MATH 1400 Examples 5

1(a) • $\ddot{x} - \dot{x} - 2x = 0 \qquad x(0) = 0 \quad \dot{x}(0) = 1$

$\lambda^2 - \lambda - 2 = 0 = (\lambda - 2)(\lambda + 1)$

roots are $\lambda = 2, \lambda = -1$

$\rightarrow x(t) = C_1 e^{+2t} + C_2 e^{-t} \qquad \dot{x}(t) = +2C_1 e^{+2t} - C_2 e^{-t}$

$$\left.\begin{array}{l} x(0) = 0 = C_1 + C_2 \\ \dot{x}(0) = 1 = +2C_1 - C_2 \end{array}\right\} \rightarrow C_1 = +\tfrac{1}{3}, \quad C_2 = -\tfrac{1}{3}$$

$\rightarrow x(t) = +\frac{1}{3} e^{+2t} - \frac{1}{3} e^{-t}$

• write $y = \dot{x} = \frac{2}{3} e^{2t} + \frac{1}{3} e^{-t}$; $\dot{y} = \ddot{x} = \dot{x} + 2x$

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; the known solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + \frac{1}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

The equilibrium point is where

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \rightarrow \begin{array}{l} y = 0 \\ 2x + y = 0 \end{array} \qquad y = 0, x = 0 \text{ – the origin.}$$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0 = \lambda^2 - \lambda - 2$

$\rightarrow \lambda = 2, -1$ as above.

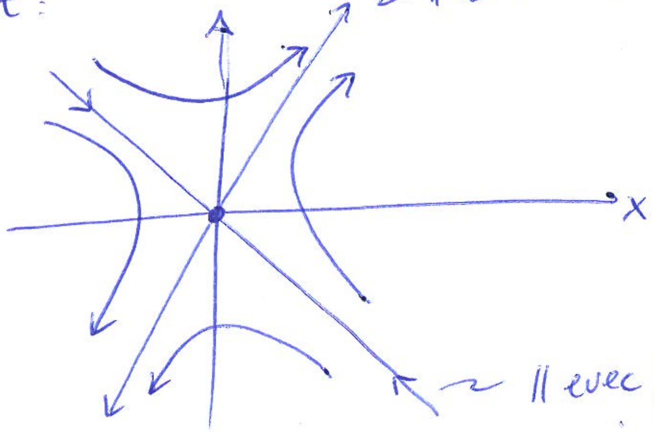
Eigenvectors: $\lambda = 2 : \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \rightarrow$ evec is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = -1 : \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \rightarrow$ evec is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvalues are real, opposite sign $\rightarrow$ <u>saddle</u>.

phase portrait:

$\sim \parallel$ evec $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $e^{2t} \rightarrow$ outwards



$\sim \parallel$ evec $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $e^{-t} \Rightarrow$ inwards

1b • $\ddot{x} + 4\dot{x} + 8x = 0$

$\lambda^2 + 4\lambda + 8 = 0$

$\rightarrow \lambda = \dfrac{-4 \pm \sqrt{16 - 4\times 8}}{2} = -2 \pm \tfrac{1}{2}\sqrt{-16} = -2 \pm 2i$

$\rightarrow x(t) = e^{-2t}(C_1 \cos 2t + C_2 \sin 2t)$

$\dot{x} = -2e^{-2t}(C_1 \cos 2t + C_2 \sin 2t) + e^{-2t}(-2C_1 \sin 2t + 2C_2 \cos 2t)$

