n \rightarrow \infty} M e^{-S_n} = 0. \tag{7.11}$$

Moreover, using the estimate (7.10) and recalling that ζ_i are i.i.d. random variables (see Eq. (5.3)), it is easy to show (e.g., using Kolmogorov's "three series" theorem, see Ref. [10]) that the random series

$$\eta := \sum_{n=1}^{\infty} \zeta_n e^{-S_{n-1}} \quad (7.12)$$

converges with probability 1. Therefore, from Eqs. (7.9), (7.11) and (7.12) it follows that for any $M > 0$,

$$\mathbb{P}_x\{X_{\sigma_n} \leq M \text{ infinitely often}\} \leq \mathbb{P}\{\eta \leq -x\}.$$

Returning to Eq. (7.8), we deduce that

$$\begin{aligned} 1 \geq f_{\infty}(x) &\geq 1 - \mathbb{P}\{\eta \leq -x\} \\ &= \mathbb{P}\{\eta > -x\} \rightarrow 1 \quad (x \rightarrow +\infty). \end{aligned}$$

On the other hand, writing the left-hand side of Eq. (7.9) as $x + \eta + \delta_n$, where $\delta_n \rightarrow 0$ a.s., we have, for any $\varepsilon > 0$, $M > 0$,

$$\begin{aligned} \mathbb{P}\{x + \eta \leq -\varepsilon\} &\leq \mathbb{P}\{x + \eta + \delta_n \leq 0 \text{ for all } n \text{ large enough}\} \\ &\leq \mathbb{P}\{x + \eta + \delta_n < M e^{-S_n} \text{ infinitely often}\}, \end{aligned}$$

which in view of Eq. (7.8) implies

$$\begin{aligned} 0 \leq f_{\infty}(x) &\leq 1 - \mathbb{P}\{\eta \leq -x - \varepsilon\} \\ &= \mathbb{P}\{\eta > -x - \varepsilon\} \rightarrow 0 \quad (x \rightarrow -\infty). \end{aligned}$$

Thus, the proof is completed. \square

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