Exact Solutions of Nonlinear Differential Equations*

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In most elementary courses on ordinary or partial differential equations we learn a few specific examples of equations which can actually be solved and get the impression that these are oddments, solved by ad hoc methods, and of no general interest. The only class of equation for which general methods are presented are *linear*, for which we have *separation of variables*, *Laplace* and *Fourier transforms* and so on.

However, all those "ad hoc" methods of a first year course in ordinary differential equations are just examples of a *general* method introduced by the Norwegian mathematician Sophus Lie (1842–99), who published his ideas in "The Theory of Transformation Groups (1880)" and "On Differential Invariants (1884)". These same methods can be applied to partial differential equations, giving rise to "similarity solutions". A particular case of the latter are *travelling wave solutions*.

In recent years, new methods have been discovered. An "accidental discovery" in the late 1960s showed that the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}$$

has remarkable properties with *exact solutions* representing the interaction of several travelling wave solutions. These are known as *multi-soliton solutions* and can be derived by a variety of methods, such as the *inverse scattering transform* (a nonlinear version of the Fourier transform, invented in 1968 in the context of the KdV equation), *Darboux transformation* and *Bäcklund transformation*. This was the birth of modern **integrable systems theory**, which has had tremendous impact in a wide variety of areas of mathematics and mathematical physics, far beyond the original chance discovery and the KdV equation.

This project can be built from a selection of topics from the general area described above, allowing the student to mould the project to suit his/her personal interests and abilities. Specific reading matter would depend upon the choice of topics, but many of the general ideas can be found in the books listed below.

References

[1] P.G. Drazin and R.S. Johnson. *Solitons : an introduction.* CUP, Cambridge, 1989.

[2] A.P. Fordy. A historical introduction to solitons and Bäcklund transformations. In A.P. Fordy and J.C. Wood, editors, *Harmonic Maps and Integrable Systems*, pages 7–28. Vieweg, Wiesbaden, 1994.

[3] P.E. Hydon. *Symmetry Methods for Differential Equations.* CUP, Cambridge, 2000.

[4] N.H. Ibragimov. *Elementary Lie Group Analysis and Ordinary Differential Equations.* Wiley, Chichester, 1999.

*Assignment in Applied Mathematics