$x(0) = 0 = C_1$

$\dot{x}(0) = 1 = 2C_2 \rightarrow C_2 = \tfrac{1}{2}$

$x(t) = \tfrac{1}{2} e^{-2t} \sin 2t$

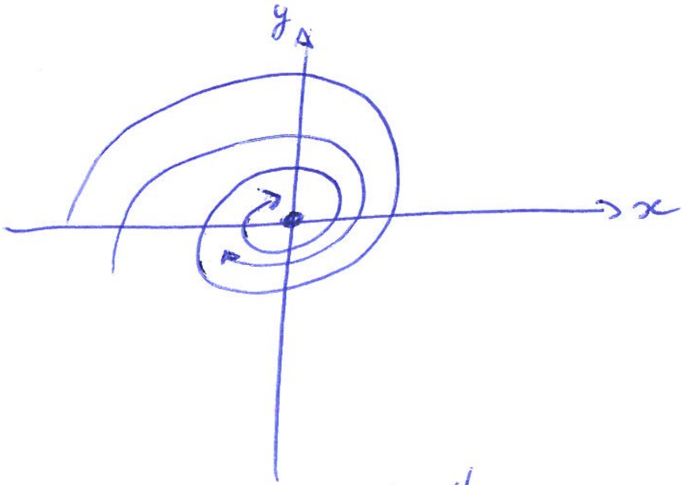
• $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; known solution is

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2} e^{-2t} \sin 2t \\ e^{-2t}(-\sin 2t + \cos 2t) \end{pmatrix}$

The equilibrium point is (0,0), as in 1(a)

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -8 & -4-\lambda \end{vmatrix} = 0 = \lambda^2 + 4\lambda + 8 \rightarrow \lambda = -2 \pm 2i$.

Complex, negative real part $\rightarrow$ stable focus



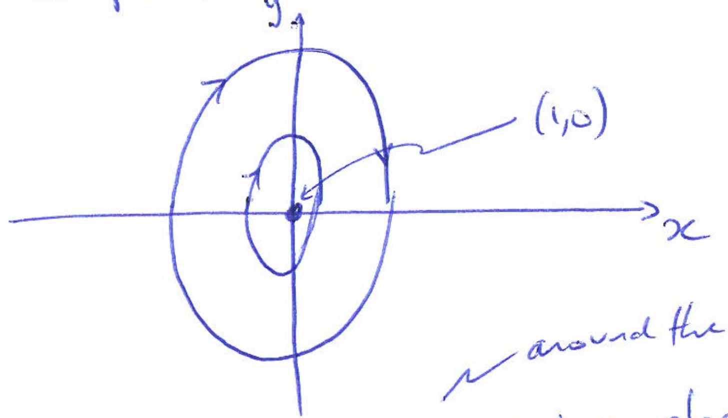
Trajectories spiral in to the eqm point in a clockwise direction since $\dot{x}(=y) > 0$ whenever $y = 0$ (traj move to the right in the upper half-plane).

2 (a) $\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{bmatrix}0 & 1\\ -4 & 0\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}0\\ 4\end{pmatrix}$

The equilibrium point is found by setting $\left.\begin{matrix}\dot x = y = 0\\ \dot y = -4x+4 = 0\end{matrix}\right) \to \begin{matrix}y=0\\ x=1\end{matrix}$

Eigenvalues: $\begin{vmatrix}-\lambda & 1\\ -4 & -\lambda\end{vmatrix} = 0 = \lambda^2+4 \to \lambda = \pm 2i.$

These are complex, zero real part $\Rightarrow$ a <u>centre</u>



Trajectories are closed loops, going clockwise around the point $(1,0)$.

(b) $\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{bmatrix}-1 & 2\\ 6 & 3\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \cancel{0}$. Eqm point is the origin!

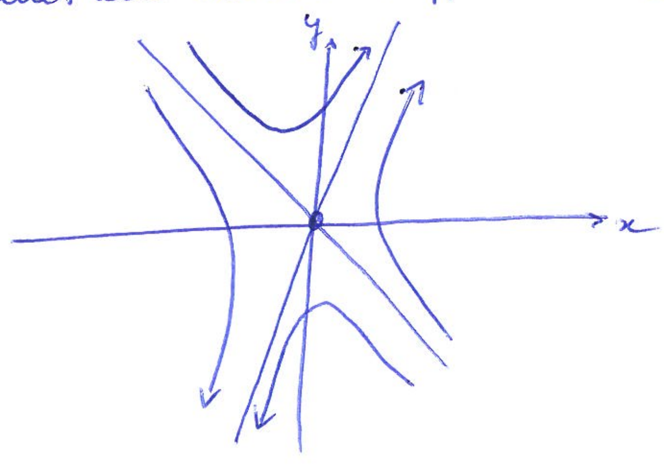
Eigenvalues: $\begin{vmatrix}-1-\lambda & 2\\ 6 & 3-\lambda\end{vmatrix} = 0 = \lambda^2 - 2\lambda - 3 - 12 = \lambda^2 - 2\lambda - 15$
$= (\lambda-5)(\lambda+3)$

