MATH4421 – PROJECT ON INTEGRABLE DYNAMICAL MAPPINGS

Frank Nijhoff, room 9.20c

In the theory of dynamical systems one often investigates dynamical *mappings*, i.e. systems of the form

$$\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n)$$

(where $\boldsymbol{x}$ respresents some set of variables, e.g. a scalar variable x or a 2-vector $\boldsymbol{x} = (x, y)$ or a collection of more variables). The function f describes the iteration step from n to $n + 1$, and this is like a discrete step in time. Often the dynamical mappings of this type have very complicated behaviour, like the famous *logistic map*, i.e.

$$x_{n+1} = \mu x_n(1 - x_n) \quad , \quad \mu \text{ a parameter,}$$

which exhibits complicated behaviour (period doubling, Feigenbaum universality, chaos, etc.). This is, in fact, the generic situation and most dynamical mappings, consequently, cannot be exactly solved. However, there exist exceptional circumstances where dynamical mappings do become amenable to exact and rigorous methods, and these are mostly the cases in which they become *exactly integrable*. Nonetheless, even in these cases the dynamics is far from trivial and hides very interesting mathematical structures.

In 1988 a very general family of *integrable* dynamical mappings of the (x, y)-plane was constructed by Quispel, Roberts and Thompson (the socalled QRT mappings), which are given by the system of equations:

$$\begin{aligned} x_{n+1} &= \frac{f_1(y_n) - f_2(y_n)x_n}{f_2(y_n) - f_3(y_n)x_n} \\ y_{n+1} &= \frac{g_1(x_{n+1}) - g_2(x_{n+1})y_n}{g_2(x_{n+1}) - g_3(x_{n+1})y_n} \end{aligned}$$

where f_1, f_2, f_3 and g_1, g_2, g_3 are all some quartic polynomials of their arguments. This dynamical mapping has some remarkable properties, one of the most important being the fact that there exists an *exact invariant*, i.e. a nontrivial nice function $I(x_n, y_n)$ that does not change under the mapping, i.e. for which $I(x_{n+1}, y_{n+1}) = I(x_n, y_n)$. For the mapping given above such a function $I(x, y)$ can be found (it is a rational function of x and y), and the dynamics takes place on a *general biquadratic* curve, i.e. a geometrical object given by a formula of the form:

$$ax^2y^2 + bx^2y + cxy^2 + dx^2 + ey^2 + fxy + gx + hy = \text{constant}$$

where the coefficients $a, b, c, \ldots$ are related to the coefficients in the mapping. The existence of an invariant is very special and for most dynamical mappings it is impossible to find one! The invariant helps you in principle to solve the mapping in closed form, but even that is nontrivial and requires some really interesting mathematics.

In spite of the fact that the QRT map has generated a considerable amount of interest, and its importance as a starting point for the construction of other types of integrable mappings (the so-called *discrete Painlevé equations*) is well-established, many of its features are still unknown.

In the project we are going to study exactly integrable mappings, in particular the QRT mapping given above. Doing this analytically requires some theory of integrable systems in general (i.e. connections with the theory of *solitons*). Furthermore, one needs to learn something about a special class of functions, the so-called *elliptic functions* which have very beautiful properties. In addition, in studying the dynamics of mappings like the QRT map, some programs can be used to calculate the orbits and phase picture of the system, as well as some algebraic manipulation packages (Mathematica or MAPLE) to perform some of the theoretical calculations involved. Some aspects of the general theory of dynamical systems and of function theory (with connections to geometry) will come into play as well. However, in order to do the project, any of this will be studied from scratch and no particular pre-knowledge (except the material from standard Mathematics modules) is needed.

Some literature:

1. A. Veselov, Integrable Maps, *Russ. Math. Surv* **46** (1991) 1-55.
2. G.R.W. Quispel, J.A.G. Roberts and C. Thompson, Integrable Mappings and Soliton Equations, *Phys. Lett.* **A126** (1988) 419–421; *Physica* **D34** (1989) 183–192.
3. V. Papageorgiou, F.W. Nijhoff and H.W. Capel, Integrable Mappings and Nonlinear Integrable Lattice Equations, *Phys. Lett.* **147A** (1990) 106–114.
4. B. Grammaticos, F. Nijhoff and A. Ramani, Discrete Painlevé Equations, in: *The Painlevé property: one century later*, (Springer Verlag, 1999) pp. 413–516.
5. H. McKean and V. Moll, Elliptic Curves, (Cambridge Univ. Press, 1997).
6. A. Iatrou and J.A.G. Roberts, Integrable mappings of the plane preserving biquadratic invariant curves, J. Phys. A: Math. Gen. **34** (2001) 6617–6636.