PROPERLY $\Sigma_2$ MINIMAL DEGREES AND $0''$ COMPLEMENTATION.

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Abstract. We show that there exists a properly $\Sigma_2$ minimal (Turing) degree $b$, and moreover that $b$ can be chosen to join with $0'$ to $0''$ – so that $b$ is a $0''$ complement for every degree $a$ such that $0' \leq a < 0''$.

1. Introduction.

Jockusch has been able to show that for every $n \geq 3$ there exists a properly $\Sigma_n$ minimal degree, and has asked whether or not there exists a properly $\Sigma_2$ minimal degree. We are able to answer the question positively\(^1\) by using a direct construction relative to $0'$. We shall use this direct construction to produce a properly $\Sigma_2$ minimal degree $b$ such that $0' \lor b = 0''$. For any degree $a$ such that $0' \leq a < 0''$, then, $0'' = a \lor b$ and since $b$ is minimal $0 = a \land b$ so that $b$ is a $0''$ complement for every such degree.

Our notation agrees almost entirely with that of Soare [1987]. We suppose given an oracle for $\emptyset'$ and an enumeration of $\emptyset''$ and we shall enumerate an approximation $\{B_s\}_{s \geq 0}$ to a set $B$ of minimal degree. Our method will consist of what is basically the standard construction of a minimal degree below $0'$ (see Sacks [1963], Shoenfield [1966] and especially Soare [1987]) but with some simple modifications. At any stage $s + 1$ of the construction we shall proceed as follows. At stage $s$ we will have defined a finite nested sequence of computable partial trees $T^*_e$, $e \leq s$, and a finite binary string $B_s$. Associated with the string $B_s$ will be another, $f(B_s)$ say, which may be regarded as a ‘guessed’ initial segment of $\emptyset''$. Let us suppose first of all that $f(B_s) \subset \emptyset''_{s+1}$. Following the standard argument we then let $n_{s+1}$ be the largest integer $\leq s$ such that $B_s$ has a splitting $\tau_0, \tau_1$ on $T^*_{n_{s+1}}$. Order so that $\tau_0$ is the leftmost of $\tau_0, \tau_1$ and define $f(\tau_0) = f(B_s) \cdot 0$, $f(\tau_1) = f(B_s) \cdot 1$. Let $i \in \{0, 1\}$ be such that $f(\tau_i) \subset \emptyset''_{s+1}$ and define $B_{s+1} = \tau_i$. Now suppose that $f(B_s) \not\subset \emptyset''_{s+1}$. Then let $\tau$ be the longest string such that $f(\tau) \downarrow \subset \emptyset''_{s+1}$ and define $B_{s+1} = \tau$.

\(^1\)The existence of a properly $\Sigma_2$ minimal degree was observed some years ago by the first author — see Odifreddi [1999], p.769. At the time that the first and third authors had embarked on writing up this result, Richard Shore (Shore [2004]) informed them that he had also found a solution, and was writing it up for publication. It was only when the second author observed that with almost no extra work one could actually complement with properly $\Sigma_2$ minimal degrees that the present short paper came into existence.
Let \( \{ \Psi_e \}_{e \geq 0} \) be an effective listing of the Turing functionals. Having defined \( B_{s+1} \) we then proceed to define the trees \( T_e^{s+1} \), \( e \leq s + 1 \), in almost the standard fashion except that we must now take account of the fact that possibly \( B_{s+1} \) is incompatible with \( B_s \). It may be the case, for example, that at stage \( s + 1 \) we are able to define \( T_e^{s+1} = T_e^s \), that we had previously found some initial segment of \( B_s \) above which there are no \( \Psi_e \) splittings lying on \( T_e^s \) (if possible we shall define \( T_{e+1}^s \) to be a \( \Psi_e \) splitting tree), but that this is not true of \( B_{s+1} \).

**Definition 1.1.** We assume that for all \( s \), \( |\Psi^{s+1}_e - \Psi^s_e| = 1 \). Let \( t_s, s \geq 1 \), be the \( s \)th true stage in the enumeration of \( \Psi'' \). Let \( t'_s, s \geq 1 \), be the \( s \)th stage \( s' \) such that \( B_s \) or \( B_{s+1} \) is strictly to the right of \( B_s \).

2. The construction.

Stage \( s = 0 \).

Set \( B_0 = \emptyset \). Define \( T_0^0 = \text{Id} \) and \( f(\emptyset) = \emptyset \).