$\to \lambda = 5$ evec: $\begin{bmatrix}-6 & 2\\ 6 & -2\end{bmatrix}\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \to \begin{pmatrix}1\\ 3\end{pmatrix}$

$\lambda = -3$ evec: $\begin{bmatrix}2 & 2\\ 6 & 6\end{bmatrix} \to \begin{pmatrix}1\\ -1\end{pmatrix}$

general solution: $\begin{pmatrix}x\\ y\end{pmatrix} = c_1\begin{pmatrix}1\\ 3\end{pmatrix}e^{5t} + c_2\begin{pmatrix}1\\ -1\end{pmatrix}e^{-3t}$

eigenvalues are real and opposite sign $\to$ <u>saddle</u>



3. $\ddot{I} + 2\dot{I} + 5I = 0$

$\lambda^2 + 2\lambda + 5 = 0 \quad \rightarrow \quad \lambda = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

general solution: $I(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$

as $t \to \infty$, $e^{-t} \to 0 \Rightarrow I(t) \to 0$.

$I'' + 2I' + 5I = \omega \sin \omega t$

Particular integral: $I = A \sin \omega t + B \cos \omega t$.

$\rightarrow -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t$
$+ 2(A\omega \cos \omega t - B\omega \sin \omega t) + 5(A \sin \omega t + B \cos \omega t) = \omega \sin \omega t$

$\sin \omega t: \quad (5-\omega^2)A - 2\omega B = \omega$

$\cos \omega t: \quad 2\omega A + (5-\omega^2)B = 0$

① $\times (5-\omega^2)$ + ② $\times 2\omega$: $(5-\omega^2)^2 A + (2\omega)^2 A = \omega(5-\omega^2)$
$= (\omega^4 - 10\omega^2 + 25 + 4\omega^2)A$

$A = \dfrac{\omega(5-\omega^2)}{\omega^4 - 6\omega^2 + 25}$; $B = \dfrac{-2\omega}{5-\omega^2} A = \dfrac{-2\omega^2}{\omega^4 - 6\omega^2 + 25}$

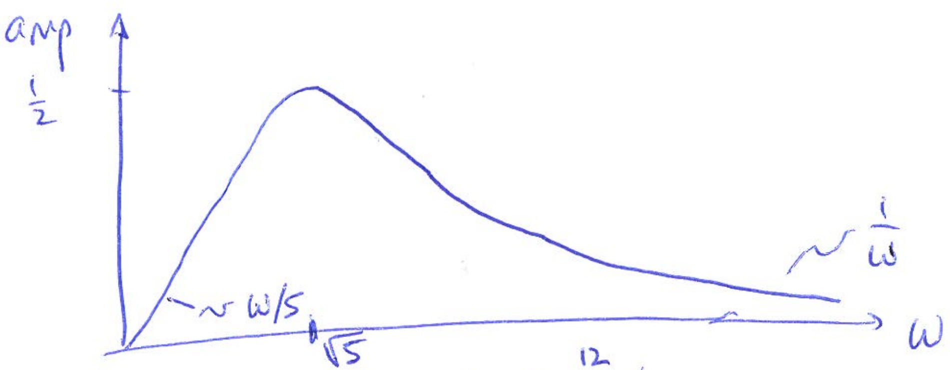
$\rightarrow$ General solution:

