MATH4421 – PROJECT QUANTUM DISCRETE SYSTEMS

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This project does not pre-suppose knowledge of Quantum Mechanics, learning the essentials of which forms part of the project.

In recent years the dynamics of integrable discrete-time dynamical systems has undergone a revolutionary development. Dynamical systems theory often deals with discrete evolutionary systems of the form:

$$\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n)$$

(where $\boldsymbol{x}$ respresents some set of variables, e.g. a scalar variable x or a 2-vector $\boldsymbol{x} = (x, y)$ or a collection of more variables). The function f describes the iteration step from n to $n+1$, and this is like a discrete step in time. Often the dynamical mappings of this type have very complicated behaviour, like the famous *logistic map*, i.e.

$$x_{n+1} = \mu x_n(1 - x_n) \quad , \quad \mu \ \text{ a parameter},$$

which exhibits complicated behaviour (period doubling, Feigenbaum universality, chaos, etc.). This is, in fact, the generic situation and most dynamical mappings, consequently, cannot be exactly solved. However, there exist exceptional circumstances where nonlinear dynamical mappings do become amenable to exact and rigorous methods, and these are mostly the cases in which they become *exactly integrable*.

A specific example is given by the McMillan map, defined as the dynamical map from the (x, y)-plane to itself given by the iteration:

$$x_{n+1} = y_n \ , \tag{1a}$$

$$y_{n+1} = -x_n + \frac{\alpha y_n}{1 - y_n^2} \ , \tag{1b}$$

(α being a parameter). Exceptionally, this dynamical map possesses an exact *invariant*, namely the function:

$$I_n = (1 - x_n^2)(1 - y_n^2) + \alpha x_n y_n \ , \tag{2}$$

for which it holds that $I_{n+1} = I_n$ as a consequence of the mapping given by (1). Furthermore, the mapping is area-preserving, which in this case implies that the Poisson brackets of any two function $f(x_n, y_n)$, $g(x_n, y_n)$, which is the expression given by

$$\{f\,,\,g\} = \frac{\partial f}{\partial x_n}\frac{\partial g}{\partial y_n} - \frac{\partial g}{\partial x_n}\frac{\partial f}{\partial y_n}$$

is preserved under the transition $(x_n, y_n) \mapsto (x_{n+1}, y_{n+1})$. In particular, setting $f(x, y) = x, g(x, y) = y$ this implies that setting the Poisson brackets:

$$\{x_n\,,\,y_n\} = 1 \ ,$$

remains true for all values of n, consistently with the dynamics given by the map (1).

Remarkably, it turns out that this dynamical map has an exact analogue in Quantum Mechanics. To cut a long story short, this means that the coordinates of the classical mechanical "phase space", (i.e. the space of variables (x_n, y_n)) can be replaced by certain (differential) operators or (infinitely-sized) matrices, whilst the Poisson brackets are replaced by so-called commutator brackets:

$$[\, x_n \, , \, y_n \,] = x_n y_n - y_n x_n = i\hbar \ ,$$

($\hbar$ being the famous Planck constant of quantum theory), in such a way that not only do we have a dynamics in terms of these operators consistent with these commutator brackets, but also there is a "quantum analogue" of the invariant I_n.

These examples of what we nowadays call *integrable quantum mappings*, cf. [Nijhoff et al.], open up a whole new line of investigation within Quantum Mechanics, developing a (so far in its infancy) quantum theory of "integrable" discrete-time systems. Such a theory is expected to have important applications in nano-science and quantum computation theory.

The project is divided into two parts:

1. First part is to learn the basics of Quantum Mechanics. No pre-knowledge is required, as the learning of the essentials of quantum theory forms part of the project. Obviously, there are many textbooks on Quantum Mechanics, cf. e.g. [Schiff, Schwabl] or [Gustafson et al.], but a set of Lecture notes of the module MATH3383 (not taught this year) will be provided as a short and quick introduction to the basic ideas.

2. The second part is to use and apply the required knowledge of Quantum Mechanics to the context of discrete-time quantum systems. This will involve topics such as: Heisenberg and Schrödinger picture, unitary operator of the time-evolution, time-slicing, quantum Yang-Baxter structure and quantum Lax pairs.

Those students who already possess a sufficient knowledge of Quantum Mechanics can embark directly on part 2, in which case later in the project we can look into additional topics such as: connection to path integrals, cf. [Dittrich et al.], and applications in quantum computation theory, cf. [Nielsen et al.].

Some literature:

1. L.I. Schiff, *Quantum Mechanics*, (McGraw-Hill, 1968).

2. F. Schwabl, *Quantum Mechanics*, (Springer Verlag, 2002).

3. S.J. Gustafson and I.M. Sigal. *Mathematical Concepts of Quantum Mechanics*, (Springer, 2003).

4. F.W. Nijhoff, Lecture Notes MATH3383

5. F W Nijhoff, H W Capel and V Papageorgiou, Integrable Quantum Mappings, Phys Rev **A46** (1992) 2155-8.

6. F.W. Nijhoff and H.W. Capel, Quantization of integrable mappings, Springer Lect. Notes in Physics **424** (1993) 187–211.

7. W. Dittrich and M. Reuter, Classical and Quantum Dynamics, (Springer Verlag, 2001).

8. M.A. Nielsen and I.L. Chuang, Quantum Computation and Information, (Cambridge Univ. Press, 2001).