Stage \( s + 1 \).

a) If \( f(B_s) \subset \Psi''_s \). Let \( n_{s+1} \) be the largest integer \( \leq s \) such that \( B_s \) has a splitting \( \tau_0, \tau_1 \) on \( T_n_{s+1}^s \). Order so that \( \tau_0 \) is the leftmost of \( \tau_0, \tau_1 \) and define \( f(\tau_0) = f(B_s) \ast 0, f(\tau_1) = f(B_s) \ast 1 \). Let \( i \in \{ 0, 1 \} \) be such that \( f(\tau_i) \subset \Psi''_{s+1} \) and define \( B_{s+1} = \tau_i \).

Set \( T_{e+1}^s = T_e^s \) for all \( e \leq n_{s+1} \). If \( n_{s+1} = s \) then let \( T_{e+1}^{s+1} \) be the \( \Psi_s \) splitting subtree of \( T_{e+1}^s \) above \( B_{s+1} \). Otherwise (if \( n_{s+1} < s \)) proceed as follows. Choose \( \alpha \) such that \( T_{n_{s+1}}^s(\alpha) = B_{s+1} \), define \( T_{n_{s+1}+1}^s(\gamma) = T_{n_{s+1}+1}^s(\alpha \ast \gamma) \) for all \( \gamma \in 2^{\omega} \) and define \( T_{e+1}^{s+1} \) to be the \( \Psi_s \) splitting subtree of \( T_{e+1}^s \) above \( B_{s+1} \) for all \( e \) such that \( n_{s+1} < e \leq s \).

b) If \( f(B_s) \not\subset \Psi''_{s+1} \). Let \( \tau \) be the longest string such that \( f(\tau) \not\subset \Psi''_{s+1} \) and define \( B_{s+1} = \tau \). Let \( s' > 0 \) be the stage at which we defined \( f(\tau) \). Set \( T_{e+1}^s = T_e^{s'} \) for all \( e \leq n_{s'} \). If \( n_{s'} = s' - 1 \) then define \( T_{e+1}^{s+1} \) to be the \( \Psi_{s'} \) splitting subtree of \( T_{e+1}^{s'} \) above \( B_{s+1} \). If \( n_{s'} < s' - 1 \), choose \( \alpha \) such that \( T_{n_{s'+1}}^{s'}(\alpha) = B_{s+1} \) and define \( T_{n_{s'+1}+1}^{s'}(\gamma) = T_{n_{s'+1}+1}^{s'}(\alpha \ast \gamma) \) for all \( \gamma \in 2^{\omega} \). For all \( e \) such that \( n_{s'} < e \leq s \) define \( T_{e+1}^{s+1} \) to be the \( \Psi_s \) splitting subtree of \( T_{e+1}^s \) above \( B_{s+1} \). Define \( n_{s+1} = n_{s'} \).

3. The verification.

In order to show that \( B \) is properly \( \Sigma_2 \) and \( \emptyset'' \equiv_T \emptyset' \oplus B \) it suffices to prove by induction on \( s \) that the following statements hold:

1. If we define a value \( f(\tau) \) at stage \( s \) then, until the point at which this definition is made, we have not defined any values \( f(\tau') \) for \( \tau' \geq \tau \). If \( \tau' \subset \tau \) and \( f(\tau') \downarrow \) then \( f(\tau') \subset f(\tau) \).
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(2) If $s = t_{s'}$ for some $s' \geq 1$ then $f(B_s)$ is an initial segment of $\emptyset''$ of length $\geq s' - 1$ and $B_s \subseteq B_{s'}$ for all $s'' > s$ so that there are arbitrarily long strings in $\Lambda$.

(3) For all $s > 0$, either $B_s$ extends $B_{s-1}$ or $B_s$ is strictly to the right of $B_{s-1}$.

(4) At the end of stage $s$ we have not defined any values $f(\tau)$ for $\tau \supset B_s$.

That statements (1) to (4) are true for $s = 0$ is clear, so suppose that $s > 0$ and that the result holds for all $s' < s$. First consider the possibility that case a) of the construction applies at stage $s$. That (3) is true at stage $s$ is then immediate. That (1) is true at stage $s$ follows directly from the induction hypothesis on statements (1) and (4) since, if we define a value $f(\tau)$ for any string $\tau$ at stage $s$ then $\tau \supset B_{s-1}$. That (4) is true at stage $s$ follows immediately from the fact that (1) is true at stage $s$. Now suppose that $s = t_{s'}$ for some $s' \geq 1$. There exists a unique $n \in \emptyset''_s - \emptyset''_{s-1}$ and, since $f(B_{s-1}) \subseteq \emptyset''_s$ we have that $n \geq f(B_{s-1})$. Thus $f(B_s) \subseteq \emptyset''$. If we let $s_0 = t_{s'-1}$ then we have by the induction hypothesis on (2) that $B_{s_0} \subseteq B_s$ and that $f(B_{s_0})$ is of length $\geq s' - 2$ so that $f(B_s)$ is of length $\geq s' - 1$. It is clear that for all $s'' > s$, $B_s \subseteq B_{s''}$.