$I(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + \dfrac{\omega(5-\omega^2)\sin \omega t - 2\omega^2 \cos \omega t}{\omega^4 - 6\omega^2 + 25}$

as $t \to \infty$, $I \to A \sin \omega t + B \cos \omega t$

this is an oscillation with amplitude $\sqrt{A^2+B^2} = \dfrac{\sqrt{\omega^2(5-\omega^2)^2 + 4\omega^4}}{\omega^4 - 6\omega^2 + \cdots}$

amp. $= \dfrac{\omega}{\sqrt{\omega^4 - 6\omega^2 + 25}}$



$\dfrac{d\,\text{amp}}{d\omega} = \dfrac{\sqrt{\cdots} - \omega \cdot \frac{1}{2\sqrt{\cdots}} \cdot (4\omega^3 - \overset{12}{\cancel{6}}\omega)}{\omega^4 - 6\omega^2 + 25} = 0$

$\Rightarrow \omega^4 - 6\omega^2 + 25 - \cancel{4}(2\omega^4 - \cancel{6}\omega^2) = 0 = -\omega^4 + 25$

$\rightarrow \omega = \sqrt{5}$ at max.

4. $T_1 = 87.7$ $\quad T_2 = 246{,}000$ years $\quad k_1 = \frac{\log 2}{T_1}$ $\quad k_2 = \frac{\log 2}{T_2}$

$$\begin{aligned} \dot{x} &= -k_1 x \\ \dot{y} &= k_1 x + -k_2 y \end{aligned} = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

note $k_1 \gg k_2$

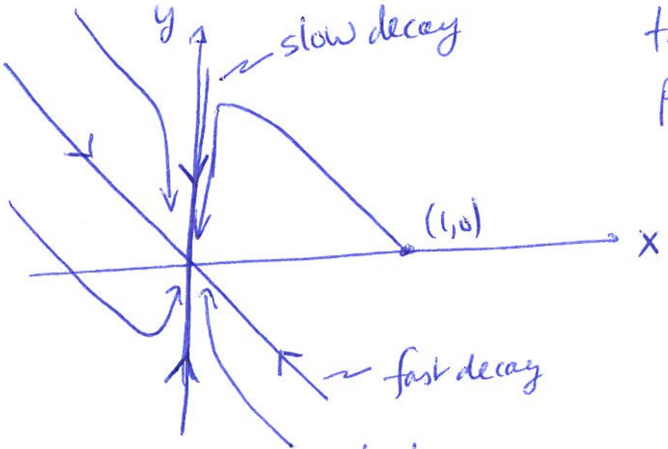
(a) equilibrium point: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: eigenvalues are $-k_1, -k_2$

eigenvectors: $-k_1$: $\begin{bmatrix} 0 & 0 \\ k_1 & -k_2+k_1 \end{bmatrix} \to \begin{pmatrix} k_1 - k_2 \\ -k_1 \end{pmatrix}$

$-k_2$: $\begin{bmatrix} k_2-k_1 & 0 \\ k_1 & 0 \end{bmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

general solution: $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} +k_1 - k_2 \\ -k_1 \end{pmatrix} e^{-k_1 t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-k_2 t}$

The eqm point is a stable node (two real negative eigenvalues).



trajectories aproach parallel to the slow eigenvector.

(b) $\dot{x} = -k_1 x \to x(t) = e^{-k_1 t}$ (with $x(0) = 1$)

$\dot{y} = k_1 e^{-k_1 t} + -k_2 y \to \dot{y} + k_2 y = k_1 e^{-k_1 t}$ First order linear

$IF = \exp\left(\int k_2 dt\right) = e^{k_2 t}$

$\Rightarrow (y e^{k_2 t})' = k_1 e^{(k_2 - k_1)t}$

$y = \frac{k_1}{k_2 - k_1} e^{-k_1 t} + C_2 e^{-k_2 t}$

$y(0) = 0 = \frac{k_1}{k_2 - k_1} + C_2$

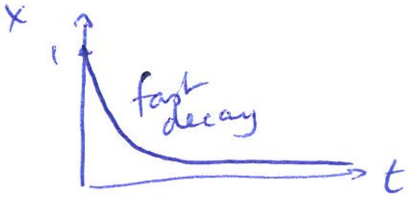
$\to y(t) = \frac{k_1}{k_2 - k_1}\left(e^{-k_1 t} - e^{-k_2 t}\right) = \frac{k_1}{k_1 - k_2}\left(e^{-k_2 t} - e^{-k_1 t}\right)$

