Exactly Solvable Potentials in Quantum Mechanics*

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Quantum mechanics is the theory developed in the early 20^{th} century in order to understand and describe processes on the atomic scale (photo-electric effect, atomic spectra, wave-particle duality, etc). Some basic principles of classical physics (such as classical mechanics, Maxwell's electro-magnetism) break down at this scale. For the student with no previous knowledge of quantum mechanics, this project can be a mechanism for learning the basic physical and mathematical principles.

The main thrust of this project is to study the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + V(\mathbf{r})\psi$$

We can *separate out* the time to get the *time independent Schrödinger equation*:

$$\psi(\mathbf{r}, t) = \varphi(\mathbf{r})e^{-iEt/\hbar} \quad \Rightarrow \quad \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\varphi(\mathbf{r}) = E\varphi(\mathbf{r}),$$

where E is the *energy* of the particle. Each quantum mechanical system is characterised by the *potential energy function* $V(\mathbf{r})$. Mathematically, the time independent Schrödinger equation (together with some *boundary conditions*) is an *eigenvalue problem*, whose *spectrum* should coincide with the measured energy spectrum of the *physical* quantum mechanical system. For most potentials $V(\mathbf{r})$, it is not possible to calculate *exact* formulae for eigenvalues and eigenfunctions, so we need to resort to perturbation techniques or numerical evaluations. However, there are physically relevant potential functions for which exact formulae *can* be calculated, such as the *quantum harmonic oscillator* and the *hydrogen atom*. These examples possess *symmetries* and *ladder operators*, which are used to calculate these formulae.

These are simple examples of *exactly solvable* Schrödinger equations, which are, in turn, examples of **quantum integrable systems**. This project is concerned with general techniques for studying such systems. In particular the student will study such topics as Schrödinger's "factorisation method", ladder operators, commuting operators (symmetries) and the separation of variables.

This project is closely related to the various "Integrable Systems" modules (Soliton Theory, Discrete Integrable Systems, etc), one of which is available as a 4MM module each year. However, the project can be tailored to the background of most students, regardless of whether they have studied any of these.

There are numerous books on Quantum Mechanics. The ones below are relatively simple to read. For physical motivation and non-technical background reading, try [2].

References

[1] S. Brandt and H.D. Dahmen. *The Picture Book of Quantum Mechanics.* Springer, Berlin, 1995.

[2] John Gribbin. *In Search of Schrödinger's Cat. Quantum Physics and Reality.* Corgi Books, London, 1984.

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