Now suppose that case b) of the construction applies at stage $s$. That (1) holds at stage $s$ is immediate since we do not define any values $f(\tau)$ at stage $s$. Let $\tau'$ be the longest initial segment of $B_{s-1}$ such that $f(\tau') \subseteq \emptyset''$. Let $s_0$ be such that $\tau' = B_{s_0}$. Let $t_0$ and $\tau_1$ be as defined at stage $s_0 + 1$, where $t_0$ is the leftmost of these two strings. At stage $s_0 + 1$ we defined $f(\tau_1) = f(B_{s_0}) \cdot 1$ and, in performing the instructions for the construction at stage $s$, we shall find that $\tau_1$ is the longest string $\tau$ such that $f(\tau) \subseteq \emptyset''$. Thus $B_s$ is strictly to the right of $B_{s-1}$. That (4) holds at stage $s$ then follows by the induction hypothesis on (1) and since we shall not have defined values $f(\tau''_s)$ for $\tau''_s$ compatible with $\tau_1$ subsequent to stage $s_0 + 1$. Now suppose that $s = t_{s'}$ for some $s' \geq 1$. If $\{n\} = \emptyset''_s - \emptyset''_{s-1}$ then $n \geq f(B_{s_0})$. Thus $f(B_s) \subseteq \emptyset''$. We have by the induction hypothesis on (2) that $B_{s'-1} \subseteq B_s$ and that $f(B_{s'-1})$ is of length $\geq s' - 2$ so that $f(B_s)$ is of length $\geq s' - 1$. It is clear that for all $s'' > s$, $B_s \subseteq B_{s''}$.

In order to show that $B$ is a set of minimal degree we prove, by induction on $e$, that for all $e$ the following statement holds:

\[(\dagger_e) \quad T_e = \lim_{s \to \infty} T_e^{s(e)} \text{ exists and (a.e. } s)[n_{t_e} \geq e].\]

Clearly the statement holds for $e = 0$. Suppose that $e \geq 0$ and that the result holds for all $e' \leq e$. Let $s(e)$ be the least integer such that $(\forall s \geq s(e))[T_{e'}^{s(e)} = T_{e'}^{s(e)}]$ and $(\forall s > s(e))[n_{t_{e'}} \geq e]$. Clearly $(\dagger_{e+1})$ holds if there do not exist at least two $s > s(e)$ such that $n_{t_e} = e$ so suppose otherwise and let $s(e+1)$ be the second such. Choose $\alpha$ such that $T_{e'}^{s(e)}(\alpha) = B_{t_{s(e+1)}}$. Then $T_{e+1}^{s(e+1)}(\alpha) = T_{e+1}^{s(e+1)}(\alpha \ast \gamma)$ for all $\gamma \in 2^{<\omega}$ and, for all $s \geq s(e+1)$, any splitting on $T_{e+1}^{s(e)}$ extending $B_{t_e}$ is also on $T_{e+1}^{s(e+1)}$. Thus for all $s > s(e+1)$, $n_{t_e} \geq e + 1$ and statement $(\dagger_{e+1})$ holds.

Given $e \geq 0$ let $s(e+1)$ be the least integer such that $(\forall s \geq s(e+1))[T_{e+1}^{s(e+1)} = T_{e+1}^{s(e+1)}]$ and $(\forall s > s(e+1))[n_{t_e} \geq e + 1]$. For all $s \geq s(e+1)$, $B_{t_e}$ lies on $T_{e+1}$ so that $B$ lies on $T_{e+1}$. If $T_{e+1}$ is a $\Psi_e$ splitting tree and $\Psi^B_e$ is total then, by the standard argument,
\( B \leq_T \Psi^B_e \). So suppose otherwise. Then there does not exist any \( \Psi_e \) splitting extending \( B_{t(e+1)-1} \) on \( T_e \). If \( \Psi^B_e \) is total it is therefore computable.

**Remark:** the same argument shows that for every degree \( a \geq 0' \) there is a minimal degree \( b \) such that \( b \) is c.e. in \( a \) and \( a \vee b = a' \). By taking \( a = 0^{(n)} (n > 0) \), it follows that for every \( n > 0 \) there is a properly \( \Sigma_{n+1} \) minimal degree.

**References**


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