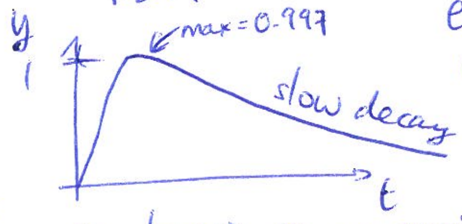
$k_1 \gg k_2$

$e^{-k_1 t}$ decays quickly $\to \max(y) \simeq 1$

or, $\frac{dy}{dt} = \left(\frac{k_1}{k_1 - k_2}\right)\left(-k_2 e^{-k_2 t} + k_1 e^{-k_1 t}\right) = 0$

$t = \frac{1}{k_1 - k_2} \log\left(\frac{k_1}{k_2}\right)$; $y_{max} = \exp\left(\frac{k_2}{k_1 - k_3} \log\left(\frac{k_2}{k_1}\right)\right) = 0.997$





as $t \to \infty$, $x \to 0$ quickly. all of x is converted to y, which $\to 0$ slowly.

5. Cutting a long story short:

$$\ddot{p} - 7\dot{p} + 6p = 12$$

define $q = \dot{p} \rightarrow \quad \dot{q} = \ddot{p} = 7\dot{p} - 6p + 12$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 7 \end{bmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} ; \quad \text{eqm point: } \left.\begin{array}{l} q = 0 \\ -6p = 12 \end{array}\right\} \rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -6 & 7-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6 = 0$

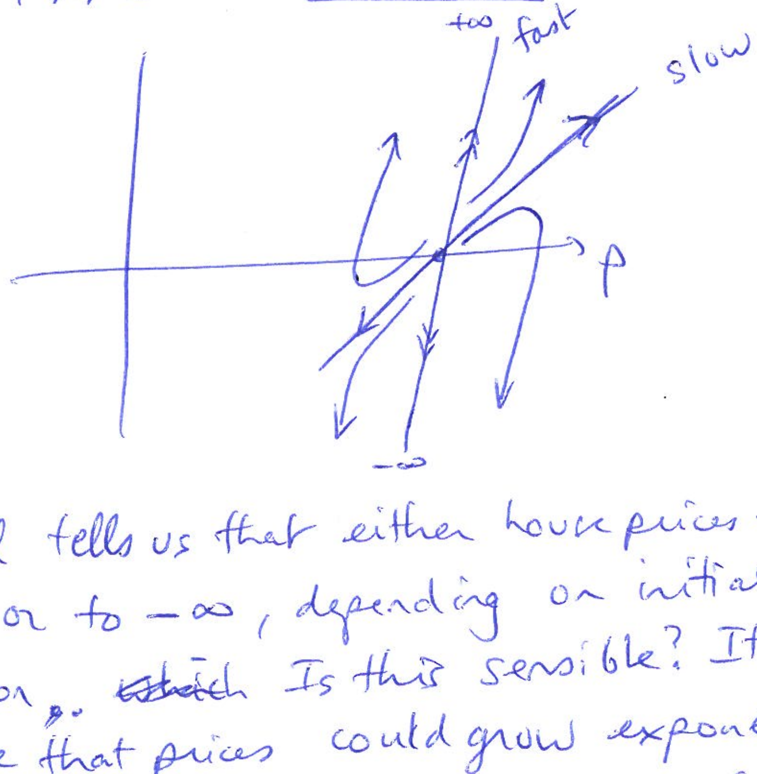
$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 1, \lambda = 6$$

eigenvectors: $\lambda = 1$: $\begin{bmatrix} -1 & 1 \\ -6 & 6 \end{bmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 6$: $\begin{bmatrix} -6 & 1 \\ -6 & 1 \end{bmatrix} \rightarrow \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

eqm point $(2,0)$ is an <u>unstable node</u>



The model tells us that either house prices tend to $+\infty$ or to $-\infty$, depending on initial condition. ~~which~~ Is this sensible? It's plausible that prices could grow exponentially, but once p hits zero, it can't go on (presumably the market has crashed).